X. *On the Resistance to Torsion of Certain Forms of Shafting, with Special Reference to the Effect of Keyways.*

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§ 1. *Object and Methods of the Investigation.*

The object of the present paper is to obtain solutions of the problem of torsion for cylinders whose cross-sections are bounded by confocal conics. It is mainly an extension of de Saint-Venant's investigations, and is based upon his general equations of torsion.

The method employed depends upon the use of conjugate functions ξ and η, such that $\xi =$ const. represents confocal ellipses and $\eta =$ const. confocal hyperbolas.

The use of conjugate functions for the torsion problem has been suggested by Thomson and Tait ('Natural Philosophy'), by Clebsch ('Theorie der Elasticität fester Körper,' §§ 33–35), and by Boussinesq ('Journal de Mathématiques,' pp. 177–186, Série III., vol. 6). Clebsch has used such elliptic coordinates to solve the torsion problem for hollow cylinders bounded by confocal ellipses, and de Saint-Venant has applied conjugate functions to the same problem for shafts whose sections are sectors of circles; curvilinear coordinates have also been employed by Mr. H. M. MacDonald ("On the Torsional Strength of Hollow Shafts," 'Proc. Camb. Phil. Soc.,' vol. 8, 1893, p. 62, *et seq.*), but I am not aware that the actual solution has yet been obtained for sections bounded by both ellipses and hyperbolas.

The work proceeds on lines analogous to those developed by Saint-Venant himself, in his solution of the problem of torsion for the cylinder of rectangular cross-section. The strains and stresses are expressible in terms of infinite series involving circular and hyperbolic functions.

The boundaries of the section are given by constant values of ξ and η. The values of ξ are taken to be $\pm\alpha$.

The conditions from which the unknown quantity w (the shift parallel to the axis) is determined are

$$d^2w/dx^2 + d^2w/dy^2 = 0$$

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throughout the section; and

$$dw/dn + (mx - ly)\tau = 0$$

along the boundary, where dn = an element of the outwards normal to the boundary, τ is the angle of torsion per unit length, and l, m are the direction-cosines of dn.

Now in the present case

$$dn = \pm\, d\xi \times (c\sqrt{\mathrm{J}})$$

where

$$\mathrm{J} = \partial\left(\frac{x}{c}, \frac{y}{c}\right) \Big/ \partial(\xi, \eta)$$

at the boundary where ξ = const., and

$$dn = \pm\, d\eta \times (c\sqrt{\mathrm{J}})$$

at the boundary where η = const., the sign being so determined that dn is positive.

By adding suitable terms to w, we can reduce one or other of the boundary conditions to the form

$$dw_1/dn = 0,$$

where

$$w = w_1 + \text{suitable terms.}$$

Suppose we make

$$\left(\frac{dw_1}{d\xi}\right)_{\xi=\pm\alpha} = 0.$$

Expanding now w_1 in the form of a series,

$$w_1 = \sum_{n=0}^{n=\infty} \mathrm{A}_n \sinh\left\{\frac{2n+1}{2\alpha}\pi(\eta+\kappa)\right\} \sin\frac{\widehat{2n+1}\pi\xi}{2\alpha},$$

the differential equation and the first boundary condition are identically satisfied.

When this value is substituted in the second boundary condition, we get an equation expressing a given function of ξ in a series of sines of odd multiples of $\frac{\pi\xi}{2\alpha}$, between the limits $+\alpha$ and $-\alpha$.

But such an expression can be definitely obtained by a method analogous to that for FOURIER'S series. Comparing coefficients, we obtain relations which determine completely all the constants in the expression of w_1.

w is then known. The shears and torsion moment are then deduced by differentiation and a double integration.

§ 2. *Summary of the Results.*

The cross-sections which are dealt with in the present paper are of very great generality, and they include as special cases many of the cross-sections which SAINT-VENANT has worked out, for instance the rectangle and the sector of a circle.

The first section of which I treat is that bounded by an ellipse and two confocal

hyperbolas. Although the analysis is worked out for the case where the two hyperbolic segments are not symmetrical, I have not given any numerical examples of this case, as the sections obtained by taking two hyperbolas curved the same way, as in fig. 1, do not correspond to any intersecting practical case: the section is too broad at the ends and too narrow at the bend to be any fair representation of the angle iron.

The section (fig. 2) bounded by an ellipse and the two branches of a confocal hyperbola is, on the other hand, an approximate representation of a well-known section, much used in engineering practice, the rail section.

This section I have worked out for various values of the eccentricity of the ellipse and of the angle between the asymptotes of the hyperbola.

The four sections in fig. 2, where this angle is 120°, give the best representation of the rail section.

The numerical results are tabulated so as to show the ratio of the torsional rigidity of this section to that of the circular section of the same area, and also the same ratio for the maximum stress.

The ratio of these two ratios gives us a kind of measure of the usefulness or "efficiency" of the section.

In the case of the sections of fig. 2 I have investigated at length the position of the *fail-points*, or points of maximum strain and stress, the maximum strain, in the case of torsion, being coincident with the maximum stress. It is found that for the two smaller ellipses the maximum stress occurs at the point B where the section is thinnest. For the two larger ellipses the maximum stress occurs at points F, F, F, F, symmetrically distributed round the contour, and lying on the broad sides of the section. The critical section, when these two cases pass into one another, can be calculated and is shown as qq, qq in fig. 2. In figs. 2–5 the corresponding points belonging to the different sections are distinguished by suffixes.

The changes in the stresses are shown by the curves in fig. 9, (p. 340) in which the abscissa represents the quantity α whose hyperbolic cosine and sine are proportional to the major and minor axes of the ellipse respectively, and in which the ordinates represent the stresses at A, B, F, divided by the maximum stress of the circular section of equal area. The curves are in certain parts only roughly drawn, but they suffice to show the manner in which the stresses vary. It is seen that the stress at B separates from the maximum stress after the critical value $\alpha = 1{\cdot}225$, and gradually diminishes, compared with the stresses at A and F.

This result might have been expected from the investigations of DE SAINT-VENANT upon certain sections bounded by curves of the fourth degree. These investigations appear, however, not to have been sufficiently noticed. THOMSON and TAIT, in their 'Natural Philosophy,' and BOUSSINESQ, in his researches on torsion ('Journal de Mathématiques,' Série II., vol. 16, p. 200), both conclude that the fail-points are at the points of the cross-section nearest to the centre, and BOUSSINESQ even gives an

apparently general proof of this proposition. His proof, however, is subject to certain restrictions which I point out, and which prevent it from being applied to the sections I am dealing with.

The sections are sensibly less useful than the circular section, their torsional rigidity

Fig. 1.

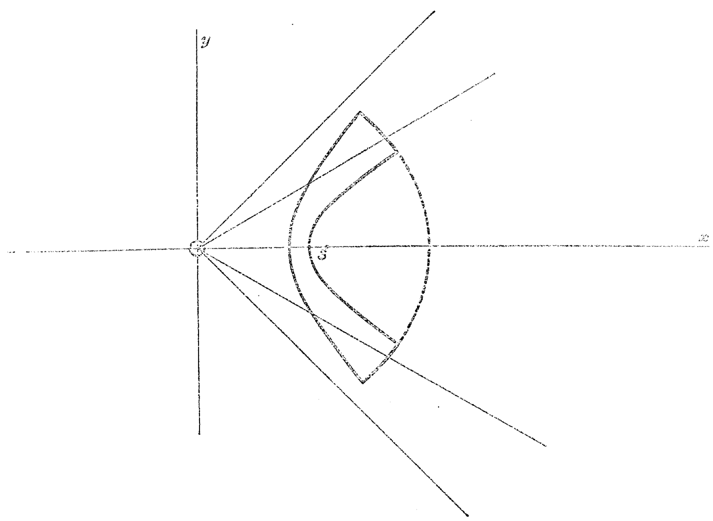
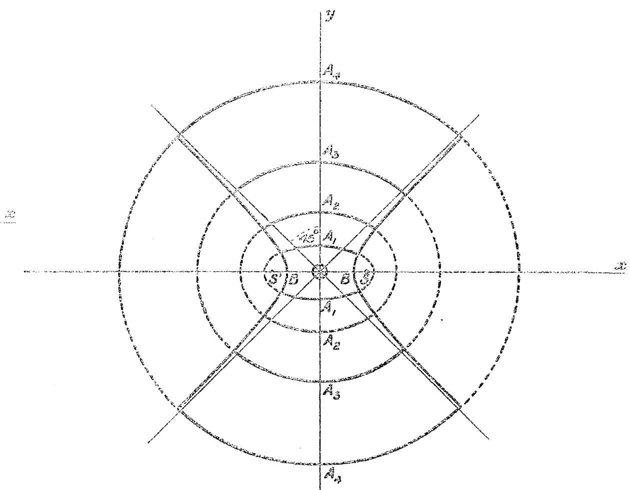


Fig. 3.



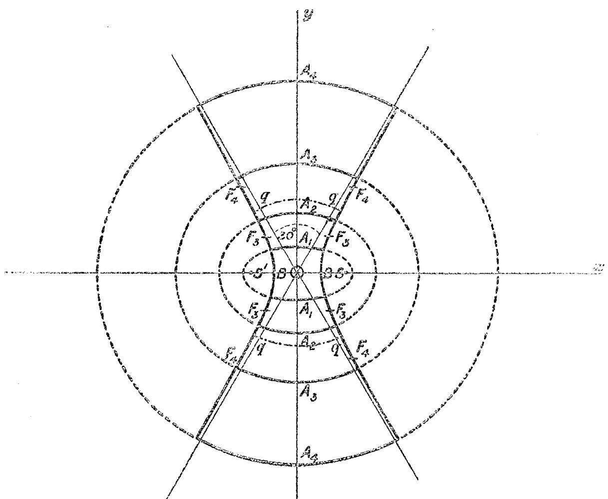


Fig. 2.

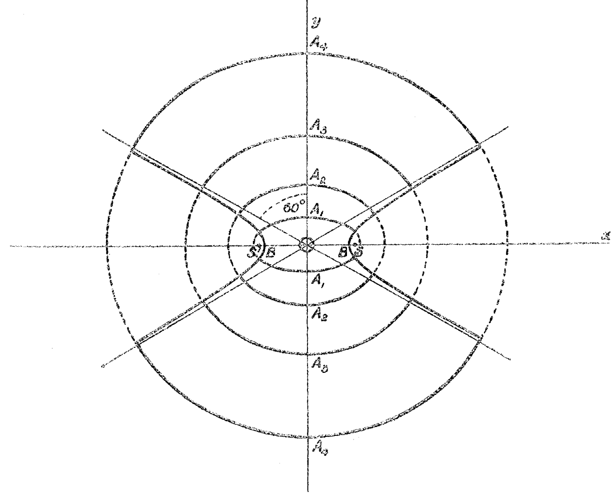


Fig. 4.

being always diminished and the maximum stress very often increased. This remark, I may add, applies to all the sections dealt with in this paper.

This usefulness or efficiency decreases as the neck of the section becomes more narrow, as, indeed, might have been anticipated.

Other sections worked out are those corresponding to angles between the asymptotes of 90° (fig. 3), 60° (fig. 4), and 0° (fig. 5); in the latter case the sections degenerate into ordinary elliptic sections with two straight slits, or indefinitely thin

keyways, cut into them along the major axis, as far as the foci. The stress at the foci, however, is then theoretically infinite.

It is interesting to see how, as we make the bend round the foci sharper, the values of α, for which the two fail-points break up into four, become larger and larger,

Fig. 5.

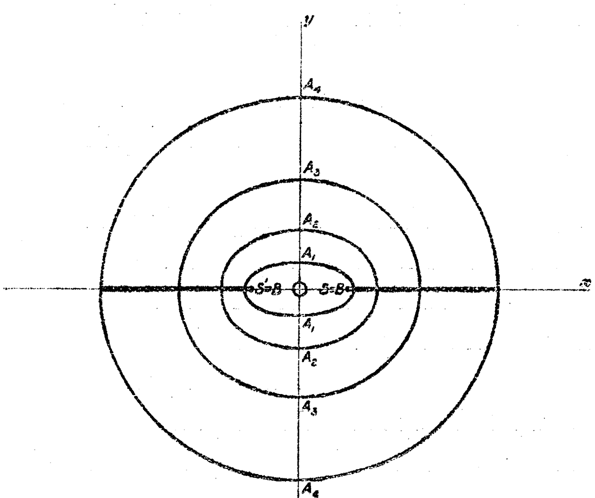
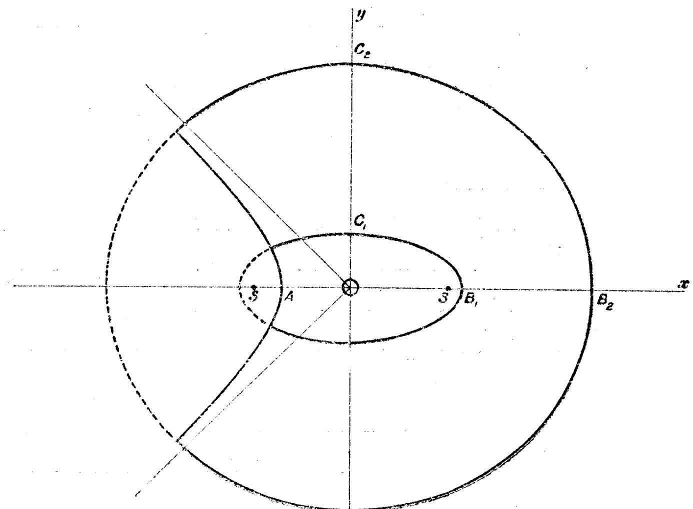


Fig. 7.



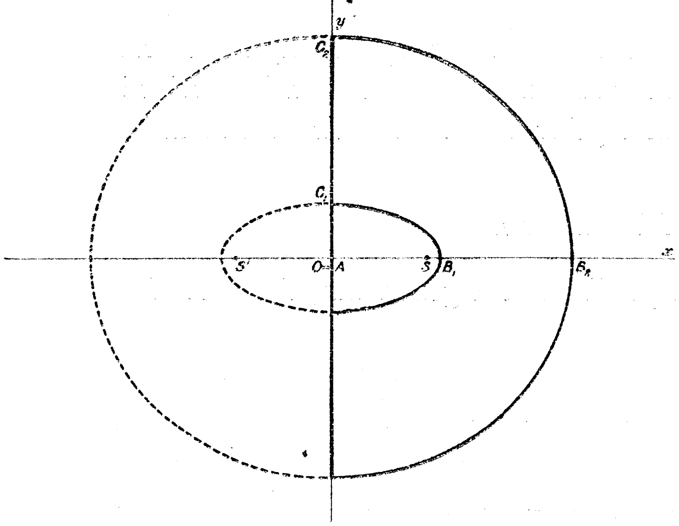


Fig. 6.

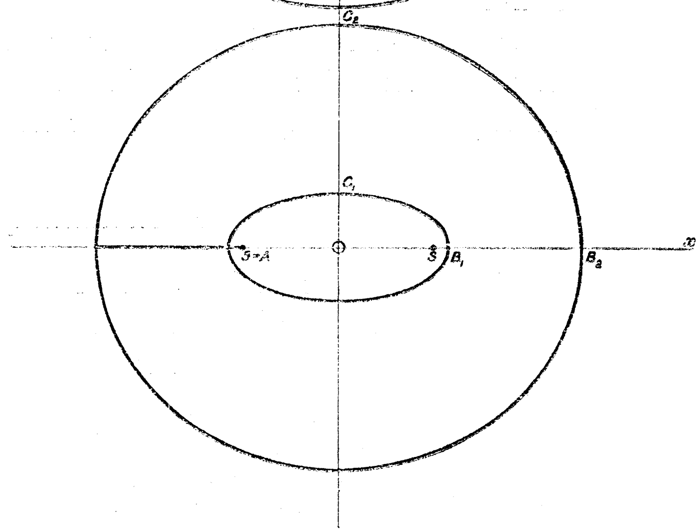


Fig. 8.

until, when the angle between the asymptotes of the hyperbolas is less than 73°, the greatest stress always occurs at the neck of the section.

