II. *On Boomerangs.*

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The attempts that have hitherto been made to explain the flight of a boomerang have in general been of a somewhat fanciful nature.

Exception must be made in the case of such papers as those of Werner Stille, "Versuche und Rechnungen zur Bestimmung der Bahnen des Bumerangs" (Poggendorff, 'Annalen der Physik,' Bd. 147, 1872), and of Edmund Gerlach, "Ableitung gewisser Bewegungsformen geworfener Scheiben aus dem Luftwiderstandsgesetze" ('Zeitschrift des Deutschen Vereins zur Förderung der Luftschifffahrt,' Heft 3, 1886). In the latter, which is the most noticeable contribution to the subject with which I am acquainted, the author gives an explanation in general terms of some of the effects of the air-resistance upon a symmetrical boomerang: he introduces, however, no analytical treatment of the dynamics of the rotating body and neglects entirely all consequences of the important deviations from symmetry which I have subsequently described as "twisting" and "rounding." Without one of these a return flight is, I believe, impossible.

For an account of the native Australian weapons, and in particular those of Victoria, reference should be made to the very complete descriptions given in Brough Smyth's book, 'The Aborigines of Victoria,' vol. 1, pp. 311–318; shorter notices are to be found in books of travel, such as that of Karl Lumholtz, 'Among Cannibals,' p. 50.

Boomerangs may at the outset be divided into two classes—returning and non-returning; it is rather on weapons of the latter of these types that the natives of Australia rely when engaged in war or the chase. A typical returning boomerang (see fig. 1) resembles in general outline an arc of a hyperbola, and is about 80 centims. in length measured along the curve. At the centre, where the dimensions of the cross section (fig. 1′) are greatest, the width is about 7 centims., and the thickness 1 centim.; these dimensions become smaller as the ends are approached.

As a rule two properties are present. In the first place, the transverse section at any point would show that one surface possesses distinctly greater curvature than the other; secondly, the arms of the implement must be slightly twisted (from coincidence with the plane through each of them) after the fashion of the blades of a

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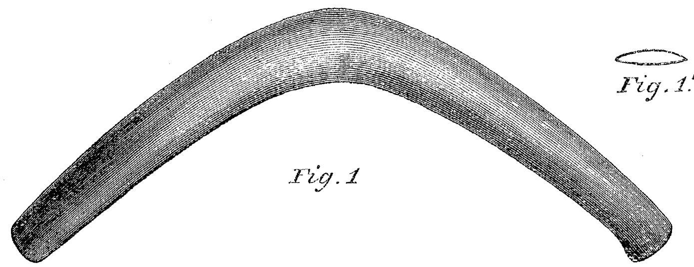
screw propeller or a windmill. The direction of the twist is such that rotation about a normal to the plane tends to set up linear velocity of the boomerang in the direction of the vector representing that rotation. These two peculiarities will in future be referred to as the "rounding" and the "twisting."



A weapon of this type is thrown in a horizontal direction in such a way as to impart considerable rotation in the vertical plane containing its initial direction of motion; the more convex surface is towards the thrower. The plane of rotation leans slowly over to the right (*i.e.*, the vector representing the spin begins to point slightly upwards) and the path curls to the left. The projectile proceeds to describe a loop whose longer diameter is about fifty yards; it gradually rises until it reaches a height which is usually about thirty feet from the ground, travels horizontally for a time, and then gradually sinks to the earth.

The change in the angular motion has throughout the flight continued unaltered in character; the inclination of the plane of rotation to the horizon has steadily diminished from a right angle to zero, and the axis of the spin has veered continually to the left (as seen from above) in such a manner that as long as the linear velocity remains large, the angle between the direction of motion and the plane of rotation is small.

In the accompanying diagram (figs. 2, 3) a plan and elevation of this, the simplest form of path, is given. An attempt is made to indicate the inclination of the axis of rotation by representing at intervals the projection of a line of constant length drawn along that axis.

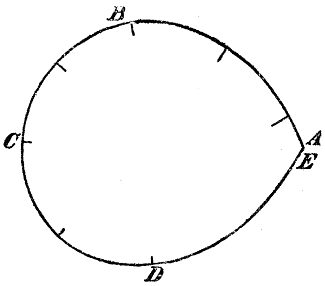
If it be not desired to make so large a loop as that described, it is fairly easy to get the boomerang to describe a circle of thirty-five yards in diameter, without ever rising to more than twelve feet from the earth.

In the more complicated paths, as long as the velocity remains considerable, the manner in which the plane of rotation and the direction of motion change is precisely the same as in the simpler cases; it is the rates of change that differ. The graceful gyrations that a boomerang performs on its downward course, if the linear velocity dies out while it is high in the air, present little or nothing that is new in principle. It is in the explanation of the earlier motion that the problem really lies, and the observation of actual flights makes it clear that their character is deducible when the

two components of angular velocity (denoted subsequently by Ω_1, Ω_2), whose axes lie in the plane of the boomerang, are determined.

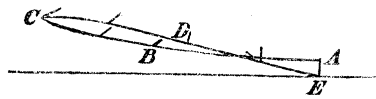
The flight may be regarded as a case of steady motion of which the circumstances gradually vary. It is only with very badly made instruments that small oscillations are at times perceptible; with ordinary boomerangs, the accident of grazing the ground or meeting a sudden puff of wind will not cause visible vibrations.

Fig. 2. Plan.



The scale of this and the following diagrams is 1 : 1000, or 28 yards to 1 inch, approximately.

Fig. 3.



Elevation upon a vertical plane through AC.

Let the plane containing the arms of the boomerang (in future called the primary plane) be taken as that of XY, with the centre of gravity as the origin, and the projection upon this plane of the resultant velocity as OX; OZ is drawn on the more convex side. If then the rectangular components of linear and angular velocity of the body be U, O, W, and Ω_1, Ω_2, Ω_3, it may be observed that W is always small compared with U, and Ω_1, Ω_2 compared with Ω_3. Throughout the motion Ω_1 is positive, Ω_2 negative, and Ω_3 positive. The time of flight is about nine seconds, and the greatest distance fifty yards; the mean values of Ω_1 and Ω_2 may be estimated at one-sixth and *minus* one-third respectively, while in C.G.S. units U is two thousand and Ω_3 is thirty.

The angular velocity θ_3 of the axes is small and positive throughout the motion, except near the conclusion, when it sometimes vanishes and becomes negative.

For theoretical purposes I have regarded the body as replaced by one of extremely thin material with the same general shape and twist; the transverse section will be a circular arc with its convex surface on the same side as the more rounded surface of the wooden weapon.