The limiting case of such sections, when the angle between the asymptotes is very small and the eccentricity of the ellipse nearly unity, the distance between the foci being very great, gives us the rectangle.

I then pass on to the section bounded by one ellipse and one confocal hyperbola. In the limiting case when the foci coincide, we obtain the sector of a circle.

Of this I have worked out numerically three cases, in each case taking two ellipses

(1.) The semi-ellipse (fig. 6).

(2.) The ellipse with a keyway cut into it of the shape of a rectangular confocal hyperbola (fig. 7).

(3.) The ellipse with a single slit cut into it (fig. 8).

The most striking of the results is in reference to the reduction of the torsional rigidity of the ellipse in case (3). This reduction of rigidity decreases rapidly as the depth of the notch decreases.

The rigidity, which is reduced by as much as 23 per cent. when the depth of the keyway is as great as ·6 (semi-major axis), falls only about 1 per cent. when this depth is ·12 (semi-major axis).

Possibly this may throw some light on the fact that the effect of cutting such slits into the material does not always give in practice the reduction in the torsional rigidity which should have been expected from SAINT-VENANT'S results for the circle. Clearly the depth of the keyway is a factor of the very first importance, and keyways of moderate depth will produce a comparatively small effect on the torsional rigidity.

It is also shown that the effect of cutting two equal and opposite slits is practically equal, in the two cases which I have calculated (namely, $\alpha = \pi/6$ and $\alpha = \pi/2$), to twice the effect of a single slit.

It seems, therefore, that the study of these sections brings to light several interesting facts in the theory of elasticity, and will well repay the trouble involved in dealing with the long and somewhat tedious algebra and arithmetic which lead to these results.

§ 3. *Statement of Notation, &c.*

In what follows the axis of z will be taken parallel to the generators, the axes of x and y in one of the terminal cross-sections.

The origin, however, will not necessarily be at the centroid of the cross-section.

The shifts of any point of the material parallel to the axis will be denoted as usual by u, v, w, and for the stresses I shall use the notation of TODHUNTER and PEARSON'S 'History of Elasticity,' $\widehat{rs}$ denoting the stress, parallel to r, across a plane element perpendicular to s.

Then if, following SAINT-VENANT, we suppose the terminal cross-section $z = 0$ to be fixed (that is to say, $u = 0$, $v = 0$, but $w \neq 0$), then if τ be the angle of torsion per unit length,

$$v = \tau xz, \qquad u = -\tau yz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

If μ be the modulus of rigidity,

$$\frac{\widehat{yz}}{\mu} = \frac{dw}{dy} + \tau x, \qquad \frac{\widehat{xz}}{\mu} = \frac{dw}{dx} - \tau y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and all the other stresses are zero.

w is then determined from the body stress equation

$$\frac{d\widehat{xz}}{dx} + \frac{d\widehat{yz}}{dy} = 0,$$

or

$$\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

which holds at all points of the cross-section, and from the surface-stress equation

$$l\,(\widehat{xz}) + m\,(\widehat{yz}) = 0,$$

that is,

$$\frac{dw}{dn} + (mx - ly)\,\tau = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

at the boundary, where dn is an element of the outwards normal, and (l, m) are its direction-cosines.

The above conditions allow us to determine w uniquely. They are associated with the condition that the parallel to the generators through the origin remains fixed. If, however, we take any other parallel to the generators through the point (a, b) to remain fixed, then

$$u = -\tau\,(y - b)\,z, \qquad v = \tau\,(x - a)\,z,$$

$$\frac{\widehat{yz}}{\mu} = \frac{dw}{dy} + \tau\,(x - a), \qquad \frac{\widehat{xz}}{\mu} = \frac{dw}{dx} - \tau\,(y - b),$$

and the equation at the boundary becomes

$$\frac{dw}{dn} + (mx - ly)\,\tau - \tau\,(am - lb) = 0.$$

Now, if instead of w we write

$$w = w' + \tau\,(ay - bx),$$

then

$$\frac{\widehat{yz}}{\mu} = \frac{dw'}{dy} + \tau x, \qquad \frac{\widehat{xz}}{\mu} = \frac{dw'}{dx} - \tau y,$$

and the equations to determine w' are the same which we had before for w. It follows that the *stresses* in the cylinder are unaltered, whatever be the parallel to the generators about which it is twisted, the effect of the change being merely to introduce a term $\tau\,(ay - bx)$ into w, which corresponds to a rigid rotation about an axis joining (a, b) to the origin.

It follows from this that in dealing with stresses due to torsion we may take our origin wherever it is most convenient.

§ 4. *Analytical Work for Sections Bounded by One Elliptic and Two Hyperbolic Arcs.*

Consider now the transformation

$$x = c \cosh \xi \sin \eta,$$

$$y = c \sinh \xi \cos \eta.$$

If we allow η to vary between β and β', and ξ between $+\alpha$ and $-\alpha$, the point (x, y) will move within the space contained between the ellipse

$$\frac{x^2}{c^2 \cosh^2 \alpha} + \frac{y^2}{c^2 \sinh^2 \alpha} = 1,$$

and the two hyperbolas

$$\frac{x^2}{c^2 \sin^2 \beta} - \frac{y^2}{c^2 \cos^2 \beta} = 1, \qquad \frac{x^2}{c^2 \sin^2 \beta'} - \frac{y^2}{c^2 \cos^2 \beta'} = 1.$$

Using then the coordinates (ξ, η) instead of (x, y) we find that our equations for w become

$$\frac{d^2 w}{d\xi^2} + \frac{d^2 w}{d\eta^2} = 0 \quad \text{for} \quad \left\{ \begin{array}{c} \beta' < \eta < \beta \\ -\alpha < \xi < \alpha \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3').$$

Also

$$\left. \begin{array}{lll} \dfrac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\eta = 0, & \text{when} \quad \xi = \pm \alpha, & \beta' < \eta < \beta \\ \dfrac{dw}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\xi = 0, & \text{when} \quad \eta = \beta \text{ or } \beta', & -\alpha < \xi < \alpha \end{array} \right\} \quad . \quad . \quad (4').$$

Write now

$$w = w_1 - \tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cosh 2\alpha}.$$

Then

$$dw_1/d\xi = 0, \qquad \xi = \pm \alpha, \qquad \beta' < \eta < \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

$$\left. \begin{array}{lll} \dfrac{dw_1}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\xi \left(1 + \dfrac{\cos 2\beta}{\cosh 2\alpha}\right) = 0, & \eta = \beta, & -\alpha < \xi < \alpha \\ \dfrac{dw_1}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\xi \left(1 + \dfrac{\cos 2\beta'}{\cosh 2\alpha}\right) = 0, & \eta = \beta', & -\alpha < \xi < \alpha \end{array} \right\} \quad . \quad . \quad (6).$$

Let us assume

$$w_1 = \sum_{n=0}^{n=\infty} \left(\mathrm{A}_n \sinh \frac{\widehat{2n+1}\pi}{2\alpha}(\eta - \epsilon) + \mathrm{B}_n \cosh \frac{\widehat{2n+1}\pi(\eta - \epsilon)}{2\alpha} \right) \sin \frac{\widehat{2n+1}\pi\xi}{2\alpha},$$

where $\epsilon = \frac{1}{2}(\beta + \beta')$.

Then conditions (5) and (3′) are identically satisfied. Let us now determine the coefficients A and B so that (6) shall also be satisfied.

We have to expand $\sinh 2\xi$ between the limits $\pm\alpha$ in a series of sines as follows:

$$\sinh 2\xi = a_0 \sin\frac{\pi\xi}{2\alpha} + a_1 \sin\frac{3\pi\xi}{2\alpha} + \ldots + a_n \sin\frac{\widehat{2n+1}\pi\xi}{2\alpha} + \ldots$$

The coefficients $a_0, a_1, \ldots a_n \ldots$ are found in the usual way

$$a_n = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} \sinh 2\xi \sin\frac{\widehat{2n+1}\pi\xi}{2\alpha}\, d\xi = (-1)^n \frac{16\alpha\cosh 2\alpha}{\pi^2(2n+1)^2+16\alpha^2},$$

also we have, substituting in (6), equating coefficients of $\sin\dfrac{\widehat{2n+1}\pi\xi}{2\alpha}$, and writing $\beta - \beta' = 2\gamma$,

$$A_n \cosh\frac{\widehat{2n+1}\pi\gamma}{2\alpha} + B_n \sinh\frac{\widehat{2n+1}\pi\gamma}{2\alpha} = \frac{\tau c^2\alpha}{(2n+1)\pi} a_n\left(1+\frac{\cos 2\beta}{\cosh 2\alpha}\right),$$

$$A_n \cosh\frac{\widehat{2n+1}\pi\gamma}{2\alpha} - B_n \sinh\frac{\widehat{2n+1}\pi\gamma}{2\alpha} = \frac{\tau c^2\alpha}{(2n+1)\pi} a_n\left(1+\frac{\cos 2\beta'}{\cosh 2\alpha}\right),$$

whence, solving for A_n and B_n, we find

$$A_n = \frac{16\tau c^2\alpha^2}{(2n+1)\pi}(-1)^n \frac{\operatorname{sech}\dfrac{\widehat{2n+1}\pi\gamma}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}(\cosh 2\alpha + \cos 2\epsilon\cos 2\gamma) \quad . \quad . \quad (7),$$

$$B_n = -\frac{16\tau c^2\alpha^2}{(2n+1)\pi}(-1)^n \frac{\operatorname{cosech}\dfrac{\widehat{2n+1}\pi\gamma}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}\sin 2\epsilon\sin 2\gamma \quad . \quad . \quad . \quad . \quad . \quad (8),$$

whence

$$w = -\tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cosh 2\alpha}$$

$$+16\tau c^2\alpha^2\{\cosh 2\alpha + \cos 2\epsilon\cos 2\gamma\}\sum_{n=0}^{\infty}(-1)^n \frac{\sinh\dfrac{\widehat{2n+1}\pi}{2\alpha}(\eta-\epsilon)\sin\dfrac{\widehat{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2+16\alpha^2]\cosh\dfrac{\widehat{2n+1}\pi\gamma}{2\alpha}}$$

$$-16\tau c^2\alpha^2\sin 2\gamma\sin 2\epsilon\sum_{n=0}^{\infty}(-1)^n \frac{\cosh\dfrac{\widehat{2n+1}\pi}{2\alpha}(\eta-\epsilon)\sin\dfrac{\widehat{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2+16\alpha^2]\sinh\dfrac{\widehat{2n+1}\pi\gamma}{2\alpha}} \quad . \quad . \quad (9).$$

Having obtained w, the shears are easily deduced by simple differentiation

$$\left.\begin{aligned}\frac{\widehat{xz}}{\mu} &= \frac{1}{c\mathrm{J}}\left(\sinh\xi\sin\eta\frac{dw}{d\xi} + \cosh\xi\cos\eta\frac{dw}{d\eta}\right) - \tau c\sinh\xi\cos\eta \\ \frac{\widehat{yz}}{\mu} &= \frac{1}{c\mathrm{J}}\left(\cosh\xi\cos\eta\frac{dw}{d\xi} - \sinh\xi\sin\eta\frac{dw}{d\eta}\right) + \tau c\cosh\xi\sin\eta\end{aligned}\right\} \quad . \quad . \quad (10),$$

where $\mathrm{J} = \cosh^2\xi\cos^2\eta + \sinh^2\xi\sin^2\eta$.

These again may be put into the slightly different form

$$\frac{\widehat{xz}}{\mu} = \frac{1}{c\mathrm{J}}\left(\sinh\xi\sin\eta\frac{dw_1}{d\xi} + \cosh\xi\cos\eta\frac{dw_1}{d\eta}\right) - \tau c\sinh\xi\cos\eta\,(1 + \operatorname{sech} 2\alpha)$$

$$\frac{\widehat{yz}}{\mu} = \frac{1}{c\mathrm{J}}\left(\cosh\xi\cos\eta\frac{dw_1}{d\xi} - \sinh\xi\sin\eta\frac{dw_1}{d\eta}\right) + \tau c\cosh\xi\sin\eta\,(1 - \operatorname{sech} 2\alpha).$$

The next quantity which we require is the moment of the shears

$$\begin{aligned}\mathrm{M} &= \int(x\widehat{yz} - y\widehat{xz})\,dx\,dy \\ &= \frac{\mu\tau c^4}{2}\int_{\beta'}^{\beta}d\eta\int_{-\alpha}^{\alpha}d\xi\,[(\cosh 2\xi - \cos 2\eta) - \operatorname{sech} 2\alpha\,(1 - \cosh 2\xi\cos 2\eta)] \times \mathrm{J} \\ &\quad + \frac{\mu c^2}{2}\int_{\beta'}^{\beta}d\eta\int_{-\alpha}^{\alpha}d\xi\left[\frac{dw_1}{d\xi}\sin 2\eta - \frac{dw_1}{d\eta}\sinh 2\xi\right]\cdot \\ &= \frac{\mu\tau c^4}{8}\int_{\beta'}^{\beta}d\eta\int_{-\alpha}^{\alpha}d\xi\,(\cosh 4\xi - \cos 4\eta - \operatorname{sech} 2\alpha\,\{\cosh 2\xi + \cos 2\eta\} \\ &\qquad\qquad + \operatorname{sech} 2\alpha\,\{\cosh 4\xi\cos 2\eta + \cosh 2\xi\cos 4\eta\}) \\ &\quad + \frac{\mu c^2}{2}\int_{\beta'}^{\beta}d\eta\sin 2\eta\left[w_1\right]_{-\alpha}^{\alpha} - \frac{\mu c^2}{2}\int_{-\alpha}^{\alpha}d\xi\sinh 2\xi\left[w_1\right]_{\beta'}^{\beta}\cdot \\ &= \frac{\mu\tau c^4}{8}\left[\begin{aligned}&\gamma\sinh 4\alpha - \alpha\sin 4\gamma\cos 4\epsilon - \operatorname{sech} 2\alpha\,(2\gamma\sinh 2\alpha + 2\alpha\sin 2\gamma\cos 2\epsilon) \\ &+ \frac{\operatorname{sech} 2\alpha}{2}(\sinh 4\alpha\cos 2\epsilon\sin 2\gamma + \sinh 2\alpha\cos 4\epsilon\sin 4\gamma)\end{aligned}\right] \\ &\quad + \sum_{n=0}^{n=\infty}\frac{16\mu\tau c^4\alpha^2(\cosh 2\alpha + \cos 2\epsilon\cos 2\gamma)}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]\cosh\frac{\widehat{2n+1}\pi\gamma}{2\alpha}}\int_{\beta'}^{\beta}\sinh\frac{\widehat{2n+1}\pi}{2\alpha}(\eta - \epsilon)\sin 2\eta\,d\eta \\ &\quad - \sum_{n=0}^{n=\infty}\frac{16\mu\tau c^4\alpha^2\sin 2\gamma\sin 2\epsilon}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]\sinh\frac{\widehat{2n+1}\pi\gamma}{2\alpha}}\int_{\beta'}^{\beta}\cosh\frac{\widehat{2n+1}\pi}{2\alpha}(\eta - \epsilon)\sin 2\eta\,d\eta \\ &\quad - \sum_{n=0}^{n=\infty}\frac{16\mu\tau c^4\alpha^2(\cosh 2\alpha + \cos 2\epsilon\cos 2\gamma)(-1)^n\tanh\frac{\widehat{2n+1}\pi\gamma}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]}\int_{-\alpha}^{\alpha}\sin\frac{\widehat{2n+1}\pi\xi}{2\alpha}\sinh 2\xi\,d\xi.\end{aligned}$$

$$= \frac{\mu\tau c^4}{8}\left[\begin{array}{l}\gamma \sinh 4\alpha - \alpha \sin 4\gamma \cos 4\epsilon - \operatorname{sech} 2\alpha\,\{2\gamma \sinh 2\alpha + 2\alpha \sin 2\gamma \cos 2\epsilon\} \\ + \dfrac{\operatorname{sech} 2\alpha}{2}\{\sinh 4\alpha \sin 2\gamma \cos 2\epsilon + \sinh 2\alpha \sin 4\gamma \cos 4\epsilon\}\end{array}\right]$$

$$- 256\mu\tau c^4\alpha^4 (\cosh 2\alpha + \cos 2\epsilon \cos 2\gamma)^2 \sum_{n=0}^{n=\infty} \frac{\tanh \dfrac{\overline{2n+1}\pi\gamma}{2\alpha}}{\pi (2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]^2}$$

$$- 256\mu\tau c^4\alpha^4 (\sin 2\epsilon \sin 2\gamma)^2 \sum_{n=0}^{n=\infty} \frac{\coth \left(\dfrac{\overline{2n+1}\pi\gamma}{2\alpha}\right)}{\pi (2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]^2}$$

$$+ 64\mu\tau c^4\alpha^3 \sin 2\gamma (\cosh 2\alpha \cos 2\epsilon + \cos 2\gamma \cos 4\epsilon) \sum_{n=0}^{n=\infty} \frac{1}{[(2n+1)^2\pi^2 + 16\alpha^2]^2}.$$

But the series $\sum_0^\infty \frac{1}{[(2n+1)^2\pi^2 + 16\alpha^2]^2}$ can be summed in finite terms,* and it is found to be

$$-\frac{1}{512\alpha^3}(2\alpha \operatorname{sech}^2 2\alpha - \tanh 2\alpha).$$

Substituting this value in the expression for the torsion moment we find

$$M = \frac{\mu\tau c^4}{8}\left[\begin{array}{l}(\gamma \sinh 4\alpha - \alpha \sin 4\gamma \cos 4\epsilon)(1 - \operatorname{sech}^2 2\alpha) \\ - \alpha \sin 4\gamma \cos 4\epsilon \operatorname{sech}^2 2\alpha - 2\alpha \operatorname{sech} 2\alpha \sin 2\gamma \cos 2\epsilon \\ + \operatorname{sech} 2\alpha\,[\sinh 2\alpha \sin 2\gamma](\cosh 2\alpha \cos 2\epsilon + \cos 2\gamma \cos 4\epsilon) \\ - \sin 2\gamma\,(2\alpha \operatorname{sech}^2 2\alpha - \tanh 2\alpha)(\cosh 2\alpha \cos 2\epsilon + \cos 2\gamma \cos 4\epsilon)\end{array}\right]$$

— the two series terms.

Whence, after some obvious reductions

$$\frac{M}{\mu\tau c^4} = \frac{\tanh^2 2\alpha}{8}(\gamma \sinh 4\alpha - \alpha \sin 4\gamma \cos 4\epsilon)$$

$$- \tfrac{1}{4} \sin 2\gamma (\cosh 2\alpha \cos 2\epsilon + \cos 2\gamma \cos 4\epsilon)(2\alpha \operatorname{sech}^2 2\alpha - \tanh 2\alpha)$$

$$- 256\alpha^4 (\cosh 2\alpha + \cos 2\epsilon \cos 2\gamma)^2 \sum_{n=0}^{n=\infty} \frac{\tanh \dfrac{\overline{2n+1}\pi\gamma}{2\alpha}}{\pi (2n+1)[16\alpha^2 + \overline{2n+1}^2\pi^2]^2}$$

$$- 256\alpha^4 (\sin 2\gamma \sin 2\epsilon)^2 \sum_{n=0}^{n=\infty} \frac{\coth \dfrac{\overline{2n+1}\pi\gamma}{2\alpha}}{\pi (2n+1)[16\alpha^2 + \overline{2n+1}^2\pi^2]^2} \qquad (11).$$

* See CHRYSTAL'S 'Algebra,' vol. 2 (Differentiating the result marked (8) on p. 338).

§ 5. *Alternative Solution for the same Sections.*

There exists also an alternative solution; it is generally of a less convenient form than the one last given. It may, however, be useful in certain cases.

Return now to the boundary conditions (4′) and write

$$w = w_1 + \tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2(\eta - \epsilon)}{\cos 2\gamma}$$

we then obtain

$$\frac{dw_1}{d\eta} = 0 \quad \text{when} \quad \eta = \beta \quad \text{or} \quad \eta = \beta', \quad -\alpha < \xi < +\alpha$$

and

$$\frac{dw_1}{d\xi} + \tfrac{1}{2}\tau c^2 \left\{\sin 2\eta + \frac{\cosh 2\alpha}{\cos 2\gamma} \sin 2(\eta - \epsilon)\right\} = 0,$$

when

$$\xi = \pm\alpha, \beta' < \eta < \beta.$$

The latter condition may be written

$$\frac{dw_1}{d\xi} + \tfrac{1}{2}\tau c^2 \left[\sin 2(\eta - \epsilon)\frac{(\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon)}{\cos 2\gamma} + \cos 2(\eta - \epsilon) . \sin 2\epsilon\right] = 0.$$

Now let us write

$$w_1 = \varpi_1 + \varpi_2$$

where

$$\frac{d^2\varpi_1}{d\xi^2} + \frac{d^2\varpi_1}{d\eta^2} = 0, \quad \frac{d^2\varpi_2}{d\xi^2} + \frac{d^2\varpi_2}{d\eta^2} = 0,$$

and

$$d\varpi_1/d\eta = 0, \quad \eta - \epsilon = \pm\gamma \qquad d\varpi_2/d\eta = 0, \quad \eta - \epsilon = \pm\gamma.$$

But

$$\frac{d\varpi_1}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2(\eta - \epsilon)\left(\frac{\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon}{\cos 2\gamma}\right) = 0 \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{d\varpi_2}{d\xi} + \tfrac{1}{2}\tau c^2 \cos 2(\eta - \epsilon) \sin 2\epsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

when

$$\xi = \pm\alpha \quad \text{and} \quad -\gamma < \eta - \epsilon < +\gamma.$$

Assume

$$\varpi_1 = \sum_{n=0}^{\infty} A_n \sinh \frac{\overline{2n+1}\pi\xi}{2\gamma} \sin \frac{\overline{2n+1}\pi(\eta - \epsilon)}{2\gamma}$$

$$\varpi_2 = B_0\xi + \sum_{n=1}^{n=\infty} B_n \sinh \frac{n\pi\xi}{\gamma} \cos \frac{n\pi(\eta - \epsilon)}{\gamma},$$

we have now to express between limits $\pm\gamma$

$$\cos 2\theta = b_0 + b_1 \cos \frac{\pi\theta}{\gamma} + b_2 \cos \frac{2\pi\theta}{\gamma} + \ldots + b_n \cos \frac{n\pi\theta}{\gamma} + \ldots$$

$$\sin 2\theta = a_1 \sin \frac{\pi\theta}{2\gamma} + a_2 \sin \frac{3\pi\theta}{2\gamma} + \ldots + a_n \sin \frac{\widehat{2n+1}\pi\theta}{2\gamma} + \ldots$$

We find

$$b_n = (-1)^{n+1} \frac{4\gamma \sin 2\gamma}{n^2\pi^2 - 4\gamma^2}, \quad a_n = (-1)^n \frac{16\gamma \cos 2\gamma}{(2n+1)^2\pi^2 - 16\gamma^2}$$

$$b_0 = \frac{1}{2\gamma} \sin 2\gamma$$

Substituting in (12) and (13) and equating coefficients we obtain

$$A_n = -\frac{16\tau c^2\gamma^2 (\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon)(-1)^n}{(2n+1)\pi[(2n+1)^2\pi^2 - 16\gamma^2]} \operatorname{sech} \frac{\widehat{2n+1}\pi\alpha}{2\gamma},$$

$$B_n = \frac{2\tau c^2\gamma^2 \sin 2\epsilon \sin 2\gamma (-1)^n}{n\pi (n^2\pi^2 - 4\gamma^2)} \operatorname{sech} \frac{n\pi\alpha}{\gamma},$$

$$B_0 = -\frac{1}{4\gamma} \tau c^2 \sin 2\gamma \sin 2\epsilon,$$

whence

$$\begin{aligned} w = {} & \tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2(\eta - \epsilon)}{\cos 2\gamma} \\ & + 16\tau c^2\gamma^2 \sin 2\epsilon \sin 2\gamma \left[-\frac{\xi}{64\gamma^3} + \sum_{n=1}^{n=\infty} (-1)^n \frac{\sinh \frac{n\pi\xi}{\gamma} \cos \frac{n\pi(\eta-\epsilon)}{\gamma}}{2n\pi(4n^2\pi^2 - 16\gamma^2)\cosh \frac{n\pi\alpha}{\gamma}} \right] \\ & - 16\tau c^2\gamma^2 (\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon) \\ & \times \left[\sum_{n=0}^{n=\infty} (-1)^n \frac{\sinh \frac{\widehat{2n+1}\pi\xi}{2\gamma} \sin \frac{\widehat{2n+1}\pi(\eta-\epsilon)}{2\gamma}}{(2n+1)\pi[(2n+1)^2\pi^2 - 16\gamma^2]\cosh \frac{\widehat{2n+1}\pi\alpha}{2\gamma}} \right] \quad \ldots \quad (14). \end{aligned}$$

It may be noted here that γ may vary between 0 and $\pi/2$. It follows that γ may have the value $\pi/2$, and in that case the denominator of the first term under the first Σ becomes zero. The same happens to the denominator of the first term of the second Σ when γ has the value $\pi/4$. Further, in this latter case the first term in w also becomes infinite, so that the expression (14) is apparently no longer applicable.

It is easy to see, however, that the terms, which are apparently infinite in (14), exactly cancel each other. If we write $\gamma = \pi/4 - \zeta$, where ζ is small, simplify and proceed to the limit, we find that when $\gamma = \pi/4$ the two infinite terms reduce to:

$$\begin{aligned} & \tfrac{1}{4}\tau c^2 \sinh 2\xi \sin 2(\eta - \epsilon) \left[\frac{\cos 2\epsilon}{\cosh 2\alpha} - \frac{4}{\pi} - \frac{4\alpha}{\pi} \tanh 2\alpha \right] \\ & \quad + \frac{\tau c^2}{\pi} [\xi \cosh 2\xi \sin 2(\eta - \epsilon) + (\eta - \epsilon) \cos 2(\eta - \epsilon) \sinh 2\xi] \quad . \quad . \quad . \quad (15), \end{aligned}$$

which is finite.

In like manner when $\gamma = \pi/2$ the limit of w is easily evaluated.

For all *other* values of γ, ϵ and α the series in (14) are absolutely convergent as they stand, for all points (ξ, η) *within* or *on* the boundary of the section. The same holds of all the series in the last paragraph for *all* values of α, γ, ϵ.

The expression for w being given, the shears and torsion moment are obtained as before

$$M = \frac{\mu\tau c^4}{8}\int_{\beta'}^{\beta} d\eta \int_{-\alpha}^{\alpha} d\xi\,(\cosh 4\xi - \cos 4\eta) + \frac{\mu c^2}{2}\int_{\beta'}^{\beta}\Big[w\Big]_{-\alpha}^{\alpha}\sin 2\eta\, d\eta - \frac{\mu c^2}{2}\int_{-\alpha}^{\alpha}\Big[w\Big]_{\beta'}^{\beta}\sinh 2\xi\, d\xi.$$

The integrations are all easily effected, and we find

$$\begin{aligned}
M = {} & \frac{\mu\tau c^4}{8}(\gamma\sinh 4\alpha - \alpha\sin 4\gamma\cos 4\epsilon) \\
& + \frac{\mu\tau c^4}{4}\frac{(\alpha\sin 2\gamma + \gamma\sinh 2\alpha\cos 2\epsilon)}{\cos 2\gamma} - \frac{1}{16}\frac{\mu\tau c^4(\sinh 4\alpha\sin 2\gamma + \sinh 2\alpha\sin 4\gamma\cos 2\epsilon)}{\cos 2\gamma} \\
& + 64\mu\tau c^4\gamma^3(\cosh 2\alpha + \cos 2\gamma\cos 2\epsilon)\sinh 2\alpha \sum_{n=0}^{n=\infty}\frac{1}{[(2n+1)^2\pi^2 - 16\gamma^2]^2} \\
& - 256\mu\tau c^4\gamma^4(\cosh 2\alpha + \cos 2\gamma\cos 2\epsilon)^2 \sum_{n=0}^{n=\infty}\frac{\tanh\dfrac{(2n+1)\pi\alpha}{2\gamma}}{(2n+1)\pi[\pi^2(2n+1)^2 - 16\gamma^2]^2} \\
& - 256\mu\tau c^4\gamma^4(\sin 2\epsilon\sin 2\gamma)^2\left\{\sum_{n=1}^{n=\infty}\frac{\tanh\dfrac{n\pi\alpha}{\gamma}}{2n\pi(4n^2\pi^2 - 16\gamma^2)^2} + \frac{\alpha}{1024\gamma^5}\right\}.
\end{aligned}$$

Remembering that $\sum_{n=0}^{n=\infty}\dfrac{1}{\{(2n+1)^2\pi^2 - 16\gamma^2\}^2} = \dfrac{2\gamma\sec^2 2\gamma - \tan 2\gamma}{512\gamma^3}$, reducing, and re-grouping the terms, we find finally

$$\begin{aligned}
\frac{M}{\mu\tau c^4} = {} & \tfrac{1}{8}(\alpha\sin 4\gamma - \gamma\sinh 4\alpha)\tan^2 2\gamma \\
& + \tfrac{1}{4}(\cosh 2\alpha + \cos 2\gamma\cos 2\epsilon)(2\gamma\sec^2 2\gamma - \tan 2\gamma)\sinh 2\alpha \\
& + \frac{\alpha}{8\gamma}\sin 4\gamma\sin^2 2\epsilon\,(2\gamma - \tan 2\gamma) \\
& - 256\gamma^4(\cosh 2\alpha + \cos 2\gamma\cos 2\epsilon)^2\sum_{n=0}^{n=\infty}\frac{\tanh\dfrac{\overline{2n+1}\pi\alpha}{2\gamma}}{\overline{2n+1}|\pi[\pi^2(2n+1)^2 - 16\gamma^2]^2} \\
& - 256\gamma^4(\sin 2\epsilon\sin 2\gamma)^2\sum_{n=1}^{n=\infty}\frac{\tanh\dfrac{n\pi\alpha}{\gamma}}{2n\pi(4n^2\pi^2 - 16\gamma^2)^2} \qquad (16).
\end{aligned}$$

We may test the correctness of the expressions (11) and (16) by remembering that

when we make the distance c between the foci very great, and γ and α very small, the section reduces to a rectangle, of which the half sides a and b are given by

$$a = c\alpha \cos \epsilon, \qquad b = c\gamma \cos \epsilon.$$

The first and last terms in (16) are ultimately of negligible order, when multiplied by c^4.

The second and third reduce to

$$\tfrac{8}{3}\gamma^3\alpha\,(1 + \cos 2\epsilon) - \tfrac{2}{3}\gamma^3\alpha\,(1 - \cos 4\epsilon) = \tfrac{2}{3}\gamma^3\alpha\,(3 + 4\cos 2\epsilon + \cos 4\epsilon)$$
$$= \tfrac{16}{3}\gamma^3\alpha \cos^4 \epsilon.$$

The fourth term gives

$$-\,1024\,(\gamma \cos \epsilon)^4 \sum_{n=0}^{n=\infty} \frac{\tanh \dfrac{\overbrace{2n+1}\pi a}{2b}}{(2n+1)^5\pi^5}\,.$$

Hence

$$\frac{\mathrm{M}}{\mu\tau b^4} = \tfrac{16}{3}\frac{a}{b} - \left(\frac{4}{\pi}\right)^5 \sum_{n=0}^{n=\infty} \frac{\tanh \dfrac{\overbrace{2n+1}\pi a}{2b}}{(2n+1)^5} \quad . \quad . \quad .,$$

which is one of SAINT-VENANT'S expressions for the torsion moment of a rectangle of sides $2a$, $2b$.

If we treat in a similar manner expression (11), neglecting terms of order greater than four in α, γ, we get the other expression for the torsion moment of the rectangle.

§ 6. *Recapitulation of Results for the Symmetrical Case.*

By far the most important case we have to deal with is that in which the sections are symmetrical.