Experiment shows that if a thin rectangular plate be advancing with velocity v in a direction that is inclined at a small angle α to its plane, the air-pressure produces a

force and a couple about the centre; the mean normal pressure per unit area may be denoted by $\lambda v^{\mu}\alpha$, where λ is a constant that depends on the proportions of the rectangle and μ is a number which has been often assumed to be equal to 2. In his "Experiments in Aerodynamics" ('Smithsonian Contributions to Knowledge,' 1891), S. P. LANGLEY makes this assumption, but if from Table XIV. the value of μ be calculated by comparing the soaring velocities of 24×6 planes, weighing 250 and 1,000 grms. at inclinations of 10° (the smallest inclination quoted), it proves to be 2·7; for square planes of the same weight, inclined at 5°, the index is 2·5. From comparison of the cases of inclination of 2° and 5° of Table XII., the value 3·3 of μ may be deduced. In the course of the following analysis it will be seen that progress is attended with extreme difficulty unless $\mu = 3$, and inasmuch as the constant λ is at our disposal, we shall be justified in taking $\mu = 3$ and choosing λ so as to agree with LANGLEY'S experiments at the mean value of the velocity under discussion. Any error introduced by an incorrect value of μ will be quantitative rather than qualitative.

In addition to the uniform pressure acting on the rectangular plate, there will be a couple whose amount may be taken as $\kappa v^{\nu}\alpha$ per unit area, κ being a constant depending on the dimensions of the plate.* The assumption of a velocity potential would lead to the value $\nu = 2$,† while $\nu = 3$ is suggested by the previous assumption. In order to simplify subsequent proceedings we shall choose the smaller value and deduce the value of κ from LANGLEY'S experiments.

We have now to consider the effect of the air on the slightly distorted thin surface which represents the boomerang, and in order to surmount the difficulties introduced by the fact that the velocities at different points vary, as well as the directions of the normals to the surface, we are driven to make some hypothesis.

Now the effect of the air-pressure upon a plane surface in uniform motion may be obtained by integrating over it, provided that we regard the effect due to any small portion as proportional to the area of that portion.

We therefore assume, as a first approximation, that the contribution from any element of the distorted surface is the same as if the rest of the surface were in the same plane as the element and had the same velocity; that this assumption, in the case of simple distortions, leads to results of the right character, is easily verified.

The determination of κ depends on the fact that if the width of an arm measured in the direction of the velocity of the point in question be c, and if f stand for the

* See THOMSON and TAIT, 'Natural Philosophy,' § 325. The existence of this couple is often stated in the form that the resultant thrust on the plate does not act at the centre of figure. LANGLEY finds (chap. viii., pp. 89–93), that in the case of a square plate the point of application of the resultant pressure, when α does not exceed ten degrees, is at a distance from the centre of figure equal to about one-sixth of the length of the side. He quotes JOËSSEL and KUMMER as having obtained a fifth and a sixth respectively as the value of this ratio.

† LAMB'S 'Hydrodynamics,' p. 185 (3); BASSET'S 'Hydrodynamics,' vol. 1, § 190.

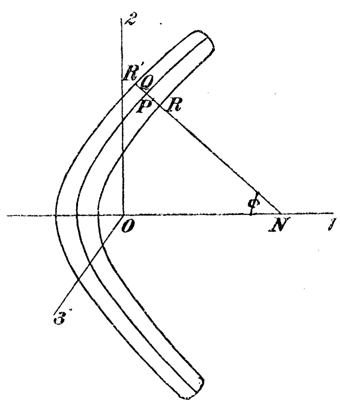
ratio 1 : 5 or 1 : 6 as deduced from experiment, then the couples $\lambda U^3 . cf$ and κU^2 should be the same. This would give a value of κ varying from point to point. We therefore, for convenience, treat κ as constant, and give it the magnitude corresponding to the mean value of c.

It must be realised at the outset that the following analysis does not claim to be more than a first approximation, in which the quantities neglected may be of a tenth of the magnitude of those retained. Our knowledge of the laws of the resistance of the air is not at present great enough for accurate results to be attainable, and I have accordingly not hesitated to neglect small terms in order to effect a material simplication in the mathematical analysis.

It may appear that such processes reduce the method to little more than a qualitative one, but though much may be done by qualitative methods applied to this subject, and all the chief terms may be traced to their source without the use of algebraical symbols, yet, as will soon become clear, the effects of the forces in action are conflicting. It is therefore necessary, in order to obtain results which are qualitatively right, to adopt methods which, although not accurate, have at any rate some approach to quantitative correctness.

We now take axes fixed in the body, 1 and 2 being along and perpendicular to the axis of symmetry in the primary plane.

Fig. 4.



If the velocity at any point xyz have components u, v, w, and the direction cosines of the normal on the convex side there be l, m, n, then the normal pressure in the direction $-l$, $-m$, $-n$ will be

$$\lambda q^{\mu-1}(lu + mv + nw),$$

where l, m, w are small quantities and $q^2 = u^2 + v^2 + w^2$.

The couple per unit area will have moment

$$\kappa q^{\nu-1}(lu + mv + nw)$$

about an axis whose direction cosines are

$$mw - nv, \quad nu - lw, \quad lv - mu,$$

each divided by a quantity differing from q by squares of small quantities. Hence, if $\mu = 3$, $\nu = 2$, the normal force at any point will be

$$\lambda q^2 (lu + mv + w),$$

and the component couples

$$-\kappa v (lu + mv + w), \quad \kappa u (lu + mv + w), \quad 0,$$

where squares of small quantities are omitted.

Now if, due to the "rounding," a transverse line RR′ (fig. 4) through any point P be a circular arc of radius ρ, and if its middle point Q have co-ordinates x, y, then, denoting QP as measured along the inward normal QN by s, the direction cosines of the normal at

$$x + s \cos \phi, \quad y - s \sin \phi$$

will be

$$\frac{s \cos \phi}{\rho}, \quad -\frac{s \sin \phi}{\rho}, \quad 1.$$

In addition to this the line RR′ will, owing to the twisting, be turned about the tangent at Q through an angle which may be taken as $\frac{y}{\tau}$, where τ is a constant length. The superposition of this small distortion on the former will add to the direction cosines terms

$$\frac{y \cos \phi}{\tau}, \quad -\frac{y \sin \phi}{\tau}, \quad 0.$$

Let the linear and angular velocities of the body referred to the axes 1, 2, 3 be

$$u, \ v, \ w, \ \omega_1, \ \omega_2, \ \omega_3,$$

where w is small compared with $(u^2 + v^2)^{\frac{1}{2}}$ and ω_1, ω_2 compared with ω_3. The component velocities u_1, v_1, w_1 of P will then be

$$\begin{aligned} &u - (y - s \sin \phi)\, \omega_3, \\ &v + (x + s \cos \phi)\, \omega_3, \\ &w - (x + s \cos \phi)\, \omega_2 + (y - s \sin \phi)\, \omega_1. \end{aligned}$$

The resultant force due to the pressures will have negligible components parallel to the axes 1, 2, and the force Z along the third axis is the integral over the surface of

$$- \lambda [u^2 + v^2 + 2(x + s \cos\phi) v\omega_3 - 2(y - s\sin\phi) u\omega_3 + \{x^2 + y^2 + 2s(x\cos\phi - y\sin\phi) + s^2\}\omega_3^2]$$
$$\times \left[\left(\frac{s}{\rho} + \frac{y}{\tau}\right)\cos\phi\,\{u - (y - s\sin\phi)\omega_3\} - \left(\frac{s}{\rho} + \frac{y}{\tau}\right)\sin\phi\,\{v + (x + s\cos\phi)\omega_3\} + w - (x + s\cos\phi)\omega_2 + (y - s\sin\phi)\omega_1\right].$$

This may be regarded as the sum of three forces: (1) Z_0 the force which is exerted on a boomerang without distortion, (2) Z_1 due to the rounding, (3) Z_2 due to the twisting.