We have then $\beta' = -\beta$, and therefore $\epsilon = 0$, $\gamma = \beta$.

Both solutions then simplify a good deal, and we have the equivalent expressions

$$w = -\tfrac{1}{4}\tau c^2 \sinh 2\xi \sin 2\eta \operatorname{sech} 2\alpha$$
$$+\, 16\tau c^2\alpha^2 (\cosh 2\alpha + \cos 2\beta) \sum_{n=0}^{n=\infty} \frac{(-1)^n \sinh \dfrac{\overbrace{2n+1}\pi\eta}{2\alpha} \sin \dfrac{\overbrace{2n+1}\pi\xi}{2\alpha}}{\pi\,(2n+1)\,[(2n+1)^2\pi^2 + 16\alpha^2] \cosh \dfrac{\overbrace{2n+1}\pi\beta}{2\alpha}}$$
$$= \tfrac{1}{4}\tau c^2 \sinh 2\xi \sin 2\eta \sec 2\beta$$
$$-\, 16\tau c^2\beta^2 (\cosh 2\alpha + \cos 2\beta) \sum_{n=0}^{n=\infty} \frac{(-1)^n \sinh \dfrac{\overbrace{2n+1}\pi\xi}{2\beta} \sin \dfrac{\overbrace{2n+1}\pi\eta}{2\beta}}{\pi\,(2n+1)\,[(2n+1)^2\pi^2 - 16\beta^2] \cosh \dfrac{\overbrace{2n+1}\pi\alpha}{2\beta}}. \quad (17).$$

$$\frac{\widehat{xz}}{\mu} = -\tau c \sinh\xi \cos\eta\,(1 + \operatorname{sech} 2\alpha) + 32\tau c\alpha^2 \frac{(\cosh 2\alpha + \cos 2\beta)}{(\cosh 2\xi + \cos 2\eta)}$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n \left(\sinh\xi \sin\eta \dfrac{d}{d\xi} + \cosh\xi\cos\eta\dfrac{d}{d\eta}\right) \sinh\dfrac{\overline{2n+1}\pi\eta}{2\alpha} \sin\dfrac{\overline{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)[(2n+1)^2\pi^2 + 16\alpha^2]\cosh\dfrac{\overline{2n+1}\pi\beta}{2\alpha}}$$

$$= -\tau c \sinh\xi\cos\eta\,(1 - \sec 2\beta) - 32\tau c\beta^2 \frac{(\cosh 2\alpha + \cos 2\beta)}{(\cosh 2\xi + \cos 2\eta)}$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n \left(\sinh\xi \sin\eta \dfrac{d}{d\xi} + \cosh\xi\cos\eta\dfrac{d}{d\eta}\right) \sinh\dfrac{\overline{2n+1}\pi\xi}{2\beta} \sin\dfrac{\overline{2n+1}\pi\eta}{2\beta}}{\pi(2n+1)[(2n+1)^2\pi^2 - 16\beta^2]\cosh\dfrac{\overline{2n+1}\pi\alpha}{2\beta}} \quad . \quad (18).$$

$$\frac{\widehat{yz}}{\mu} = \tau c \cosh\xi \sin\eta\,(1 - \operatorname{sech} 2\alpha) + 32\tau c\alpha^2 \left(\frac{\cosh 2\alpha + \cos 2\beta}{\cosh 2\xi + \cos 2\eta}\right)$$

$$\times \sum_{n=0}^{n=\infty} \frac{(-1)^n \left(\cosh\xi \cos\eta \dfrac{d}{d\xi} - \sinh\xi\sin\eta\dfrac{d}{d\eta}\right) \sinh\dfrac{\overline{2n+1}\pi\eta}{2\alpha} \sin\dfrac{\overline{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]\cosh\dfrac{\overline{2n+1}\pi\beta}{2\alpha}}$$

$$= \tau c \cosh\xi\sin\eta\,(1 + \sec 2\beta) - 32\tau c\beta^2 \left(\frac{\cosh 2\alpha + \cos 2\beta}{\cosh 2\xi + \cos 2\eta}\right)$$

$$\times \sum_{n=0}^{n=\infty} \frac{(-1)^n \left(\cosh\xi \cos\eta \dfrac{d}{d\xi} - \sinh\xi\sin\eta\dfrac{d}{d\eta}\right) \sinh\dfrac{\overline{2n+1}\pi\xi}{2\beta} \sin\dfrac{\overline{2n+1}\pi\eta}{2\beta}}{\pi(2n+1)[\pi^2(2n+1)^2 - 16\beta^2]\cosh\dfrac{\overline{2n+1}\pi\alpha}{2\beta}} \quad . \quad (19).$$

$$\frac{M}{\mu\tau c^4} = \frac{\tanh^2 2\alpha}{8}(\beta\sinh 4\alpha - \alpha\sin 4\beta)$$

$$- \tfrac{1}{4}\sin 2\beta\,(\cosh 2\alpha + \cos 2\beta)\,(2\alpha\operatorname{sech}^2 2\alpha - \tanh 2\alpha)$$

$$- 256\alpha^4(\cosh 2\alpha + \cos 2\beta)^2 \sum_{n=0}^{n=\infty} \frac{\tanh\dfrac{\overline{2n+1}\pi\beta}{2\alpha}}{\pi(2n+1)[16\alpha^2 + \overline{2n+1}|^2\pi^2]^2}$$

$$= \frac{\tan^2 2\beta}{8}(\alpha\sin 4\beta - \beta\sinh 4\alpha)$$

$$+ \tfrac{1}{4}\sinh 2\alpha\,(\cosh 2\alpha + \cos 2\beta)\,(2\beta\sec^2 2\beta - \tan 2\beta)$$

$$- 256\beta^4(\cosh 2\alpha + \cos 2\beta)^2 \sum_{n=0}^{n=\infty} \frac{\tanh\dfrac{\overline{2n+1}\pi\alpha}{2\beta}}{\pi(2n+1)[\overline{2n+1}|^2\pi^2 - 16\beta^2]^2} \quad . \quad . \quad (20).$$

§ 7. *Importance of the Maximum Stress. Application to Rupture.*

I now pass on to the numerical determination of the torsion moment and stresses, and in particular of the maximum stress.

SAINT-VENANT has shown, in his memoir on torsion,* that if we assume an ellipsoidal distribution of *limiting* stretch (*i.e.*, stretch such that, when it is exceeded, the elasticity of the material is impaired), then, in the case of a shaft under torsion, we have at any point

$$s/\bar{s} = \sqrt{(\sigma_{xz}^2 + \sigma_{yz}^2)/\bar{\sigma}^2},$$

where σ_{yz}, σ_{xz} are the shearing strains in the planes yz, xz respectively, $\bar{\sigma}$ is the value of the *limiting shearing strain* of the material, and $s/\bar{s}$ is the maximum value of the ratio of the stretch in any direction to the *limiting* stretch in that direction.

The condition that there should be no failure of elasticity is therefore that $s/\bar{s} \nless 1$. Therefore

$$\sigma_{yz}^2 + \sigma_{zx}^2 < \bar{\sigma}_0^2,$$

and, since $\sigma_{yz} = \widehat{yz}/\mu$, $\sigma_{zx} = \widehat{xz}/\mu$,

$$\widehat{xz}^2 + \widehat{yz}^2 < \mu^2\bar{\sigma}_0^2.$$

The points where this condition will first be broken are called by SAINT-VENANT the *fail-points* ("*points dangereux*"). They are clearly the points where $\widehat{xz}^2 + \widehat{yz}^2$ is a maximum, *i.e.*, where the resultant stress across an element of the section is a maximum. Hence the importance of determining the points of maximum stress. Strictly speaking, the latter give us no certain information as to where, or how, rupture will actually take place: all that they tell us is where linear elasticity begins to fail. But they will, in general, give us a useful clue to the regions where breaking may be expected to occur, and, in the absence of any definite theory of plastic deformation and rupture, we must be content to be guided by the results of elastic theory.

I have worked out numerically the values of the stresses at the points $\xi = \pm\alpha$, $\eta = 0$ and $\xi = 0$, $\eta = \pm\beta$. These give the four points in which the axes meet the boundary of the cross-section. I have denoted them by A and B respectively. The boundary is convex at A and concave at B From considerations of symmetry it follows that A and B must be points of *maximum* or *minimum* stress, and the stress being zero both at the corners and at the centre, it will often happen that they are points of *maximum* stress. When this is the case those of the points A and B, where the stress is numerically greater, will give us the *fail-points.* But there is an obvious exception, when there are two points of maximum stress on either side of the mid-point, and this is a case which, we shall see, does occur in these sections.

* 'Mémoires des Savants Étrangers,' 1855, vol. xiv., pp. 278–288. See also TODHUNTER and PEARSON, Hist. Elast.,' vol. ii., part i., pp. 7–10.

§ 8. *Methods of Calculation.*

The symmetrical sections selected for numerical treatment are those for which $\beta = \pi/6, \pi/4, \pi/3, \pi/2$ and $\alpha = \pi/6, \pi/3, \pi/2$, and $2\pi/3$, sixteen in all. These sections are shown in figs. 2 to 5. β of course is the complement of the half angle between the asymptotes, and all sections having the same β have been collected in one figure.

The numerical calculations were generally based upon the formulæ of § 4, as they did not require modification for the value $\pi/4$ of β. In many cases, however, the alternative series were used also, in order to test the results obtained.

The calculation of the terms of the series in the expression for the torsion moment was carried on until $\frac{\widehat{2n+1}\pi\alpha}{2\beta}$ was so great that $\tanh\frac{\widehat{2n+1}\pi\alpha}{2\beta}$ could be taken sensibly equal to unity. The remainder of the series, namely,

$$\sum_{n}^{\infty}\frac{1}{\pi(2n+1)\left[\pi^2(2n+1)^2+16\alpha^2\right]^2},$$

was then obtained by expanding the denominator by the binomial theorem, thus

$$\sum_{n}^{\infty}\frac{1}{\pi^5(2n+1)^5}-2\left(16\alpha^2\right)\sum_{n}^{\infty}\frac{1}{\pi^7(2n+1)^7}+3\left(16\alpha^2\right)^2\sum_{n}^{\infty}\frac{1}{\pi^9(2n+1)^9}-\ldots$$

The successive terms were easily calculated from the values of $\sum_{0}\frac{1}{n^{2s+1}}$ given in CHRYSTAL's 'Algebra,' vol. 2, chapter XXX, § 15.

The stresses at A and B were calculated from formulæ (18) and (19). If S_A, S_B denote these stresses, we find easily

$$\frac{S_A}{\mu} = -\tau c\tanh 2\alpha\cosh\alpha + 8\tau c\alpha\left(\frac{\cosh 2\alpha+\cos 2\beta}{\cosh\alpha}\right)\sum_{n=0}^{n=\infty}\frac{\operatorname{sech}\frac{\widehat{2n+1}\pi\beta}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}$$

$$= \tau c\sinh\alpha\left(\sec 2\beta-1\right) - 8\tau c\beta\left(\frac{\cosh 2\alpha+\cos 2\beta}{\cosh\alpha}\right)\sum_{n=0}^{n=\infty}\frac{(-1)^n\tanh\frac{\widehat{2n+1}\pi\alpha}{2\beta}}{\pi^2(2n+1)^2-16\beta^2} \quad (21),$$

$$\frac{S_B}{\mu} = \tau c\sin\beta\left(1-\operatorname{sech}2\alpha\right) + 8\tau c\alpha\left(\frac{\cosh 2\alpha+\cos 2\beta}{\cos\beta}\right)\sum_{n=0}^{n=\infty}\frac{(-1)^n\tanh\frac{\widehat{2n+1}\pi\beta}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}$$

$$= \tau c\tan 2\beta\cos\beta - 8\tau c\beta\left(\frac{\cosh 2\alpha+\cos 2\beta}{\cos\beta}\right)\sum_{n=0}^{n=\infty}\frac{\operatorname{sech}\frac{\widehat{2n+1}\pi\alpha}{2\beta}}{(2n+1)^2\pi^2-16\beta^2} \quad . \quad . \quad . \quad (22).$$

The first expressions for S_A, S_B were used in each case as the main basis of the calculation, but the results were partly verified by means of the second expressions.

With regard to the accuracy obtained, I may say that, where I was able to verify, I found the values of the stresses correct to five, and sometimes to six, significant figures. In the values of the torsion moment the first five figures generally agreed, so that, on the whole, the results may be considered correct to the number of figures given.

The first series for S_B and the second series for S_A are very slowly convergent indeed. The method adopted in dealing with them was the following:

The series

$$\sum_{n=0}^{n=\infty} \frac{(-1)^n \tanh \dfrac{\overline{2n+1}\pi\beta}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}$$

was broken up into two series

$$\sum_{n=0}^{n=\infty} \frac{(-1)^n}{\pi^2(2n+1)^2+16\alpha^2} - \sum_{n=0}^{n=\infty} \frac{(-1)^n\left(1-\tanh \dfrac{\overline{2n+1}\pi\beta}{2\alpha}\right)}{\pi^2(2n+1)^2+16\alpha^2}.$$

The latter series converges rapidly, and was easily calculated. The former was calculated to an even number of terms, and the remainder obtained by means of the Euler-Maclaurin sum-formula, thus:

$$\sum_{n=2x-1}^{n=\infty} \frac{(-1)^n}{\pi^2(2n+1)^2+16\alpha^2} = \sum_{x}^{\infty} \frac{1}{\pi^2(4x+1)^2+16\alpha^2} - \sum_{x}^{\infty} \frac{1}{\pi^2(4x-1)^2+16\alpha^2}.$$

Now

$$\Sigma u_x = C + \int u_x\,dx - \tfrac{1}{2}u_x + \frac{B_1}{2!}\frac{du_x}{dx} - \frac{B_2}{4!}\frac{d^3u_x}{dx^3} + \ldots,$$

where $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, &c.

Let me write

$$\rho^2 = (4x+1)^2\pi^2 + 16\alpha^2,$$

$$\theta = \tan^{-1}\frac{4\alpha}{(4x+1)\pi}, \qquad u_x = \frac{1}{\rho^2}.$$

Then

$$\int u_x\,dx = -\frac{\theta}{16\alpha\pi} \qquad \frac{d^n u_x}{dx^n} = \frac{(-1)^n \sin \overline{n+1}\theta}{\rho^{n+1}} \frac{(4\pi)^n\, n!}{4\alpha},$$

whence

$$\sum^{x-1} u_x = C - \frac{\theta}{16\alpha\pi} - \frac{1}{2\rho^2} - \frac{B_1}{2}\frac{\sin 2\theta}{\rho^2}\cdot\frac{(4\pi)}{4\alpha} + \frac{B_2}{4}\frac{\sin 4\theta}{\rho^4}\frac{(4\pi)^3}{4\alpha} - \ldots$$

Put $x = \infty$, $\theta = 0$, $\rho = \infty$,

$$\sum^{\infty} u_x = C,$$

therefore,

$$\sum_{x}^{\infty} u_x = \frac{\theta}{16\alpha\pi} + \frac{1}{2\rho^2} + \frac{B_1}{2}\frac{\sin 2\theta}{\rho^2}\left(\frac{\pi}{\alpha}\right) - \frac{B_2}{4}\cdot\frac{\pi}{\alpha}\cdot\frac{\sin 4\theta}{\rho^4}\,16\pi^2 + \ldots$$

In like manner if $v_x = \dfrac{1}{\rho'^2}$,

$$\rho'^2 = \pi^2 (4x - 1)^2 + 16\alpha^2,$$

$$\theta' = \tan^{-1} \frac{4\alpha}{(4x-1)\pi},$$

$$\sum_x^\infty v_x = \frac{\theta'}{16\alpha\pi} + \frac{1}{2\rho'^2} + \frac{B_1}{2}\frac{\sin 2\theta'}{\rho'^2}\left(\frac{\pi}{\alpha}\right) - \frac{B_2}{4}\left(\frac{\pi}{\alpha}\right)\frac{\sin 4\theta'}{\rho'^4} 16\pi^2 + \dots$$

These series converge fairly rapidly if x is at all large, and thus the remainder can be obtained.

Even with the help of all these devices the labour of calculating the moment and stress for the sixteen sections was considerable.

The values of the hyperbolic functions were taken from GUDERMANN'S Tables ('Theorie der Potenzial oder Cyklisch-hyperbolischen Functionen'), and from GLAISHER'S and NEUMAN'S Tables of the Exponential Function ('Cambridge Phil. Trans.,' vol. 13).

§ 9. *Values of the Torsional Rigidity.*

The first quantity calculated was the torsion moment. The values found are shown in the table below.

TABLE of $M/\mu\tau c^4$.

	$\beta = \pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$\alpha = \pi/6$	·1710	·3116	·4055	·4676
$\alpha = \pi/3$	·8764	2·0317	3·2205	4·8117
$\alpha = \pi/2$	3·8798	9·4161	16·442	29·912
$\alpha = 2\pi/3$	22·898	54·824	96·411	194·18

It is interesting to compare this table with the table of values of the torsion moment, as given by DE SAINT-VENANT'S empirical formula, viz.,

$$\text{torsion moment} = \frac{\mu\tau}{40}\frac{A^4}{I},$$

where A = area, I = moment of inertia of section about its centroid.

Calling M' this value of the torsion moment, we have

$$\frac{M'}{\mu\tau c^4} = \frac{1}{5}\frac{(\beta \sinh 2\alpha + \alpha \sin 2\beta)^4}{\beta \sinh 4\alpha - \alpha \sin 4\beta},$$

whence we obtain the following set of values :—

TABLE of $M'/\mu\tau c^4$.

	$\beta = \pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$\alpha = \pi/6$	·1835	·3266	·4152	·4723
$\alpha = \pi/3$	·9914	2·3759	3·8023	6·0121
$\alpha = \pi/2$	4·3368	12·195	23·260	51·500
$\alpha = 2\pi/3$	23·289	71·826	152·96	420·95

If we compare this with the preceding table, we see at once that although the agreement is fairly good for the more compact sections, SAINT-VENANT's empirical formula utterly breaks down for deeply indented sections. That, indeed, might have been expected, since it takes no account of slits cut into the material. One rather noticeable feature in the comparison is that SAINT-VENANT's formula always gives too high a value for the torsional rigidity.

§ 10. *Comparison with the Circle. "Relative" Torsional Rigidity.*

In order, however, to compare properly the efficiency or usefulness of these various sections, it was found advisable to refer each of them to some kind of standard, or unit. The most obvious standard, as I thought, was the circular section, this being the one whose torsion obeys the most simple laws. I determined, therefore, to compare every section with the circular section of the same area.

Now if r be the radius of this circular section, its torsion moment $M_0 = \frac{1}{2}\pi\mu\tau r^4$, and the maximum stress $S_0 = \mu\tau r$. To find r, we have the equation :

$$\pi r^2 = \text{area of given section} = c^2(\beta \sinh 2\alpha + \alpha \sinh 2\beta),$$

whence

$$\frac{M_0}{\mu\tau c^4} = \frac{\pi}{2}\left(\frac{\beta \sinh 2\alpha + \alpha \sin 2\beta}{\pi}\right)^2,$$

$$\frac{S_0}{\mu\tau c} = \left(\frac{\beta \sinh 2\alpha + \alpha \sin 2\beta}{\pi}\right)^{\frac{1}{2}}.$$

The values of $\left(\frac{\beta \sinh 2\alpha + \alpha \sin 2\beta}{\pi}\right)$ for the various sections are easily found.

TABLE of $\left(\dfrac{\beta \sinh 2\alpha + \alpha \sin 2\beta}{\pi}\right)$.

	$\beta = \pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$\alpha = \pi/6$	·3526	·4790	·5608	·6247
$\alpha = \pi/3$	·9551	1·3330	1·6216	1·9993
$\alpha = \pi/2$	2·3578	3·3872	4·2826	5·7744
$\alpha = 2\pi/3$	6·0713	8·9076	11·565	16·482

whence we obtain the following table giving us M/M_0:—

	$\beta = \pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$\alpha = \pi/6$	·8756	·8644	·8208	·7628
$\alpha = \pi/3$	·6116	·7279	·7797	·7663
$\alpha = \pi/2$	·4443	·5225	·5707	·5711
$\alpha = 2\pi/3$	·3955	·4399	4590	·4551

When we look at this table, we observe immediately that the torsional rigidity decreases, compared with the torsional rigidity of the circular section, as we increase α, that is to say, as we decrease the thickness of the neck with regard to the other linear dimension. This indeed might have been expected, for it is clear that such a process must weaken the rigidity enormously, inasmuch as it tends to render the two halves of the section independent of each other.

When $\alpha = \pi/6$, $\beta = \pi/6$, the ratio M/M_0 is greatest. In this case the section does not deviate very much from a square. (For very small values of α and β the section is, of course, a rectangle.) This result shows us therefore that, so far as torsional rigidity is concerned, the square is a more efficient form of section than any one of those dealt with in the present paragraph.

That the circle is a more efficient type of section for rigidity is quite evident from the table, since all the values in it are less than unity.

It may be interesting to note what are the values of M/M_0 for the full ellipse. When we use the values given by SAINT-VENANT in his memoir on torsion

$$M = \frac{\mu\tau\pi a^3 b^3}{a^2 + b^2} \quad \text{and} \quad M_0 = \frac{\mu\tau\pi a^2 b^2}{2},$$

$$\frac{M}{M_0} = \frac{2ab}{a^2 + b^2} = \tanh 2\alpha, \quad \text{if} \quad b/a = \tanh \alpha.$$

Hence we have, for the full ellipse

	$\alpha = \pi/6$.	$\alpha = \pi/3$.	$\alpha = \pi/2$.	$\alpha = 2\pi/3$.
M/M_0	·7807	·9701	·9963	·9995

If we compare this with the previous table, we see that for the flattest ellipse, $\alpha = \pi/6$, the ratio of the torsional rigidity to the torsional rigidity of the circular section of equal area, which I propose to call for brevity the *relative* torsional rigidity, is greater for the truncated than for the full ellipse, except in the last case, $\beta = \pi/2$. This last must necessarily be, since the strength of the section should be reduced by cutting two slits into it along the major axis. For the higher values of α we see that the relative torsional rigidity is always greater for the full than for the truncated ellipse.

§ 11. *Values of the Stresses at the Points of Symmetry.*

Passing now to the values of the stress, the values of $S_B/\mu\tau c$ are given, for the sixteen symmetrical sections, in the table below.

TABLE of $S_B/\mu\tau c$.

	$\beta = \pi/6$.	$\beta = \pi/4$.	$\beta = \pi/3$.	$\beta = \pi/2$.
$\alpha = \pi/6$	·7594	·8421	·8949	∞
$\alpha = \pi/3$	1·1482	1·6735	2·2690	∞
$\alpha = \pi/2$	1·3084	2·2798	3·6780	∞
$\alpha = 2\pi/3$	1·3897	2·7955	5·2484	∞

The values of $S_A/\mu\tau c$ were only calculated for $\beta = \pi/6$ and $\beta = \pi/4$, it being clear that for the given values of α, S_A would be less than S_B for $\beta = \pi/3$. The values of $S_A/(\mu\tau c)$ are given in the following table:—

TABLE of $S_A/\mu\tau c$.

	$\beta = \pi/6$.	$\pi/4$.
$\alpha = \pi/6$	·6730	·8126
$\alpha = \pi/3$	·8394	1·0975
$\alpha = \pi/2$	1·1971	1·5176
$\alpha = 2\pi/3$	1·8987	2·3650

The values of $S_A/\mu\tau c$, as calculated from the formula (21) above, actually turn out to be negative. The sign is, however, of no importance.

Several important results are seen to follow at once from these tables.

In the first place, when a keyway cut into a shaft of elliptic cross-section reduces to a mere slit, the stress at the inner extremity of the keyway is seen to be infinite, although this keyway is the limit of a single continuous curve and not of two curves making a sharp angle, as is the case for a slit along a radius of a circle, obtained as the limit of a keyway of the shape of a sector of the circle.

Such slits are thus bound to produce rupture or plastic flow of the material at their deepest points, whatever be the manner in which we approximate to them in practice.

The second point of importance, which these tables bring out clearly, is that the maximum strain and stress do not always occur, as most of the results obtained by DE SAINT-VENANT would lead one to suppose, and as THOMSON and TAIT ('Natural Philosophy,' vol. 1, Part II., § 710), and BOUSSINESQ ('Journal de Mathématiques,' Série II., vol. 16, p. 200) assumed, at the point of the boundary nearest the centre.

Indeed SAINT-VENANT himself, in his edition of NAVIER'S 'Leçons de Mécanique' (§ 33, p. 313), has given an example to the contrary, and it happens that the section dealt with in this example is closely analogous to the sections of fig. 2 in this paper. The shape of the section is reproduced in fig. 11 (p. 342), from SAINT-VENANT'S 'Leçons de NAVIER.' He calls it a "*section en double spatule analogue à celle d'un rail de chemin de fer.*" He gives two numerical examples in which the ratio of breadth to length of the section is ·20 and ·14, corresponding for our sections, when $\beta = \pi/6$, to $\alpha = 1{\cdot}647$ and $\alpha = 1{\cdot}985$ respectively. He finds in these two cases that the fail-points are *not* at the point of symmetry on the contour which is closest to the centre, but at points on the contour at a distance from the axis of symmetry of ·46 and ·52 of the half-length respectively.

Now it is easy to see from the tables above that the result which one would expect according to the ordinary rule, namely $S_B > S_A$, does hold in fifteen out of the sixteen cases, but there is one exception in the case of the section $\alpha = 2\pi/3$, $\beta = \pi/6$, when the greater of the two stresses is found to occur at A, the point further from the centre.

I was much struck at first by this apparently solitary deviation from the rule, and was inclined to ascribe it to some error in the arithmetic.

In order to test this, I calculated the values of S_A and S_B for the neighbouring section, $\beta = \pi/6$, $\alpha = 3\pi/4$. I found

$$\frac{S_A}{\mu\tau c} = 2{\cdot}4333, \qquad \frac{S_B}{\mu\tau c} = 1{\cdot}4144,$$

confirming the previous exception.

I then took the expressions (21) and (22) for S_A and S_B, and tried to determine the limits to which they tended, when α was made very great.

Clearly, when we look at the second of expressions (22), we see that the second term must ultimately become vanishingly small, provided $\pi\alpha/2\beta > 2\alpha$, or $\beta < \pi/4$. For such values of β, then, S_B tends to the definite limit $\mu\tau c \tan 2\beta \cos\beta$, *i.e.*, for $\beta = \pi/6$, $S_B/\mu\tau c$ tends to the limit 1·5 for large values of α.

For values of $\beta > \pi/4$ it would seem that S_B increases numerically to an indefinite extent.

Consider now the second expression for S_A. Clearly, if α exceed a certain value, $\tanh \dfrac{\overline{2n+1}\pi\alpha}{2\beta}$ approximates so closely to unity that we may replace it in the series by unity, and the error will only be a small fraction of the series itself. We then find

$$\frac{S_A}{\mu\tau c} \text{ tends to } \sinh\alpha\,(\sec 2\beta - 1) - \frac{8\beta\,(\cosh 2\alpha + \cos 2\beta)}{\cosh\alpha}\sum_0^\infty \frac{(-1)^n}{\pi^2(2n+1)^2 - 16\beta^2},$$

when α is large.

Substituting $\beta = \pi/6$, and using the Euler-Maclaurin sum-formula to calculate the series, I find

$$\left(\frac{S_A}{\mu\tau c}\right)_{\substack{\beta=\pi/6\\ \alpha\text{ large}}} = \sinh\alpha - \frac{12}{\pi}\,\frac{(\cosh 2\alpha + \frac{1}{2})}{\cosh\alpha}\,(\cdot 190086),$$

and remembering that

$$\frac{\cosh 2\alpha + \frac{1}{2}}{\cosh\alpha} = 2\cosh\alpha - \frac{1}{2\cosh\alpha} = 2\cosh\alpha, \text{ if } \alpha \text{ large},$$

$$\sinh\alpha = \cosh\alpha - \frac{1}{\cosh\alpha + \sinh\alpha} = \cosh\alpha, \text{ if } \alpha \text{ large},$$

we see that, when α is large,

$$\left(\frac{S_A}{\mu\tau c}\right)_{\beta=\pi/6} = -\cosh\alpha\,(\cdot 452147),$$

and, therefore, increases numerically indefinitely.

On the other hand, if α be very small, we get a flat section, A being now the point nearest to the centre. Looking at expressions (21) and (22) we see easily that

$$\left(\frac{S_A}{\mu\tau c}\right) \text{ tends to } -2\alpha$$

and

$$\left(\frac{S_B}{\mu\tau c}\right) \text{ to } (1 - \operatorname{sech} 2\alpha)\sin\beta + \frac{16}{\pi^2}\,\alpha\cos\beta\sum_0^\infty \frac{(-1)^n}{(2n+1)^2},$$

or, neglecting squares of α compared with the first power,

$$\left(\frac{S_B}{\mu\tau c}\right)_{\alpha\text{ small}} = \frac{16}{\pi^2}\,\alpha\cos\beta\sum_0^\infty \frac{(-1)^n}{(2n+1)^2}.$$

Now $\displaystyle\sum_0^\alpha \frac{(-1)^n}{(2n+1)^2}$ lies between 1 and 8/9, hence $\left(\dfrac{S_B}{\mu\tau c}\right) < \left(\dfrac{16}{\pi^2}\cos\beta\right)\alpha$,

and $\frac{16}{\pi^2}\cos\beta$ being always < 2, S_A is always numerically greater than S_B for small values of α. This confirms the usual rule, which we should expect, since the section approximates in this case to a flat rectangular section.

§ 12. *Discussion of the Variations in these Stresses.*

The variations in the stresses S_A, S_B are shown in fig. 9 (p. 340). This figure gives, not the values of the stresses themselves, but the ratio of the stresses to the maximum stress S_0 in the circular section of equal area. This ratio is plotted as ordinate to the various values of α as abscissæ. Of course, S_A, as given by the expressions (21) being negative throughout, the curve shows the ratio $(-S_A/S_0)$ and not S_A/S_0.

The diagram is comparatively rough, especially near the origin, owing to the very limited number of points which I could calculate. The value of β selected was $\beta = \pi/6$.

When α is small, it is not difficult to show that, for $\beta = \pi/6$

$$S_B/S_0 = 1{\cdot}799 \times \sqrt{\alpha}, \qquad S_A/S =_0 - 2{\cdot}563 \times \sqrt{\alpha},$$

and when α is large

$$S_B/S_0 = 5{\cdot}196e^{-\alpha}, \qquad S_A/S_0 = -{\cdot}7831.$$

These last enable us to see the form of the curves near the origin and at a great distance from it. They are perpendicular to the axis of α at the origin, and at first the curve of S_B lies below that of S_A. At some point between $\alpha = 0$ and $\alpha = {\cdot}5$ they cross. This corresponds to the case of the square for rectangular cross-sections. S_A is now less than S_B, but instead of its remaining so, as we should have expected, the curves cross again near the value of $\alpha = 1{\cdot}7$. The curve of S_A/S_0 now tends to become practically a straight line parallel to the axis, at a distance ·7831 from it, and the curve of S_B/S_0 approaches the axis asymptotically.

The values of S_A/S_0, S_B/S_0 are given in the tables below.

TABLE of S_A/S_0.

	$\beta = \pi/6$.	$\beta = \pi/4$.
$\alpha = \pi/6$	1·1334	1·1741
$\alpha = \pi/3$	·8589	·9506
$\alpha = \pi/2$	·7796	·8246
$\alpha = 2\pi/3$	·7706	·7924
$\alpha = 3\pi/4$	·7724	
$\alpha = \infty$	·7831	

TABLE of S_B/S_0.