Denoting by brackets ( ) the operation of taking the mean value over the area of the boomerang, we shall adopt the notation

$$(x^2) = \kappa_2^2, \quad (y^2) = \kappa_1^2, \quad (x^3) = \kappa_3^3, \quad (xy^2) = \kappa_4^3,$$
$$(x^4) = \kappa_5^4, \quad (x^2y^2) = \kappa_6^4, \quad (y^4) = \kappa_7^4,$$
$$(y\sin\phi) = l_1, \quad (xy\sin\phi) = l_2^2, \quad (y^2\cos\phi) = l_3^2,$$
$$(x^2y\sin\phi) = l_4^3, \quad (xy^2\cos\phi) = l_5^3, \quad (y^3\sin\phi) = l_6^3,$$
$$(x^3y\sin\phi) = l_7^4, \quad (x^2y^2\cos\phi) = l_8^4, \quad (xy^3\sin\phi) = l_9^4, \quad (y^4\cos\phi) = l_{10}^4,$$
$$(x^4y\sin\phi + x^3y^2\cos\phi + x^2y^3\sin\phi + xy^4\cos\phi) = l_{11}^5,$$
$$(s^2\cos^2\phi) = m_0^2, \quad (s^2\sin^2\phi) = m'^2_0,$$
$$(s^2x\cos^2\phi) = m_1^3, \quad (s^2x\sin^2\phi) = m'^3_1, \quad (s^2y\sin\phi\cos\phi) = m_2^3,$$
$$(s^2x^2\cos^2\phi) = m_3^4, \quad (s^2x^2\sin^2\phi) = m'^4_3, \quad (s^2xy\sin\phi\cos\phi) = m_4^4,$$
$$(s^2y^2\cos^2\phi) = m_5^4, \quad (s^2y^2\sin^2\phi) = m'^4_5,$$
$$(s^2x^3\sin^2\phi) = m_6^5, \quad (s^2x^2y\sin\phi\cos\phi) = m_7^5, \quad (s^2xy^2\cos^2\phi) = m_8^5,$$
$$(s^2xy^2\sin^2\phi) = m'^5_8, \quad (s^2y^3\sin\phi\cos\phi) = m_9^5.$$

Then for a plane surface we find at once

$$Z_0 = -\lambda S[(u^2 + v^2)w - 2(\kappa_1^2 u\omega_1 + \kappa_2^2 v\omega_2)\omega_3 + \{(\kappa_1^2 + \kappa_2^2)w - (\kappa_3^3 + \kappa_4^3)\omega_2\}\omega_3^2]$$

where S is the area, and terms in s^2 have been omitted since they are multiplied by the small terms w, ω_1, ω_2.

The rounding produces a force

$$Z_1 = -2\lambda\int dS\,\frac{s^2}{\rho}\{(u - y\omega_3)\cos\phi - (v + x\omega_3)\sin\phi\}\{u\omega_3\sin\phi + v\omega_3\cos\phi + (x\cos\phi - y\sin\phi)\omega_3^2\}$$
$$= \frac{2\lambda}{\rho} Su\omega_3\{(m'^2_0 - m_0^2)v + (m'^3_1 - m_1^3 + 2m_2^3)\omega_3\}.$$

Due to the twisting we have

$$
\begin{aligned}
Z_2 = & -\frac{\lambda}{\tau}\int dS\, y\,\{u^2+v^2+2(xv-yu)\,\omega_3+(x^2+y^2)\,\omega_3^2\}\,\{(u-y\omega_3)\cos\phi \\
& \qquad\qquad -(v+x\omega_3)\sin\phi\} \\
= & \ \frac{\lambda}{\tau}\,S\Big[\{l_1 v+(l_2^2+l_3^2)\,\omega_3\}\,(u^2+v^2)+(3l_4^3+2l_5^3+l_6^3)\,\omega_3^2 v+2\,(l_2^2v^2+l_3^2u^2)\,\omega_3 \\
& \qquad\qquad +(l_7^4+l_8^4+l_9^4+l_{10}^4)\,\omega_3^3\Big].
\end{aligned}
$$

The resultant couple about the first axis will be the integral over the surface of

$$
-\kappa v_1\,(lu_1+mv_1+w_1)-(y-s\sin\phi)\,\lambda\,(u_1^2+v_1^2)\,(lu_1+mv_1+w_1).
$$

As before this may be divided into three portions, of which the first, on a plane boomerang, is

$$
\begin{aligned}
F_0 = & -\int dS\,(w-x\omega_2+y\omega_1)\,[\kappa\,(v+x\omega_3)+\lambda y\,\{u^2+v^2+2\,(xv-yu)\,\omega_3 \\
& \qquad\qquad +(x^2+y^2)\,\omega_3^2\}],
\end{aligned}
$$

in which terms in s^2 have been omitted as before.

Therefore,

$$
\begin{aligned}
F_0 = & \ Sw\,(-\kappa v+2\lambda\kappa_1^2\omega_3 u)+S\omega_2\omega_3\,(\kappa\kappa_2^2-2\lambda\kappa_4^3 u) \\
& \qquad -\lambda S\omega_1\,\{\kappa_1^2\,(u^2+v^2)+2\kappa_4^3\omega_3 v+(\kappa_6^4+\kappa_7^4)\,\omega_3^2\}.
\end{aligned}
$$

$$
\begin{aligned}
F_1 = & -\int dS\,\frac{s^2}{\rho}\Big[[\kappa\omega_3\cos\phi+\lambda\,\{2y\omega_3\,(u\sin\phi+v\cos\phi)+2\omega_3^2 y\,(x\cos\phi-y\sin\phi) \\
& \qquad -\sin\phi\,(u^2+v^2+2\omega_3 vx-2\omega_3 uy+x^2\omega_3^2+y^2\omega_3^2)\}] \\
& \quad\times[(u-y\omega_3)\cos\phi-(v+x\omega_3)\sin\phi]\Big] \\
= & -\frac{S}{\rho}\,[\kappa m_0^2 u\omega_3+2\lambda\omega_3\{2m_2^3u^2+(m'^3_1-m_2^3)v^2\}+\lambda(u^2+v^2)\{(m'^3_1+m_2^3)\omega_3+m'^2_0 v\} \\
& \qquad +\lambda v\omega_3^2(3m'^4_3-2m_4^4-2m_5^4+3m'^4_5)+\lambda\omega_3^3(m_6^5-m_7^5-2m_8^5+3m'^5_8+3m_9^5)],
\end{aligned}
$$

while

$$
\begin{aligned}
F_2 = & \ \frac{1}{\tau}\int dS\,[-uy\cos\phi\,\{\lambda y\,(u^2+v^2+2\omega_3 vx+x^2\omega_3^2+y^2\omega_3^2)\} \\
& \qquad +y\,(\omega_3 y\cos\phi+v\sin\phi+\omega_3 x\sin\phi)\,\{\kappa\,(v+\omega_3 x)-2\lambda\omega_3 uy^2\}] \\
= & \ \frac{\kappa S}{\tau}\,\{l_1 v^2+(2l_2^2+l_3^2)\,\omega_3 v+(l_4^3+l_5^3)\,\omega_3^2\} \\
& \qquad -\frac{\lambda S u}{\tau}\,\{l_3^2\,(u^2+v^2)+2\,(l_5^3+l_6^3)\,\omega_3 v+(l_8^4+2l_9^4+3l_{10}^4)\,\omega_3^2\}.
\end{aligned}
$$