	$\beta = \pi/6$.	$\beta = \pi/4$.	$\beta = \pi/3$.
$\alpha = \pi/6$	1·2790	1·2168	1·1950
$\alpha = \pi/3$	1·1749	1·4494	1·7819
$\alpha = \pi/2$	·8521	1·2388	1·7773
$\alpha = 2\pi/3$	·5640	·9366	1·2928
$\alpha = 3\pi/4$	·4490		

§ 13. *Investigation of the Fail-Points other than the Points of Symmetry.*

The question now arises, are A and B really points of *maximum slide*, and therefore shear? Assuming that the fail-points do occur on the contour, are we sure that there are not other points on the boundary where greater maxima occur, and may we not be in presence of a case like that of SAINT-VENANT'S *section en double spatule.*

To see whether this is so, it is necessary to go at some length into the equations determining the maximum stress.

Consider first the sides of the section $\eta = \pm \beta$. It is clear that the resultant stress along this side will be simply the component parallel to the contour, since the normal component must vanish in virtue of the boundary conditions. Calling S this resultant stress, we find easily

$$S = \frac{\mu}{c\sqrt{J}}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\eta\right),$$

where

$$J = \tfrac{1}{2}(\cosh 2\xi + \cos 2\eta).$$

Hence we have to make

$$\left(\frac{Sc}{\mu}\right)^2 = \frac{1}{J}\left[\left(\frac{dw}{d\xi}\right)_{\eta=\beta} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right]^2,$$

a maximum with regard to ξ.

We have therefore

$$\frac{d}{d\xi}\left(\frac{1}{J}\left\{\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right\}^2\right) = 0,$$

that is

$$\frac{1}{J}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right)\left[2\frac{d^2w}{d\xi^2} - \frac{1}{J}\frac{dJ}{d\xi}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right)\right] = 0,$$

Take now the second value of w as given by (17), and we get

$$2\frac{d^2w}{d\xi^2} - \frac{1}{J}\frac{dJ}{d\xi}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right) = \tau c^2 \sinh 2\xi \tan 2\beta - 8\tau c^2 (\cosh 2\alpha + \cos 2\beta)$$

$$\times \sum_{n=0}^{n=\infty} \left\{ \frac{(2n+1)\pi}{(2n+1)^2\pi^2 - 16\beta^2} \frac{\sinh \dfrac{\widehat{2n+1}\pi\xi}{2\beta}}{\cosh \dfrac{\widehat{2n+1}\pi\alpha}{2\beta}} - \frac{2\beta \sinh 2\xi}{\cosh 2\xi + \cos 2\beta} \frac{\cosh \dfrac{\widehat{2n+1}\pi\xi}{2\beta}}{[\overline{2n+1}|^2\pi^2 - 16\beta^2] \cosh \dfrac{\widehat{2n+1}\pi\alpha}{2\beta}} \right\}.$$

Hence the equation giving the maxima and minima is

$$\tan 2\beta \sinh 2\xi$$

$$= 4\,\frac{(\cosh 2\alpha + \cos 2\beta)(\cosh 2\xi + 2\cos 2\beta)}{\cosh 2\xi + \cos 2\beta} \sum_{n=0}^{n=\infty} \frac{(2n+1)\pi}{(2n+1)^2\pi^2 - 16\beta^2} \frac{\sinh \dfrac{\widehat{2n+1}\pi\xi}{2\beta}}{\cosh \dfrac{\widehat{2n+1}\pi\alpha}{2\beta}}$$

$$+ \frac{2(\cosh 2\alpha + \cos 2\beta)}{\cosh 2\xi + \cos 2\beta} \sum_{n=0}^{n=\infty} \left\{ \frac{\sinh [\widehat{2n+1}\pi + 4\beta]\dfrac{\xi}{2\beta}}{(2n+1)\pi + 4\beta} + \frac{\sinh [\widehat{2n+1}\pi - 4\beta]\dfrac{\xi}{\beta}}{(2n+1)\pi - 4\beta} \right\} \operatorname{sech} \frac{\widehat{2n+1}\pi\alpha}{2\beta}.$$

One root of this equation is $\xi = 0$. Now it is clear that since S^2 is zero, and therefore a minimum at $\xi = \alpha$, if it be not a maximum at $\xi = 0$, then there must be a maximum somewhere between $\xi = 0$ and $\xi = \alpha$.

To find whether $\xi = 0$ is a maximum or not, we have to investigate the sign of

$$\left[\frac{d^2}{d\xi^2}\left(\frac{1}{J}\left\{\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right\}^2\right)\right]_{\substack{\xi=0\\ \eta=\beta}},$$

i.e., we have to investigate the sign of

$$\frac{1}{J}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right)\frac{d}{d\xi}\left[2\frac{d^2w}{d\xi^2} - \frac{1}{J}\frac{dJ}{d\xi}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right)\right],$$

all the other terms vanishing when $\xi = 0$.

Now $\dfrac{1}{J}\left(\dfrac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right)$ being always positive, we have to investigate the sign of

$$\frac{d}{d\xi}\left[2\frac{d^2w}{d\xi^2} - \frac{1}{J}\frac{dJ}{d\xi}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right)\right] \text{ when } \xi = 0,$$

and therefore ultimately the sign of

$$E = \beta \tan 2\beta - \frac{(\cosh 2\alpha + \cos 2\beta)(1 + 2\cos 2\beta)}{1 + \cos 2\beta} \sum_{n=0}^{n=\infty} \frac{(2n+1)^2\pi^2}{(2n+1)^2\pi^2 - 16\beta^2} \operatorname{sech} \frac{\widehat{2n+1}\pi\alpha}{2\beta}$$

$$- \frac{\cosh 2\alpha + \cos 2\beta}{1 + \cos 2\beta} \sum_{n=0}^{n=\infty} \operatorname{sech} \frac{\widehat{2n+1}\pi\alpha}{2\beta}.$$

I may remark here that all these differentiations are permissible, provided that ξ be less than α by a *finite amount*, however small, the series being then uniformly convergent. We may not, therefore, make α actually zero in the last expression. If, however, α is small but finite, then it is easy to see that E must be negative if $\beta < \pi/4$. Also if $\beta < \pi/4$, E algebraically increases continuously, until for a certain value α_0 of α it reaches the value zero. For all higher values of α it remains steadily positive.

§ 14. *Critical Values of α and β.*

It follows, therefore, that in all sections for which $\beta < \pi/4$, the maximum stress along the sides $\eta = \pm\beta$ occurs at the point B, until a certain critical value, $\alpha = \alpha_0$, is reached, when the stress at B becomes a minimum, and we now have two points of maximum stress on either side of B.

When $\beta = \pi/4$, E is apparently infinite, but it really tends to a finite limit. If we put $\beta = \pi/4 - \epsilon$ where ϵ will ultimately be made very small, and neglect terms in ϵ, ϵ^2, &c., we find that the expression becomes

$$\frac{\pi}{8\epsilon} - \tfrac{1}{2} - 2\cosh 2\alpha \sum_{n=0}^{n=\infty} \operatorname{sech} \widehat{2n+1}\, 2\alpha - \cosh 2\alpha \sum_{n=1}^{n=\infty} \operatorname{sech}(2n+1)\, 2\alpha + \tfrac{1}{4}$$

$$- (\cosh 2\alpha + 2\epsilon)(1 + 2\epsilon)\left(\frac{\pi}{2} - 2\epsilon\right)\frac{1}{4\epsilon} \operatorname{sech}\left\{2\alpha\left(1 + \frac{4\epsilon}{\pi}\right)\right\}$$

$$= -\frac{\pi}{4} - \frac{\pi}{4}\operatorname{sech} 2\alpha + \alpha \tanh 2\alpha - \frac{7}{\pi} - 3\cosh 2\alpha \sum_{n=1}^{n=\infty} \operatorname{sech}(2n+1)\, 2\alpha.$$

When α is very great, this will ultimately be positive. Hence, here also we shall have a critical value α_0. This value, however, is easily seen to lie outside the values of α taken in this paper.

I have not investigated so carefully the cases when $\beta > \pi/4$, as, for the particular sections selected, the maximum stress certainly occurs at B.

One point, however, is clear. When α is very large, the second and third terms settle the sign of E. The sign of these terms, again, is settled by the sign of the leading terms. E is negative if

$$1 + (1 + 2\cos 2\beta)\frac{\pi^2}{\pi^2 - 16\beta^2}$$

is positive, *i.e.*, if

$$\frac{\cos^2\beta}{4\beta^2} + \frac{1 + 2\cos 2\beta}{\pi^2 - 16\beta^2} > 0.$$

This will always be the case, provided β be greater than the root of the equation

$$\frac{2\beta}{\pi} = \cos\beta.$$

This root is $53^\circ\ 31'$ nearly.

Hence, for values of $\beta > 53^\circ\ 31'$ the maximum stress always occurs at the thinnest point of the section. This is the case for the sections of fig. 4 with broad keyways, for which $\beta = 60^\circ$.

Returning to the case of sections for which $\beta < \pi/4$ we have α_0 given by the equation

$$\beta \tan 2\beta = (\cosh 2\alpha_0 + \cos 2\beta)\left[2\sum_{n=0}^{n=\infty} \operatorname{sech}\frac{\overline{2n+1}\pi\alpha_0}{2\beta} + \frac{16\beta^2(1+2\cos 2\beta)}{1+\cos 2\beta}\sum_{n=0}^{n=\infty}\frac{\operatorname{sech}\dfrac{\overline{2n+1}\pi\alpha_0}{2\beta}}{\overline{2n+1}|^2\pi^2 - 16\beta 2}\right].$$

Putting $\beta = \pi/6$ in this we have for the transcendental equation giving α_0 in this case

$$\frac{\pi}{2\sqrt{3}} = (\cosh 2\alpha_0 + \tfrac{1}{2})\left[2\sum_{n=0}^{n=\infty}\operatorname{sech}[2n+1]3\alpha_0 + \frac{16}{3}\sum_{n=0}^{n=\infty}\frac{\operatorname{sech}[2n+1]3\alpha_0}{(6n+1)(6n+5)}\right].$$

Now, for fairly large values of α_0, we may neglect all terms in this summation except the first. We then have

$$\frac{\pi}{2\sqrt{3}} = \frac{23}{15}\,\frac{2\cosh 2\alpha_0 + 1}{\cosh 3\alpha_0} = \frac{23}{15}(\operatorname{sech}\alpha_0 + 2\operatorname{sech} 3\alpha_0)$$

whence I find $\alpha_0 = 1{\cdot}225$.

Hence for values of α greater than this the maximum occurs at certain points on the contour, given by previously obtained equation in ξ.

§ 15. *Calculation of the Position of Fail-Points and the Magnitude of the Maximum Stress for the Sections $\beta = \pi/6$, $\alpha = \pi/2$ and $\beta = \pi/6$, $\alpha = 2\pi/3$.*

We have, therefore, if we wish to find the maximum stress for the sections $\alpha = \pi/2$ and $\alpha = 2\pi/3$ when $\beta = \pi/6$, to solve this transcendental equation :

$$\frac{\pi\sqrt{3}}{6}\,\frac{\sinh 2\xi\,(2\cosh 2\xi + 1)}{2\cosh 2\alpha + 1} = 6\,(\cosh 2\xi + 1)\sum_{n=0}^{n=\infty}\frac{2n+1}{(6n+1)(6n+5)}\,\frac{\sinh(6n+3)\xi}{\cosh(6n+3)\alpha}$$
$$+\sum_{n=0}^{n=\infty}\left\{\frac{\sinh(6n+5)\xi}{6n+5} + \frac{\sinh(6n+1)\xi}{6n+1}\right\}\frac{1}{\cosh(6n+3)\alpha}$$

which may be put into the slightly simpler form

$$\frac{\pi\sqrt{3}}{6(\cosh 2\alpha + \frac{1}{2})} = \frac{12\cosh\xi}{\sinh 3\xi}\sum_{n=0}^{n=\infty}\frac{(2n+1)}{(6n+1)(6n+5)}\,\frac{\sinh(6n+3)\xi}{\cosh(6n+3)\alpha}$$
$$+\frac{1}{\cosh\xi\sinh 3\xi}\sum_{n=0}^{n=\infty}\left\{\frac{\sinh(6n+5)\xi}{(6n+5)\cosh(6n+3)\alpha} + \frac{\sinh(6n+1)\xi}{(6n+1)\cosh(6n+3)\alpha}\right\}.$$

I find the roots of this to be approximately given by

$$\xi = 1{\cdot}475 \text{ when } \alpha = 2\pi/3,$$
$$\xi = \ {\cdot}821 \text{ when } \alpha = \pi/2.$$

The corresponding values of $S/\mu\tau c$ are found to be $2{\cdot}1084$ and $1{\cdot}4041$ respectively. Comparing these with the values of $S_A/\mu\tau c$ given on p. 331, we see that the values of the stresses corresponding to the maximum on the broad side are the greater, and hence we have really a case absolutely analogous to SAINT-VENANT'S *section en double spatule*, with four fail-points symmetrically distributed, all of them lying on the broad sides of the contour.

§ 16. *Case where α is made very great.*

It is interesting to see to what limit the fail-points tend, when α is made very great. We have

$$\frac{S}{\mu\tau c} = \frac{1}{\sqrt{J}}\left[\frac{1}{2}\tan 2\beta\,(\cosh 2\xi + \cos 2\beta) - 8\beta\,(\cosh 2\alpha + \cos 2\beta)\sum_{n=0}^{n=\infty}\frac{\cosh\frac{\widehat{2n+1}\pi\xi}{2\beta}}{\left[(2n+1)^2\pi^2 - 16\beta^2\right]\cosh\frac{\widehat{2n+1}\pi\alpha}{2\beta}}\right].$$

Replace now $\frac{1}{2}\tan 2\beta$ by its equivalent $8\beta\sum\limits_{n=0}^{n=\infty}\frac{1}{(2n+1)^2\pi^2 - 16\beta^2}$.

$$\frac{S}{\mu\tau c} = \frac{8\beta}{\sqrt{J}}\sum_{n=0}^{\infty}\frac{\left\{(\cosh 2\xi + \cos 2\beta) - (\cosh 2\alpha + \cos 2\beta)\dfrac{\cosh\frac{\widehat{2n+1}\pi\xi}{2\beta}}{\cosh\frac{\widehat{2n+1}\pi\alpha}{2\beta}}\right\}}{(2n+1)^2\pi^2 - 16\beta^2}.$$

Now if ξ, α be great, we have $\cosh = \frac{1}{2}$ (exp.) approximately. Hence, if we suppose $\xi = \alpha - \theta$ where θ is finite, so that we are dealing with points whose distances from the centre bear a finite ratio to the dimensions of the section, we find

$$\frac{S}{\mu\tau c} = 8\beta e^{\alpha}\sum_{n=0}^{\infty}\frac{e^{-\theta} - e^{\theta\left(1-\frac{\widehat{2n+1}\pi}{2\beta}\right)}}{\overline{2n+1}|^2\pi^2 - 16\beta^2}$$

+ terms negligible in comparison and ultimately vanishingly small when α is made infinite.

Hence we have to make

$$\sum_{n=0}^{n=\infty}\frac{e^{-\theta} - e^{\theta\left(1-\frac{(2n+1)\pi}{2\beta}\right)}}{(2n+1)^2\pi^2 - 16\beta^2}$$

a maximum.

Differentiating, the series being absolutely and uniformly convergent, we get

$$e^{-\theta} \sum_{n=0}^{n=\infty} \frac{1}{(2n+1)^2\pi^2 - 16\beta^2} - \sum_{n=0}^{n=\infty} \frac{\left(\frac{\overline{2n+1}\pi}{2\beta} - 1\right) e^{\theta\left(1-\frac{(2n+1)\pi}{2\beta}\right)}}{(2n+1)^2\pi^2 - 16\beta^2} = 0,$$

i.e.,

$$\frac{\tan 2\beta}{8} = \sum_{n=0}^{n=\infty} \frac{(2n+1)\pi - 2\beta}{(2n+1)^2\pi^2 - 16\beta^2} e^{\theta\left(2-\frac{(2n+1)\pi}{2\beta}\right)}.$$

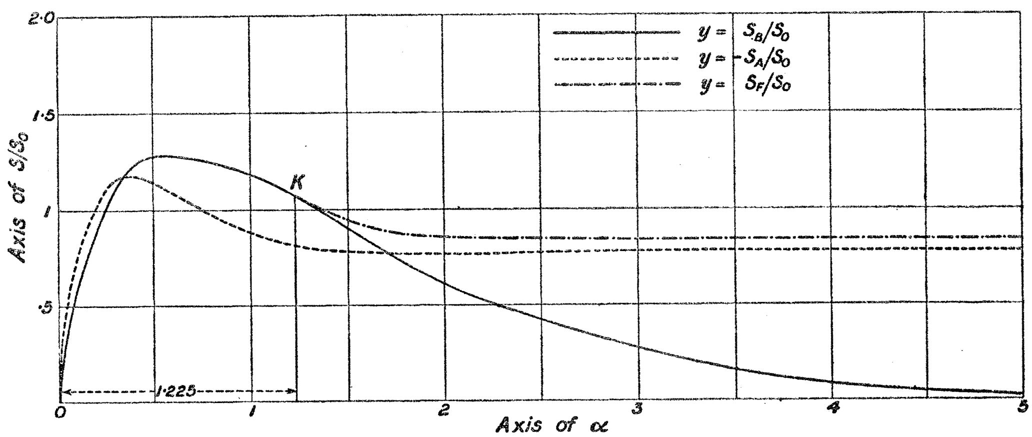
This is the equation giving θ. When we put in it $\beta = \pi/6$

$$\frac{\pi}{16\sqrt{3}} = \sum_{n=0}^{n=\infty} \frac{3n+1}{(6n+1)(6n+5)} e^{-(6n+1)\theta}.$$

I find from this $\theta = \cdot 577$ approximately, and the corresponding value of $S/\mu\tau c = \cdot 48984 \cosh \alpha$. Referring to the value of $S_A/\mu\tau c$ given on p. 333, we see that the real fail-point is on the broader side.

The curve of true maximum stress $\widehat{S}/S_0$ is shown also in fig. 9. It joins on to the

Fig. 9.



Showing the variation in the value of S/S_0 as α increases, at the three points A, B, F, of Fig. 2. β = const. = $\pi/6$.

curve of S_B/S_0 at the point corresponding to the value $\alpha = 1\cdot 225$. It then remains above the curve of S_A/S_0, tending ultimately to a straight line parallel to the axis, at a height $\cdot 8484$ above it.

§ 17. *Proof that there can be no other Maxima.*

It remains to show that the point $\eta = 0$ corresponds to a maximum along the side $\xi = \alpha$, and also that there is no maximum of the resultant stress inside the section.

The stress S along $\xi = \alpha$ is given by

$$\left(\frac{Sc}{\mu}\right)^2 = \frac{1}{J}\left(\frac{dw}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\alpha\right)^2_{\xi=\alpha}.$$

The equation giving the maxima or minima is found as before. It is

$$\left[2\frac{d^2w}{d\eta^2} - \frac{1}{J}\frac{dJ}{d\eta}\left(\frac{dw}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\alpha\right)\right]_{\xi=\alpha} = 0,$$

or, using the *first* of expressions (17) for w

$$\sin 2\eta \tanh 2\alpha + 8(\cosh 2\alpha + \cos 2\beta)\sum_{n=0}^{n=\infty} \frac{\sinh\dfrac{(2n+1)\pi\eta}{2\alpha}}{\cosh\dfrac{(2n+1)\pi\beta}{2\alpha}} \frac{(2n+1)\pi}{(2n+1)^2\pi^2 + 16\alpha^2}$$

$$+ \frac{16\alpha \sin 2\eta(\cosh 2\alpha + \cos 2\beta)}{\cosh 2\alpha + \cos 2\eta}\sum_{n=0}^{n=\infty} \frac{\cosh\dfrac{\widehat{2n+1}\pi\eta}{2\alpha}}{(2n+1)^2\pi^2 + 16\alpha} \frac{1}{\cosh\dfrac{\widehat{2n+1}\pi\beta}{2\alpha}} = 0.$$

The left-hand side is always positive. Hence there is no root except $\eta = 0$, and that corresponds to a maximum.

That no absolute *maximum* can exist inside the section can be proved as follows:—

We have

$$\widehat{xz}/\mu = dw/dx - \tau y, \qquad \widehat{yz}/\mu = dw/dy + \tau x.$$

Suppose at any point P inside the section $S^2 = \widehat{xz}^2 + \widehat{yz}^2$ is a maximum. The above forms for $\widehat{xz} + \widehat{yz}$ being independent of axes, let us take for axes of x and y the direction of the resultant stress across the section at P and the perpendicular to it. So that $\widehat{yz} = 0$ at P.

Consider a near point P′. Let $\widehat{xz}'$, $\widehat{yz}'$ be the stresses at P′. Then, since S^2 is a maximum at P,

$$\widehat{xz}^2 > \widehat{xz}'^2 + \widehat{yz}'^2,$$

but

$$yz'^2 > 0.$$

Therefore

$$xz^2 > xz'^2,$$

or $\widehat{xz}$ is a numerical maximum.

But since

$$d^2w/dx^2 + d^2w/dy^2 = 0,$$

therefore also

$$\frac{d^2(\widehat{xz})}{dx^2} + \frac{d^2(\widehat{xz})}{dy^2} = 0;$$

and it is well known that no function can have an absolute maximum or an absolute minimum inside a region where it satisfies LAPLACE'S equation. This is in fact a particular case of the general theorem that a potential cannot have a maximum in free space.

Hence we have proved that the only fail-points are those which we have already investigated.

§ 18. *Deductions from the above and Criticism of* BOUSSINESQ'S *Proof of the Position of Fail-Points.*

Thus we see the study of these symmetrical sections is extremely instructive as regards the position of the fail-points. They show us the connection between the rectangular section and the section with a neck, and they give us the limiting cases, when the four fail-points coalesce into two, and *vice versâ.* The four fail-points begin to occur after the ratio of the long to the short axis of the section exceeds a certain value, which depends upon the angle of the bounding hyperbolas. As the indented appearance of the section becomes more obvious, that is, as β increases, this limiting value becomes greater and greater until, when the angle between the asymptotes is less than 73° the fail-point is always at the vertex of the hyperbolas. But in no case are the fail-points on the convex sides of the sections, unless the ellipses are so flat that the points A are nearer to the centre than the points B.

M. BOUSSINESQ has given ('Journal de Mathématiques,' Série II., vol. 16, p. 200) a sketch of a proof that the fail-points must be sought for "sur les petits diamètres

Fig. 10.

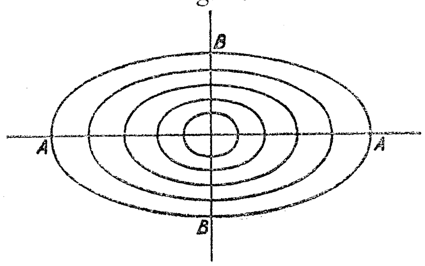
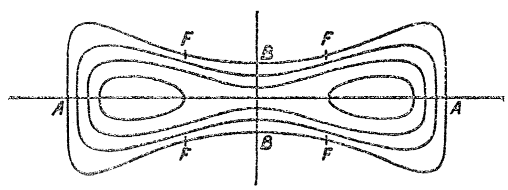


Fig. 11.



des sections." As this statement is in opposition with the results of the present paper and with DE SAINT-VENANT'S results for the rail sections already mentioned, I venture to suggest that M. BOUSSINESQ'S reasoning hardly holds in the case of sections part of whose contour is convex and part concave, for the following reason. The problem of torsion is mathematically analogous to that of a cylindrical vortex of uniform strength, whose cross-section is that of the shaft considered. The motion being in two dimensions we have a stream function ψ, and the resultant stress at any point in the torsion problem is the same as $d\psi/dn$ in the hydrodynamical analogy, dn being an element of the normal to any stream-line.

Now if we draw the stream-lines for equidistant values of ψ, they will, says M. BOUSSINESQ, "reproduce the irregularities of the contour, but more and more faintly, so that the curves are spaced at greater intervals along the large, then along the small diameters." In consequence, $d\psi/dn$, or the stress, is greater in the latter case. The above argument assumes that the same number of stream-lines cross each

diameter. This will undoubtedly be the case, and M. BOUSSINESQ's proof will hold, if the boundary of the section be everywhere convex, for then it is evident that each stream-line will consist of a single simple oval (fig. 10). But when we deal with sections like those of the present paper it is by no means so clear that this will be the case. There may be stream-lines which consist of two separate ovals (see fig. 11), and it then becomes very much open to question towards which part of the section the lines will be most crowded. In fact in some cases, as we have seen, this will occur at intermediate points, F (fig. 11), and in other cases the narrowness of the neck or the sharpness of the bend will counterbalance the limited number of the stream-lines through the neck, and the fail-points will be at B, B.

§ 19. *Comparison with* SAINT-VENANT's *Results. "Efficiency" of the various Sections.*

When we compare the results given by these sections bounded by confocal ellipses and hyperbolas with those given by DE SAINT-VENANT in his edition of the 'Leçons de NAVIER' for the "sections en double spatule," we find very good agreement. Thus the critical value of the ratio of breadth of neck to length for which the fail-points split up each into two is given by SAINT-VENANT as ·3247. For the sections of this paper, when $\beta = \pi/6$, I find this critical value to be ·3215.

The following are the principal numerical points (see 'Leçons de NAVIER,' p. 365) :—

	Section $\alpha = \pi/2$, $\beta = \pi/6$.	SAINT-VENANT'S section, $c/b = {\cdot}20$.
Ratio of breadth of neck to length	·2173	·20
M/M_0	·4443	·3921
S/S_0 (S being maximum stress)	·9144	·8668
$E = (M/M_0)/(S/S_0)$	·4859	·4524
Ratio of distance of fail-points from short axis to length of longer semi-axis	·3449	·4561

	Section $\alpha = 2\pi/3$, $\beta = \pi/6$.	SAINT-VENANT'S section, $c/b = {\cdot}14$.
Ratio of breadth of neck to length	·1250	·14
M/M_0	·3955	·3465
S/S_0	·8557	·8226
E	·4622	·4212
Ratio of distance of fail-points from short axis to length of longer semi-axis	·4486	·523

On the whole the sections of the present paper appear the more useful, M/M_0 and the quantity E being greater than for DE SAINT-VENANT's rail sections. This

quantity E I propose to call the "efficiency" or "usefulness" of the shaft. It is equal to $(S_0/M_0)/(\bar{S}/M)$, and gives us the ratio of maximum stress to torsional rigidity, compared with the same ratio for the circle of equal area, that is, it gives us a measure of how much torsion we may put into the shaft without impairing its elasticity. Thus if $\bar{M}$ be the maximum torsion moment which the given shaft will bear without failure of elasticity, $\bar{M}_0$ the maximum torsion moment which the shaft of equal circular section (made of the same material) will stand, then the corresponding stresses, $\bar{S}$, S_0 are each equal to the limiting elastic stress of the material and

$$\bar{M}/\bar{M}_0 = \text{efficiency} = E,$$

or the limiting torsion moment of a shaft of any section is obtained from that of the circular section of the same area by merely multiplying by the "efficiency" as thus defined.

TABLE of "Efficiency" of the sixteen given sections.

	$\beta=\pi/6$.	$\beta=\pi/4$.	$\beta=\pi/3$.	$\beta=\pi/2$.
$\alpha=\pi/6$	·6846	·7104	·6869	0
$\alpha=\pi/3$	·5206	·5022	·4376	0
$\alpha=\pi/2$	·4859	·4218	·3211	0
$\alpha=2\pi/3$	·4622	·4643	·3550	0

A glance at the above suffices to show that the efficiency of these sections is, in general, about one-half, that is, on the whole, this form of shaft is about half as useful as the shaft of circular section. The zero efficiency in the case of the slit ($\beta = \pi/2$) is due to the infinite stress in the keyway.

§ 20. *Analysis for the Sections bounded by one Elliptic and one Hyperbolic Arc.*

I now pass on to the consideration of cross-sections, bounded by an ellipse and a single branch of a confocal hyperbola. Such sections are shown in figs. 6–8.

In this case we shall find it more convenient to define ξ and η by the equations

$$\left.\begin{aligned} x &= c \cosh \xi \cos \eta \\ y &= c \sinh \xi \sin \eta \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (23)$$

where $-\alpha < \xi < +\alpha$, $0 < \eta < \beta$.

We find in the same way as before

$$d^2w/d\xi^2 + d^2w/d\eta^2 = 0.$$

$$\left.\begin{array}{ll} dw/d\xi + \tfrac{1}{2}\tau c^2 \sin 2\eta = 0 & \xi = \pm\alpha, \quad 0 < \eta < \beta \\ dw/d\eta + \tfrac{1}{2}\tau c^2 \sinh 2\xi = 0 & \eta = \beta, \quad -\alpha < \xi < +\alpha \end{array}\right\} \quad . \quad . \quad (24).$$

There is, however, in the present case, a further condition. For, referring to figs. 6–8, when we cross the line SB_1B_2, S being that focus of the conics which is inside the section, η is continuous, passing through the value 0, but ξ is discontinuous and changes from positive to negative. We have to ensure that the function of ξ and η, which shall represent w, shall be continuous in crossing $\eta = 0$, and that its space-differential coefficients shall also be continuous. The latter condition implies that $dw/d\eta$ must change sign; this being so, all the conditions will be satisfied.

Let us write

$$w = -\tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cosh 2\alpha} + w_1,$$

conditions (24) then become

$$\left.\begin{array}{l} dw_1/d\xi = 0 \quad \text{when} \quad \xi = \pm\alpha, \quad 0 < \eta < \beta \\ \dfrac{dw_1}{d\eta} + \tfrac{1}{2}\tau c^2\left(1 - \dfrac{\cos 2\beta}{\cosh 2\alpha}\right)\sinh 2\xi = 0 \\ \qquad \text{when} \quad \eta = \beta, \quad -\alpha < \xi < +\alpha \end{array}\right\} \quad . \quad . \quad . \quad . \quad (25).$$

If we now assume

$$w_1 = \sum_{n=0}^{n=\infty} A_n \sinh \frac{\widehat{2n+1}\pi\eta}{2\alpha} \cdot \sin \frac{\widehat{2n+1}\pi\xi}{2\alpha},$$

then the conditions of continuity and the first of (25) are identically satisfied.