The couple about the axis 2 is the integral of

$$(lu_1 + mv_1 + w_1)\,[\kappa u_1 + (x + s\cos\phi)\,\lambda\,(u_1^2 + v_1^2)],$$

leading to

$$G_0 = Sw\,[\kappa u + \lambda\omega_3\,\{2\kappa_2^2 v + (\kappa_3^3 + \kappa_4^3)\,\omega_3\}] - S\omega_1\omega_3\,(\kappa\kappa_1^2 + 2\lambda\kappa_4^3 u)$$
$$- S\omega_2\lambda\,\{\kappa_2^2\,(u^2 + v^2) + 2\kappa_3^3\omega_3 v + (\kappa_5^4 + \kappa_6^4)\,\omega_3^2\},$$

$$G_1 = \frac{S}{\rho}\,[-\kappa\omega_3\,\{m_0'^2 v + (m_1'^3 + m_2^3)\,\omega_3\} + \lambda u\,\{m_0^2\,(u^2 + v^2) + 2\omega_3 v\,(2m_1^3 + m_2^3 - m_1'^3)$$
$$+ 3\omega_3^2\,(m_3^4 + m_5^4) - 2\,(m_4^4 + m_3'^4)\,\omega_3^2\}],$$

$$G_2 = -\frac{\kappa S}{\tau}\,\{l_1 uv + (l_2^2 + 2l_3^2)\,\omega_3 u\} - \frac{\lambda S}{\tau}\,\{(u^2 + v^2)\,(l_2^2 v + l_4^3\omega_3 + l_5^3\omega_3)$$
$$+ 2\omega_3\,(l_4^3 v^2 + l_5^3 u^2) + (3l_7^4 + 2l_8^4 + l_9^4)\,\omega_3^2 v + l_{11}^5\omega_3^3\}.$$

The equations of angular motion are

$$A\dot{\omega}_1 - (B - C)\,\omega_2\omega_3 = F,$$
$$B\dot{\omega}_2 - (C - A)\,\omega_3\omega_1 = G,$$
$$C\dot{\omega}_3 - (A - B)\,\omega_1\omega_2 = 0.$$

Neglecting the product $\omega_1\omega_2$, we see that ω_3 may be replaced by n, a constant. Also, for a thin flat body, C is sensibly equal to A + B, and if m be the mass per unit area, we have

$$A = Sm\kappa_1^2, \qquad B = Sm\kappa_2^2,$$

so that our equations become

$$Sm\kappa_1^2\,(\dot{\omega}_1 + n\omega_2) = F,$$
$$Sm\kappa_2^2\,(\dot{\omega}_2 - n\omega_1) = G.$$

We shall first of all discuss the motion of an undistorted boomerang free from the action of gravity. If we revert to our former axes OX, OY, OZ, of which OX is the projection on the primary plane of the direction of motion, we shall obtain as the equations of translation,

$$\dot{U} + w\Omega_2 = 0,$$
$$-w\Omega_1 + U\theta_3 = 0,$$
$$m\,(\dot{w} - U\Omega_2) = \frac{Z_0}{S}.$$

Hence, neglecting squares, U is constant, and θ_3, the angular velocity of the axes,

is zero. Thus we are justified in replacing u, v by $\mathrm{U}\sin nt$, $\mathrm{U}\cos nt$. We shall then have

$$\left.\begin{aligned}\omega_1 &= \Omega_1 \sin nt - \Omega_2 \cos nt\\ \omega_2 &= \Omega_1 \cos nt + \Omega_2 \sin nt\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and on multiplying the former rotation equations by

$$\frac{\sin nt}{\kappa_1^2}, \quad \frac{\cos nt}{\kappa_2^2},$$

and adding, we get

$$m\,(\dot{\Omega}_1 + 2n\Omega_2) = \mathrm{F}_0 \frac{\sin nt}{\mathrm{S}\kappa_1^2} + \mathrm{G}_0 \frac{\cos nt}{\mathrm{S}\kappa_2^2} = \mathrm{P}_0 \text{ say},$$

$$m\,(\dot{\Omega}_2 - 2n\Omega_1) = -\mathrm{F}_0 \frac{\cos nt}{\mathrm{S}\kappa_1^2} + \mathrm{G}_0 \frac{\sin nt}{\mathrm{S}\kappa_2^2} = \mathrm{Q}_0,$$

$$m\,(\dot{w} - \mathrm{U}\Omega_2) = \frac{\mathrm{Z}_0}{\mathrm{S}} = \mathrm{R}_0.$$

Now

$$\begin{aligned}
\mathrm{P}_0 = {} & 2\lambda n \mathrm{U} w - \lambda \mathrm{U}^2 \Omega_1 + \frac{\kappa}{2}\left(\frac{\kappa_2^2}{\kappa_1^2} + \frac{\kappa_1^2}{\kappa_2^2}\right) n\Omega_2 - \frac{\lambda}{2}\left(\frac{\kappa_6^4 + \kappa_7^4}{\kappa_1^2} + \frac{\kappa_5^4 + \kappa_6^4}{\kappa_2^2}\right) n^2 \Omega_1 \\
& + \lambda \frac{\kappa_3^3 + \kappa_4^3}{\kappa_2^2} n^2 w \cos nt + \frac{\kappa}{2}\left(\frac{1}{\kappa_2^2} - \frac{1}{\kappa_1^2}\right) w \mathrm{U} \sin 2nt \\
& + \frac{\kappa n}{2}\left(\frac{\kappa_2^2}{\kappa_1^2} - \frac{\kappa_1^2}{\kappa_2^2}\right)(\Omega_1 \sin 2nt - \Omega_2 \cos 2nt) \\
& - \frac{\lambda}{2} \kappa_4^3 n \mathrm{U} \left[\Omega_1 (\cos nt - \cos 3nt)\left(\frac{2}{\kappa_1^2} + \frac{1}{\kappa_2^2}\right) + \Omega_2 \sin nt \left(\frac{3}{\kappa_1^2} - \frac{1}{\kappa_2^2}\right)\right. \\
& \qquad \left. - \Omega_2 \sin 3nt \left(\frac{1}{\kappa_1^2} + \frac{1}{\kappa_2^2}\right)\right] \\
& - \frac{\lambda}{2} \frac{\kappa_3^3}{\kappa_2^2} n \mathrm{U} \left[\Omega_1 (3 \cos nt + \cos 3nt) + \Omega_2 (\sin nt + \sin 3nt)\right] \\
& + \frac{\lambda n^2}{2}\left(\frac{\kappa_6^4 + \kappa_7^4}{\kappa_1^2} - \frac{\kappa_5^4 + \kappa_6^4}{\kappa_2^2}\right)(\Omega_1 \cos 2nt + \Omega_2 \sin 2nt).
\end{aligned}$$

Similarly

$$\mathrm{Q}_0 = -\lambda \mathrm{U}^2 \Omega_2 + \frac{\kappa \mathrm{U} w}{2}\left(\frac{1}{\kappa_1^2} + \frac{1}{\kappa_2^2}\right) - \frac{\kappa}{2}\left(\frac{\kappa_2^2}{\kappa_1^2} + \frac{\kappa_1^2}{\kappa_2^2}\right) n\Omega_1 - \frac{\lambda}{2}\left(\frac{\kappa_6^4 + \kappa_7^4}{\kappa_1^2} + \frac{\kappa_5^4 + \kappa_6^4}{\kappa_2^2}\right) n^2 \Omega_2,$$

together with terms whose coefficients involve circular functions of nt.

Also

$$\begin{aligned}
\mathrm{R}_0 = {} & -\lambda \{\mathrm{U}^2 + (\kappa_1^2 + \kappa_2^2) n^2\} w + \lambda (\kappa_1^2 + \kappa_2^2) n \mathrm{U} \Omega_1 \\
& - \lambda (\kappa_1^2 - \kappa_2^2) n \mathrm{U} (\Omega_1 \cos 2nt + \Omega_2 \sin 2nt) \\
& + \lambda (\kappa_3^3 + \kappa_4^3) n^2 (\Omega_1 \cos nt + \Omega_2 \sin nt).
\end{aligned}$$

If all the terms be collected on the same side there result three equations of the form

$$a_1\Omega_1 + b_1\Omega_2 + c_1 w = 0,$$
$$a_2\Omega_1 + b_2\Omega_2 + c_2 w = 0,$$
$$a_3\Omega_1 + b_3\Omega_2 + c_3 w = 0,$$

in which the coefficients may contain the operator d/dt, or circular functions of the time.

The equations giving the motion under gravity of a boomerang with the two distortions will differ from these in three ways.

First of all, if $l'm'n'$ be the direction cosines of a line drawn vertically downwards, the equations of translation will be

$$m(\dot{\mathrm{U}} + w\Omega_2) = mgl',$$
$$m(-w\Omega_1 + \mathrm{U}\theta_3) = mgm',$$
$$m(\dot{w} - \mathrm{U}\Omega_2) = \mathrm{R}_0 + \mathrm{R}_1 + \mathrm{R}_2 + mgn',$$

P_1, P_2, Q_1, Q_2, R_1, R_2 bearing to Z_1, Z_2 the same relations as P_0, Q_0, R_0 to Z_0. From the second equation

$$\theta_3 = \frac{gm'}{\mathrm{U}},$$

and from the first

$$\dot{\mathrm{U}} = gl.$$

Now θ_3 in numerical value is comparable with $\frac{1}{4}$, and n with 30, so that the additional terms in (1) due to the consideration of θ_3 will be of negligible magnitude. Again, l' is small when the path is nearly in a horizontal plane; hence, in considering the steady motion corresponding to a particular portion of the path, the change in U need not trouble us.

When the forces due to the distortions—and these will not involve Ω_1, Ω_2, w—are introduced, we obtain

$$a_1\Omega_1 + b_1\Omega_2 + c_1 w = \mathrm{P}_1 + \mathrm{P}_2,$$
$$a_2\Omega_1 + b_2\Omega_2 + c_2 w = \mathrm{Q}_1 + \mathrm{Q}_2,$$
$$a_3\Omega_1 + b_3\Omega_2 + c_3 w = \mathrm{R}_1 + \mathrm{R}_2 - mg\cos\theta,$$

where θ is the angle between the axis of revolution and the upward vertical, and $\dot{\theta}/n$ being small, we shall treat θ in the third equation as constant.

We next regard the right-hand sides of these equations as expanded in the form

$$A_1 + B_1 \sin nt + C_1 \cos nt + D_1 \sin 2nt + E_1 \cos 2nt + F'_1 \sin 3nt + G'_1 \cos 3nt,$$
$$A_2 + B_2 \sin nt + C_2 \cos nt + D_2 \sin 2nt + E_2 \cos 2nt + F'_2 \sin 3nt + G'_2 \cos 3nt,$$
$$A_3 + B_3 \sin nt + C_3 \cos nt + D_3 \sin 2nt + E_3 \cos 2nt,$$

in which A, B, C . . . are constants.

On substituting the values of F_1, F_2, we find

$$A_1 = -\frac{\kappa n U}{2\rho}\left(\frac{m_0^2}{\kappa_1^2} + \frac{m'^2_0}{\kappa_2^2}\right)$$
$$- \frac{\lambda U}{2\tau}\left\{\left(\frac{l_3^2}{\kappa_1^2} + \frac{l_2^2}{\kappa_2^2}\right) U^2 + \left(\frac{l_8^4 + 2l_9^4 + 3l_{10}^4}{\kappa_1^2} + \frac{3l_7^4 + 2l_8^4 + l_9^4}{\kappa_2^2}\right) n^2\right\},$$

$$A_2 = \frac{\lambda U}{2\rho}\left\{\left(\frac{m'^2_0}{\kappa_1^2} + \frac{m_0^2}{\kappa_2^2}\right) U^2 + \left(\frac{3m'^4_3 - 2m_4^4 - 2m_5^4 + 3m'^4_5}{\kappa_1^2} + \frac{3m_3^4 - 2m'^4_3 - 2m_4^4 + 3m_5^4}{\kappa_2^2}\right) n^2\right\}$$
$$- \frac{\kappa}{2\tau}\left\{\frac{2l_2^2 + l_3^2}{\kappa_1^2} + \frac{l_2^2 + 2l_3^2}{\kappa_2^2}\right\} nU,$$

$$A_3 = \frac{\lambda n}{\tau}\{2(l_2^2 + l_3^2) U^2 + (l_7^4 + l_8^4 + l_9^4 + l_{10}^4) n^2\} - mg \cos \theta.$$

Our equations may now be satisfied by the infinite series

$$\Omega_1 = \alpha_1 + \beta_1 \sin nt + \gamma_1 \cos nt + \delta_1 \sin 2nt + \ldots$$
$$\Omega_2 = \alpha_2 + \beta_2 \sin nt + \gamma_2 \cos nt + \ldots$$
$$w = \alpha_3 + \beta_3 \sin nt + \gamma_3 \cos nt + \ldots$$

which are convergent, since the ratio of the coefficients of $\sin \overline{r+1}\, nt$ and $\cos \overline{r+1}\, nt$ to those of $\sin rnt$ and $\cos rnt$ proves to be ultimately comparable with κ/mr.

If we adopt the notation

$$U^2 + \left(\frac{\kappa_6^4 + \kappa_7^4}{\kappa_1^2} + \frac{\kappa_5^4 + \kappa_6^4}{\kappa_2^2}\right)\frac{n^2}{2} = U_1^2,$$
$$U^2 + (\kappa_1^2 + \kappa_2^2) n^2 = U_2^2,$$
$$m - \frac{\kappa}{4}\left(\frac{\kappa_2^2}{\kappa_1^2} + \frac{\kappa_1^2}{\kappa_2^2}\right) = m_1,$$
$$\frac{1}{\kappa_1^2} + \frac{1}{\kappa_2^2} = \frac{1}{\kappa'^2},$$

the non-circular terms on the left-hand sides become

$$m\dot{\Omega}_1 + \lambda U_1^2 \Omega_1 + 2m_1 n \Omega_2 - 2\lambda n U w,$$
$$-2m_1 n \Omega_1 + m\dot{\Omega}_2 + \lambda U_1^2 \Omega_2 - \frac{\kappa U w}{\kappa'^2},$$
$$-\lambda(\kappa_1^2 + \kappa_2^2) n U \Omega_1 - mU\Omega_2 + m\dot{w} + \lambda U_2^2 w.$$

Hence the substitution of the series for Ω_1, Ω_2, w gives as the equations for α_1, α_2, α_3

$$\left.\begin{aligned}&\lambda U_1^2\alpha_1 + 2m_1 n\alpha_2 - 2\lambda n U\alpha_3 + f_1 = A_1\\ &-2m_1 n\alpha_1 + \lambda U_1^2\alpha_2 - \frac{\kappa U\alpha_3}{\kappa'^2} + f_2 = A_2\\ &-\lambda(\kappa_1^2 + \kappa_2^2) nU\alpha_1 - mU\alpha_2 + \lambda U_2^2\alpha_3 + f_3 = A_3\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2),$$

where f_1, f_2, f_3 are linear functions of $\alpha, \beta, \gamma \ldots$ in which the constant coefficients all contain λ or κ, but not the numerically more important quantity m, as a factor.

In order, then, to obtain a steady motion about which minute oscillations are going on, we neglect the terms f_1, f_2, f_3 in our first approximation. This is equivalent to taking two points on the path at an interval corresponding to a number of complete revolutions (say twelve), and asserting that the angular change in the axis of rotation is that due to the non-periodic portion or mean of the couples in action during that period.

Stability.

For steady motion to be possible it is necessary that the values of Ω_1, Ω_2, w, given by the equations

$$\begin{aligned}&m\dot{\Omega}_1 + \lambda U_1^2\Omega_1 + 2m_1 n\Omega_2 - 2\lambda n U w = 0\\ &-ln_1 n\Omega_1 + m\dot{\Omega}_2 + \lambda U_1^2\Omega_2 - \frac{\kappa U w}{\kappa'^2} = 0\\ &-\lambda(\kappa_1^2 + \kappa_2^2) nU\Omega_1 - mU\Omega_2 + m\dot{w} + \lambda U_2^2 w = 0\end{aligned}$$

shall be always small.

On inserting numerical values, it appears that this condition is satisfied if the ratio of n to U be large enough to give the determinant

$$\begin{vmatrix} \lambda U_1^2, & 2m_1 n, & -2\lambda n U \\ -2m_1 n, & \lambda U_1^2, & -\dfrac{\kappa U}{\kappa'^2} \\ -\lambda(\kappa_1^2 + \kappa_2^2) nU, & -mU, & \lambda U_2^2 \end{vmatrix}$$

a positive value.

If $2\alpha = \pi - \beta$ where β is small, the motion is unstable unless with actual values, $n > 270$.

When $2\alpha = 120°$ the critical value of n is 26, and when the arms are at right angles, stability is secured when $n = 22$.

These values are rather larger than those found necessary in practice, but their mutual relations are correct. The first time that a beginner attempts it, he can make a boomerang whose arms are at right angles travel steadily, but the more obtuse the

angle, the more difficult is the throwing of the implement, and when $2\alpha = 150^\circ$ or upwards, and the material of which the boomerang is made is light, throwing it against a wind requires skill of a high order.

The values of the constants l_2, m_3, κ_1 . . . have been calculated for boomerangs whose arms are 36 centims. in length and 5 centims. in width, the mass per unit area being five-eighths of a gramme.

When these constants are substituted in the equations of steady motion, it is found that for a boomerang whose arms are at right angles, corresponding to the values

$$\mathrm{U} = 2000, \qquad n = 40, \qquad \kappa\mathrm{U} = 7, \qquad \lambda\mathrm{U}^2 = 5,$$

are the velocities

$$\left.\begin{aligned} \Omega_1 &= -\frac{3.6}{\rho} + \frac{610}{\tau} + 1.9\cos\theta \\ \Omega_2 &= \frac{4}{\rho} - \frac{2200}{\tau} - 6.8\cos\theta \\ w &= \frac{680}{\rho} - \frac{480000}{\tau} - 1600\cos\theta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

If we make $n = 30$, the values of Ω_1, Ω_2, w given by the equations are too large; this is due to the fact that the theoretical limit of stability ($n = 22$) is not sufficiently exceeded.

If the value of $\kappa\mathrm{U}$ be taken as 5 instead of 7 (these being estimated inferior and superior limits of $\kappa\mathrm{U}$ corresponding to $c = 6$, $f = 1/6$, and $c = 7$, $f = 1/5$) there appear

$$\left.\begin{aligned} \Omega_1 &= -\frac{3.2}{\rho} + \frac{270}{\tau} + \cos\theta \\ \Omega_2 &= \frac{2.9}{\rho} - \frac{2100}{\tau} - 5.7\cos\theta \\ w &= \frac{630}{\rho} - \frac{460000}{\tau} - 1400\cos\theta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4).$$

The velocities corresponding to a larger spin

$$\mathrm{U} = 2000, \qquad n = 50, \qquad \kappa\mathrm{U} = 5, \qquad \lambda\mathrm{U}^2 = 5,$$

are

$$\left.\begin{aligned} \Omega_1 &= -\frac{2.8}{\rho} + \frac{100}{\tau} + .4\cos\theta \\ \Omega_2 &= \frac{1.2}{\rho} - \frac{1200}{\tau} - 3.2\cos\theta \\ w &- \frac{240}{\rho} - \frac{215000}{\tau} - 790\cos\theta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5).$$

It is interesting to compare these results with those belonging to a boomerang whose arms include an angle 120°.

Thus, taking

$$2\alpha = 120°, \quad U = 2000, \quad n = 50, \quad \kappa U = 7, \quad \lambda U^2 = 5,$$

we obtain

$$\left.\begin{aligned} \Omega_1 &= -\frac{8.9}{\rho} + \frac{700}{\tau} + 1.3\cos\theta \\ \Omega_2 &= \frac{3.8}{\rho} - \frac{2000}{\tau} - 3.1\cos\theta \\ w &= \frac{720}{\rho} - \frac{380000}{\tau} - 720\cos\theta \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6),$$

while the second estimate of κU, namely 5, yields

$$\left.\begin{aligned} \Omega_1 &= -\frac{10.7}{\rho} + \frac{510}{\tau} + \cos\theta \\ \Omega_2 &= \frac{4.6}{\rho} - \frac{2300}{\tau} - 3.5\cos\theta \\ w &= \frac{860}{\rho} - \frac{440000}{\tau} - 850\cos\theta \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7).$$

The values of ρ and τ in practice are usually comparable with 20 and 800 respectively.

That the form of the equations is correct, at any rate as regards a first approximation, is confirmed by the experience gained in making and throwing upwards of seventy boomerangs of different weights, shapes, and sizes.

If, for example, one of these does not curl sharply enough to the left (*i.e.*, Ω_2 is negative, but not numerically large enough), it is found that increasing the twist (*i.e.*, diminishing τ) will produce the desired effect. A further result will be an increase in Ω_1 and a consequent tendency to "sky;" this may be corrected by making the difference of curvature of the two surfaces more pronounced; a diminution in ρ will thus bring about a diminution of Ω_1.

Some of these implements were made with the express object of verifying particular terms. If there be no twist, and $\theta = \pi/2$, while ρ is not extremely large, Ω_1 is negative and Ω_2 positive; if, on the other hand, ρ is infinite, but τ is finite and positive, Ω_1 is positive and Ω_2 is negative.

From experiments made in this manner, with a somewhat smaller spin than that assumed above, I have deduced the formulæ

$$\left.\begin{aligned} \Omega_1 &= -\frac{5}{\rho} + \frac{200}{\tau} + 2\cos\theta \\ \Omega_2 &= \frac{2}{\rho} - \frac{1200}{\tau} - \frac{\cos\theta}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8),$$

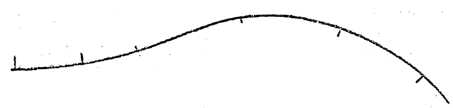
in which, owing to the experimental difficulties, the numerical values of the coefficients may be regarded as lacking in accuracy; they may however be relied upon as, at any rate, of the correct order of magnitude. Of w I have observed nothing except that it does not exceed 500, and is probably smaller and negative.

A comparison of the theoretical results (3) and (4), or (6) and (7), obtained with different data for κU will show that the formulæ, as calculated, must be looked upon as giving only a rough estimate of the motion regarded quantitatively; but, in spite of the calculated value of w being excessive (between 600 and 1200 when $\cos\theta = \frac{1}{3}$), it will be seen that the discrepancies are of the kind that might be anticipated, and that the theoretical equations are qualitatively consistent with the experimental results given in (8).

Another piece of evidence is that furnished by non-returning boomerangs. If it be desired to make an efficient missile that shall travel in as straight a path as possible, it is natural to manufacture a boomerang without twist and with the curvature of the two faces the same. It is this form that many of the cruder Australian weapons possess.

Experiment and theory alike show, however, that if initially θ have a positive value less than a right angle (*i.e.*, the natural method of throwing be adopted), then Ω_1 will be positive and Ω_2 will be negative as long as θ is less than a right angle: when the plane of rotation has reached and passed through the horizontal position Ω_2 remains negative. The shape of the path is indicated in fig. 5, and it will be seen that it is far from straight.

Fig. 5.



Plan.

A path in one vertical plane could be secured by throwing an undistorted weapon with its plane of rotation accurately vertical; the least inclination, however, would grow, and the plane of initial motion be departed from; in any case, except for the reduction in the resistance of the air, the path in a vertical plane would yield no greater range than would be afforded by a spherical missile of the same weight.

We might attain the same end by designing the shape so as to make Ω_1, Ω_2 small when $\theta = 0$, and throwing the boomerang with its plane approximately horizontal. In that case the plane would remain horizontal, and the axis OX in it would soon be pointing in a direction slightly above the tangent to the path; a much longer flight would then be maintained, as the effect of gravity would be balanced by the upward pressure of the air on the lower surface of the projectile.

It is interesting to notice that this is the method that experience has taught the blacks to adopt. Their best non-returning weapons always have strongly developed positive rounding (the more curved surface is uppermost when thrown) and often a

small negative twist; examination of the equations will show that these distortions will combine to produce the required results. An estimate of the efficiency of the shape may be made from the fact that as far as my experience goes, a boomerang of this type may be thrown more than twice as far as a spherical object of the same weight.

In figs. 6, 7 are given the plan and elevation of the path obtained with a boomerang

Fig. 6.

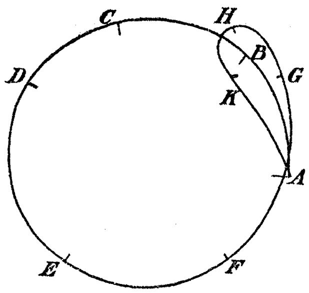
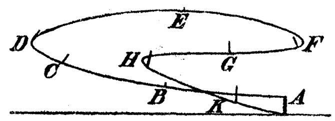


Fig. 7.



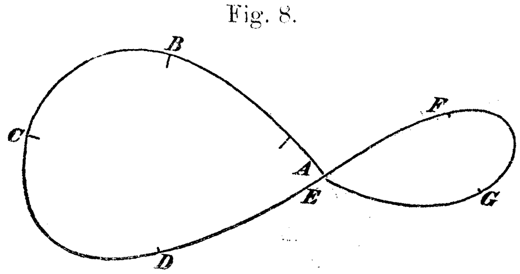
Elevation upon a vertical plane through AC.

designed to continue in its circular route as long as possible. The arms of the implement are at right angles, and the twist and rounding exaggerated a little; the initial plane of rotation is vertical, and as much energy as possible is imparted in the act of throwing, while the aim is slightly uphill.

The numerical value of Ω_2 is somewhat increased and that of Ω_1 diminished, so that when the weapon in its return journey is over the thrower's head, its axis of rotation, instead of being vertical, is inclined a little towards the inner side of the curve that it has described; the forward velocity, though reduced, being still unexpended, the original curve is continued, and the existence of Ω_2 implies that the plane of rotation will tilt slightly upwards and the tendency to fall be overcome.

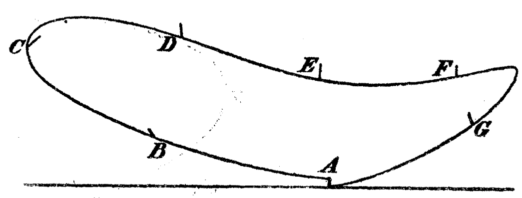
After the end H of this second loop has been reached, the forward velocity has still further diminished, and gravity brings the boomerang, still spinning fast in a nearly horizontal plane, to the ground near the starting-point. I have obtained second loops, which were thirty yards in length when measured horizontally, while, if the point H be high enough in the air, a third loop will be described before the boomerang alights.

Fig. 8.



Plan.

Fig. 9.



Elevation along CAG.

In figs. 8, 9 is represented the flight of a boomerang, of which the arms form an angle which is larger by about thirty degrees than that of the previous case. The axis

Fig. 10.

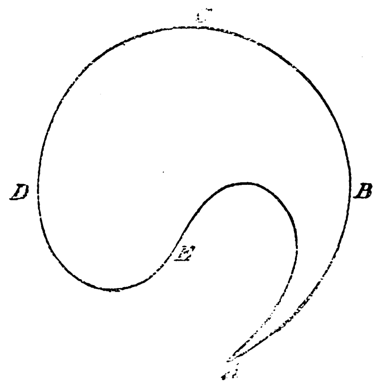
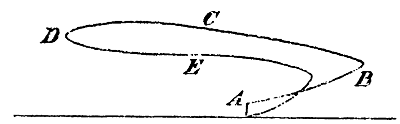
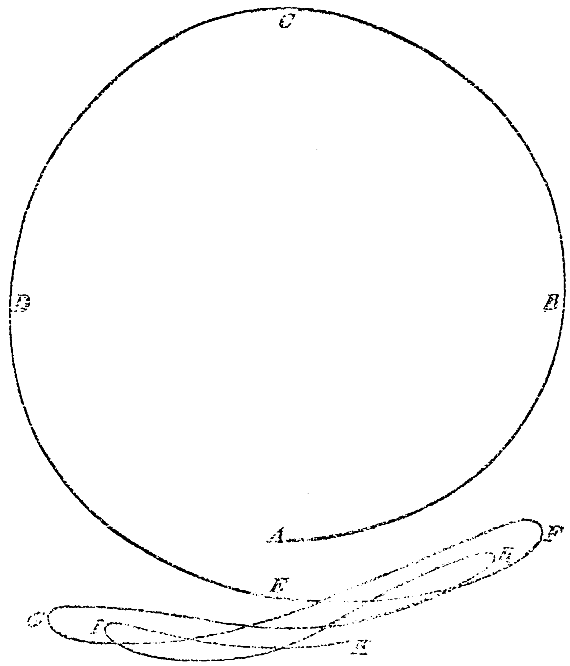


Fig. 11.



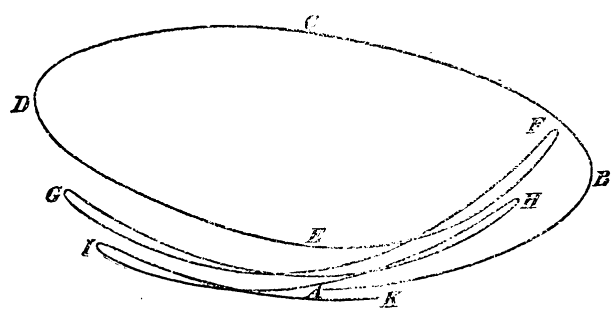
Elevation on the plane through BD.

Fig. 12



Plan. Scale $\frac{1}{1200}$.

Fig. 13.



Elevation on a plane parallel to DB.

of rotation may point at the outset rather upwards, and the initial direction of motion is slightly uphill.

As the theoretical angular velocities indicate, there will be an increase in the value of Ω_1, and this leads to the plane of rotation being horizontal when the implement passes over the thrower's head at E. The angular velocity along the axis OX will then turn the path to the right along EF, while Ω_2 implies that the body rises from E to F. After a short time the forward velocity has diminished so far that the descent from F to the ground is made quite slowly, under the influence of gravity checked by the rotation of the boomerang in a nearly horizontal plane.

Figs. 10, 11 illustrate the magnitude of the changes in the trajectory that are rendered possible by small variations in the shape of the missile. This path was traced by a boomerang which subsequently warped to a slight extent in such a way as to increase the twisting: the natural flight then became the figure of eight of the two previous diagrams.

Through the kindness of Mr. O. ECKENSTEIN, I have recently had the opportunity of seeing and throwing some boomerangs made by him, in which rounding was present, but no twisting; the angle between the arms was considerably more obtuse, the size increased, and the weight doubled.

An examination of the equations (3–7) will show that if the value of $\cos\theta$ be increased, the term due to gravity might be expected to replace for the most part that due to the twisting; further, as the angle between the arms is larger, a given amount of rounding will produce a greater effect in diminishing Ω_1.

When the proportions are rightly chosen, I have not found it difficult to obtain a return path; the plane of rotation is initially inclined at 15° instead of 90° to the horizon, and with a decidedly smaller forward velocity as much spin as possible must be imparted.

In the hands of one accustomed to its use, a boomerang of this type is capable of extremely interesting flights. For the remarkable diagrams (figs. 12, 13) which illustrate one of these, I am indebted to Mr. ECKENSTEIN.

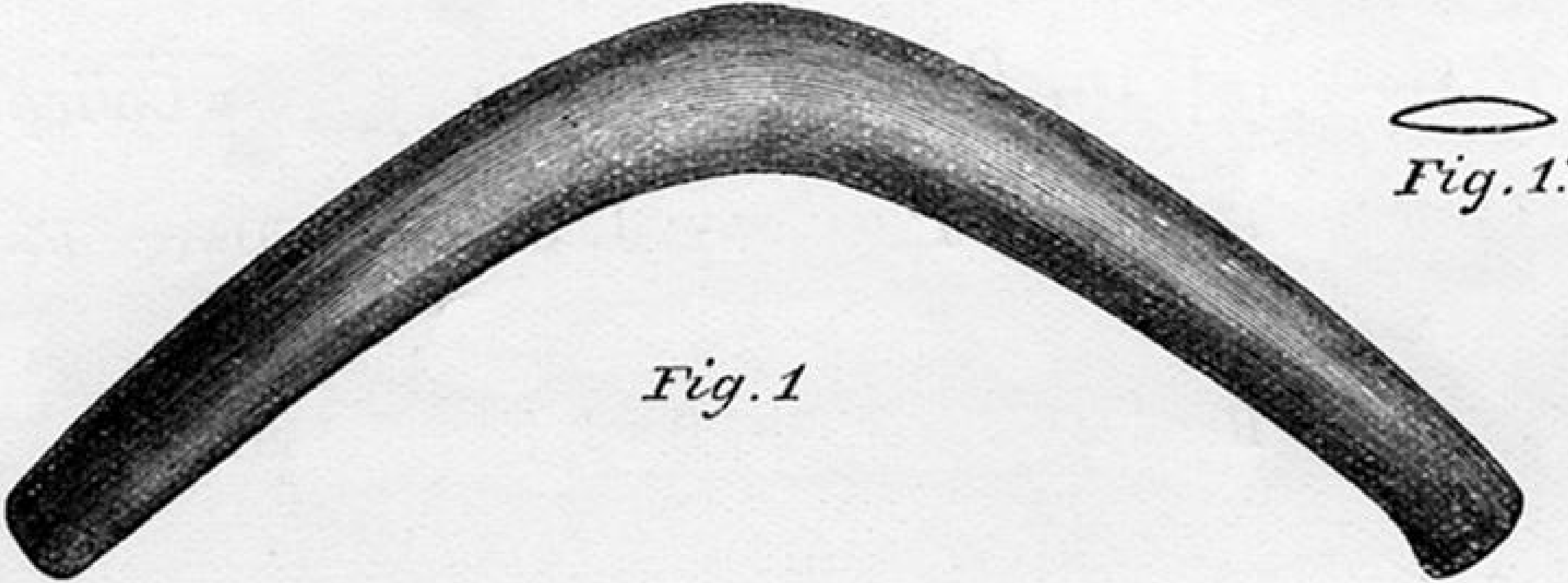


Fig. 1

Fig. 1'