A_n is found from the second equation of (25) in the same way as in § 4.

$$A_n = -\frac{16\tau c^2\alpha^2(-1)^n(\cosh 2\alpha - \cos 2\beta)}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]\cosh\dfrac{\widehat{2n+1}\pi\beta}{2\alpha}}.$$

And thus

$$w = -\tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cosh 2\alpha} - 16\tau c^2\alpha^2(\cosh 2\alpha - \cos 2\beta)$$

$$\times \sum_{n=0}^{n=\infty} \frac{(-1)^n \sinh \dfrac{\widehat{2n+1}\pi\eta}{2\alpha} \sin \dfrac{\widehat{2n+1}\pi\xi}{2\alpha}}{(2n+1)\pi[\pi^2(2n+1)^2 + 16\alpha^2]\cosh\dfrac{\widehat{2n+1}\pi\beta}{2\alpha}} \quad . \quad . \quad . \quad . \quad (26).$$

The shears are given by the equations

$$\left.\begin{aligned}\frac{\widehat{xz}}{\mu} &= \frac{1}{cJ}\left(\sinh\xi\cos\eta\frac{dw_1}{d\xi} - \cosh\xi\sin\eta\frac{dw_1}{d\eta}\right) - \tau c\sinh\xi\sin\eta\,(1+\operatorname{sech}2\alpha)\\ \frac{\widehat{yz}}{\mu} &= \frac{1}{cJ}\left(\cosh\xi\sin\eta\frac{dw_1}{d\xi} + \sinh\xi\cos\eta\frac{dw_1}{d\eta}\right) + \tau c\cosh\xi\cos\eta\,(1-\operatorname{sech}2\alpha)\end{aligned}\right\}\quad(27),$$

where J now stands for the quantity

$$\cosh^2\xi\sin^2\eta + \sinh^2\xi\cos^2\eta.$$

The torsion moment M

$$= \iint(x\widehat{yz} - y\widehat{xz})\,dx\,dy$$

$$= \mu\tau c^4\int_0^\beta d\eta\int_{-\alpha}^{\alpha} d\xi\,\{\cosh^2\xi\cos^2\eta\,(1-\operatorname{sech}2\alpha) + \sinh^2\xi\sin^2\eta\,(1+\operatorname{sech}2\alpha)\}\,J$$

$$+\frac{\mu c}{2}\int_0^\beta d\eta\int_{-\alpha}^{\alpha} d\xi\left(\sin 2\eta\frac{dw_1}{d\xi} + \sinh 2\xi\frac{dw_1}{d\eta}\right)$$

$$= \frac{\mu\tau c^4}{8}\int_0^\beta d\eta\int_{-\alpha}^{\alpha} d\xi\left[\begin{aligned}&(\cosh 4\xi - \cos 4\eta) - \operatorname{sech}2\alpha\,(\cosh 2\xi - \cos 2\eta)\\ &\qquad - \operatorname{sech}2\alpha\,(\cosh 4\xi\cos 2\eta - \cosh 2\xi\cos 4\eta)\end{aligned}\right]$$

$$+\frac{\mu c^2}{2}\int_0^\beta\Big[w_1\Big]_{-\alpha}^{\alpha}\sin 2\eta\,d\eta + \frac{\mu c^2}{2}\int_{-\alpha}^{\alpha}\Big[w_1\Big]_0^{\beta}\sinh 2\xi\,d\xi.$$

When this is integrated out and reduced as before, it is found that

$$\mathrm{M} = \frac{\mu\tau c^4}{16}\left\{\begin{aligned}&(\beta\sinh 4\alpha - \alpha\sin 4\beta)\tanh^2 2\alpha\\ &\quad + 2\sin 2\beta\,(2\alpha\operatorname{sech}^2 2\alpha - \tanh 2\alpha)(\cosh 2\alpha - \cos 2\beta)\\ &\quad - 2048\alpha^4(\cosh 2\alpha - \cos 2\beta)^2\sum_{n=0}^{n=\infty}\frac{1}{\pi(2n+1)}\frac{\tanh\dfrac{\widehat{2n+1}\,\pi\beta}{2\alpha}}{[\pi^2(2n+1)^2+16\alpha^2]^2}\end{aligned}\right\}\quad(28).$$

§ 21. *Alternative Solution for these Sections.*

For this type of section also we can find an alternative solution.

Suppose we assume

$$w = -\tfrac{1}{4}\tau c^2\frac{\sinh 2\xi\sin 2\eta}{\cos 2\beta} + w_1.$$

These conditions (24) reduce to:

$$\left.\begin{aligned}&\frac{dw_1}{d\xi} + \tfrac{1}{2}\tau c^2\sin 2\eta\left(1 - \frac{\cosh 2\alpha}{\cos 2\beta}\right) = 0,\ \ \xi = \pm\alpha,\ \ 0 < \eta < \beta\\ &dw_1/d\eta = 0 \quad \eta = \beta \quad -\alpha < \xi < \alpha\end{aligned}\right\}\quad . \quad . \quad (29).$$

Also the condition of continuity requires that w_1 must be odd in η.

We assume, therefore,

$$w_1 = \sum_{n=0}^{n=\infty} \mathrm{A}_n \sin \frac{\overline{2n+1}\pi\eta}{2\beta} \sinh \frac{\overline{2n+1}\pi\xi}{2\beta}.$$

But since

$$\sin 2\theta = a_1 \sin \frac{\pi\theta}{2\beta} + \ldots + a_n \sin \frac{\overline{2n+1}\pi\theta}{2\beta} + \ldots,$$

where

$$a_n = \frac{(-1)^n 16\beta \cos 2\beta}{(2n+1)^2\pi^2 - 16\beta^2},$$

we find easily from the first equation of (29), comparing coefficients

$$\mathrm{A}_n = \frac{(-1)^n 16\tau c^2\beta^2 (\cosh 2\alpha - \cos 2\beta)}{(2n+1)\pi\left[\overline{2n+1}^2\pi^2 - 16\beta^2\right] \cosh \dfrac{\overline{2n+1}\pi\alpha}{2\beta}},$$

and

$$w = -\tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cos 2\beta}$$

$$+ 16\tau c^2\beta^2 (\cosh 2\alpha - \cos 2\beta) \sum_{n=0}^{n=\infty} (-1)^n \frac{\sin \dfrac{\overline{2n+1}\pi\eta}{2\beta} \sinh \dfrac{\overline{2n+1}\pi\xi}{2\beta}}{(2n+1)\pi\left[\overline{2n+1}^2\pi^2 - 16\beta^2\right] \cosh \dfrac{\overline{2n+1}\pi\alpha}{2\beta}} \quad (30).$$

The stresses and torsion moment are deduced without difficulty. They are :

$$\frac{\widehat{xz}}{\mu} = -\tau c \sinh \xi \sin \eta \,(1 + \sec 2\beta) + 32\tau c\beta^2 \frac{(\cosh 2\alpha - \cos 2\beta)}{(\cosh 2\xi - \cos 2\eta)}$$

$$\times \sum_{n=0}^{n=\infty} \frac{\left(\sinh \xi \cos \eta \dfrac{d}{d\xi} - \cosh \xi \sin \eta \dfrac{d}{d\eta}\right)(-1)^n \sin \dfrac{\overline{2n+1}\pi\eta}{2\beta} \sinh \dfrac{\overline{2n+1}\pi\xi}{2\beta}}{\overline{2n+1}\,\pi\left[\overline{2n+1}^2\pi^2 - 16\beta^2\right] \cosh \dfrac{\overline{2n+1}\pi\alpha}{2\beta}} \quad (31).$$

$$\frac{\widehat{yz}}{\mu} = \tau c \cosh \xi \cos \eta \,(1 - \sec 2\beta) + 32\tau c\beta^2 \frac{(\cosh 2\alpha - \cos 2\beta)}{(\cosh 2\xi - \cos 2\eta)}$$

$$\times \sum_{n=0}^{n=\infty} \frac{\left(\cosh \xi \sin \eta \dfrac{d}{d\xi} + \sinh \xi \cos \eta \dfrac{d}{d\eta}\right)(-1)^n \sin \dfrac{\overline{2n+1}\pi\eta}{2\beta} \sinh \dfrac{\overline{2n+1}\pi\xi}{2\beta}}{\overline{2n+1}\,\pi\left[\overline{2n+1}^2\pi^2 - 16\beta^2\right] \cosh \dfrac{\overline{2n+1}\pi\alpha}{2\beta}} \quad (32).$$

$$\mathrm{M} = \frac{\mu\tau c^4}{16} \left[\begin{array}{l} -(\beta \sinh 4\alpha - \alpha \sin 4\beta) \tan^2 2\beta \\ + 2 \sinh 2\alpha (\cosh 2\alpha - \cos 2\beta)(2\beta \sec^2 2\beta - \tan 2\beta) \\ - 2048\,\beta^4 (\cosh 2\alpha - \cos 2\beta)^2 \displaystyle\sum_{n=0}^{n=\infty} \frac{\tanh \dfrac{\overline{2n+1}\pi\alpha}{2\beta}}{\overline{2n+1}\,\pi\left[\overline{2n+1}^2\pi^2 - 16\beta^2\right]^2} \end{array}\right] \quad (33).$$

The same remarks which were made concerning the solution in § 5 apply here. The critical values are $\beta = \pi/4$, $\beta = 3\pi/4$. The limits to which the values of w, M and the stresses, given in (30)-(33), tend, are easily obtained if required. In practice, however, the other solution would probably be used.

§ 22. *Numerical Results. Effects of Keyways upon Torsional Rigidity.*

I have worked out numerically six cases of this type of section. The sections selected are shown in figs. 6–8. The bounding ellipses are $\alpha = \pi/6$ and $\alpha = \pi/2$, the bounding hyperbolas are $\beta = \pi/2$, $3\pi/4$ and π, giving respectively (a) the half ellipse, (b) the ellipse with a rectangular hyperbolic keyway, (c) the ellipse with a single thin keyway or slit.

The values of the torsion moment and of its ratio to the torsion moment of the equal circular section, are shown in the tables below.

TABLE of $M/\mu\tau c^4$.

	$\beta = \pi/2$.	$\beta = 3\pi/4$.	$\beta = \pi$.
$\alpha = \pi/6$	·1365	·3944	·4731
$\alpha = \pi/2$	10·354	26·319	40·142

TABLE of M/M_0.

	$\beta = \pi/2$.	$\beta = 3\pi/4$.	$\beta = \pi$.
$\alpha = \pi/6$	·8907	·8244	·7718
$\alpha = \pi/2$	·7907	·7985	·7664

When we look at these results we see that the torsional rigidity of these sections is always less than that of the circular section. The sections consisting of a complete ellipse with one fine slit up to the focus are weaker than the half-ellipse or the sections with a broad keyway. This is more particularly shown in the case of the more elongated ellipse, $\alpha = \pi/6$.

With regard to the effects of slits, or thin keyways, it is interesting to compare the values of M/M_0 for the ellipses $\alpha = \pi/6$, $\alpha = \pi/2$, when (i.) there are no slits, (ii.) there is one slit, (iii.) there are two slits

We find

	$\alpha = \pi/6$.	$\alpha = \pi/2$.
(i.) M/M_0 = . . .	·7807	·9963
(ii.)	·7718	·7664
(iii.)	·7628	·5711

It follows that, in the first case, the cutting of one thin keyway lowers the rigidity of the section by 1·14 per cent., and of two keyways by 2·29 per cent. Hence the effect of two such keyways is slightly greater, if anything, than twice the effect of a single keyway, in the case of the more elongated ellipse. The difference is, however, practically negligible.

In the other case the result is different. The reduction of the torsional rigidity is very great: it amounts to 23·08 per cent. for one keyway and to 42·68 per cent. for two keyways. Here we see the effect of two keyways is rather less than twice the effect of one.

We may infer, however, from these two results that we may in practice, without very large error, if we have a number of keyways cut symmetrically into a section, and we know the effect on the torsional rigidity of any single keyway, assume that the effect of all the keyways is the sum of their separate effects.

Another important point which is brought out by the above results is that the effect of such a keyway upon the torsional rigidity is by no means simply proportional to the depth of the keyway, but increases according to some much more rapid law.

Thus, for the ellipse $\alpha = \pi/6$, the depth of the keyway $=$ ·123 (semi-major axis). For the ellipse $\alpha = \pi/2$, the depth $=$ ·601 (semi-major axis). Thus, when the depth of the keyway is only decreased to one-fifth of what it was before, the reduction of torsional rigidity falls from 23 per cent. to 1 per cent., or nearly in the ratio of the *squares* of the depths of the keyways.

This result may explain the fact that, when keyways of only moderate depth are cut into shafts, the decrease of torsional rigidity is by no means so great as would have been inferred from DE SAINT-VENANT's results for a circular section, with a thin keyway or slit extending right up to the centre.

If we suppose, which appears reasonable, that the effect of such a slit upon an ellipse, which is not very elongated, does not differ much from the effect on a circle, we see that a keyway, whose depth equals about one-eighth of the radius, will decrease the torsional rigidity by only about 1 per cent.

Now, when we make $\alpha = \infty$, we get the case of the circle with a keyway going right up to the centre. The reduction of torsional rigidity is then 44 per cent. about.

Hence we have, roughly,

Depth of keyway in terms of radius.	Reduction of torsional rigidity.
	per cent.
1·00	44
·60	23
·12	1

If the reduction of torsional rigidity were simply proportional to the depth of the keyway, the last two results ought to be 26·4 per cent. and 5·3 per cent. respectively.

It may be objected to these deductions that in the above sections, the fact that the stress at the vertex of the keyway is infinite, violates the physical conditions assumed by the theory of elasticity and renders our results untrustworthy.

My answer to this is that these cases should really be considered as limiting cases. If, instead of considering a section where the keyway is actually a straight line, we consider a hyperbola with a very sharp bend, we can easily ensure, provided the angle of torsion be not too great, that the physical conditions shall *not* be violated, and, on the other hand, the values of the torsional rigidity (since they tend to a finite limit) will differ but little from the values obtained above. The very fact that the torsional rigidity tends to a finite limit shows that, even in the extreme case, the area where the physical considerations are violated is infinitesimal.

Finally, in drawing conclusions from such results we shall only be following the example of SAINT-VENANT and of THOMSON and TAIT, who have not hesitated to use results found in a precisely similar way for such keyways cut into a circle.

§ 23. *Values of the Stresses.*

I have also determined the stresses at three points on the boundary of the section, in order to find where the stress was greatest. The points selected are denoted in figs. 6, 7, and 8 by the letters A, B, C.

A is the vertex of the hyperbola, B is the point opposite to A, and C is the point corresponding to $\eta = \pi/2$, $\xi = \pm\alpha$.

A is given by $\xi = 0$, $\eta = \beta$, and B by $\xi = \pm\alpha$, $\eta = 0$.

If S_A, S_B, S_C denote the corresponding stresses, I find

$$\begin{aligned}\frac{S_A}{\mu\tau c} &= \cos\beta\,(1 - \operatorname{sech} 2\alpha) \\ &\quad - \frac{8\alpha(\cosh 2\alpha - \cos 2\beta)}{\sin\beta}\sum_{n=0}^{n=\infty}\frac{(-1)^n \tanh\dfrac{\overline{2n+1}\,\pi\beta}{2\alpha}}{\pi^2(2n+1)^2 + 16\alpha^2} \\ &= -\tan 2\beta\sin\beta \\ &\quad + \frac{8\beta(\cosh 2\alpha - \cos 2\beta)}{\sin\beta}\sum_{n=0}^{n=\infty}\frac{\operatorname{sech}\dfrac{\overline{2n+1}\,\pi\alpha}{2\beta}}{(2n+1)^2\pi^2 - 16\beta^2} \quad . \quad . \quad . \quad (34),\end{aligned}$$

$$\frac{S_B}{\mu\tau c} = \tanh 2\alpha \sinh \alpha - \frac{8\alpha(\cosh 2\alpha - \cos 2\beta)}{\sinh \alpha} \sum_{n=0}^{n=\infty} \frac{\operatorname{sech} \frac{\widehat{2n+1}\pi\beta}{2\alpha}}{(2n+1)^2\pi^2 + 16\alpha^2}$$

$$= -\cosh \alpha (\sec 2\beta - 1) + \frac{8\beta(\cosh 2\alpha - \cos 2\beta)}{\sinh \alpha} \sum_{n=0}^{n=\infty} \frac{(-1)^n \tanh \frac{\widehat{2n+1}\pi\alpha}{2\beta}}{\overline{2n+1}|^2\pi^2 - 16\beta^2} \quad \ldots \ldots \quad (35),$$

$$\frac{S_C}{\mu\tau c} = -\sinh \alpha (1 + \operatorname{sech} 2\alpha) + \frac{8\alpha(\cosh 2\alpha - \cos 2\beta)}{\cosh \alpha} \sum_{n=0}^{n=\infty} \frac{\cosh \frac{\widehat{2n+1}\pi^2}{4\alpha} \Big/ \cosh \frac{\widehat{2n+1}\pi\beta}{2\alpha}}{\overline{2n+1}|^2\pi^2 + 16\alpha^2}$$

$$= -\sinh \alpha (1 + \sec 2\beta) - \frac{8\beta(\cosh 2\alpha - \cos 2\beta)}{\cosh \alpha} \sum_{n=0}^{n=\infty} \frac{(-1)^n \sin \frac{\widehat{2n+1}\pi^2}{4\beta} \tanh \frac{\widehat{2n+1}\pi\alpha}{2\beta}}{\overline{2n+1}|^2\pi^2 - 16\beta^2} \quad \ldots \quad (36),$$

whence I find—

TABLE of Stresses.

	$\beta = \pi/2$.	$\beta = 3\pi/4$.	$\beta = \pi$.	
$\alpha = \pi/6$	$-$ ·67631	$-$ ·85397	$-\infty$	$S_A/\mu\tau c$
$\alpha = \pi/2$	$-$2·0716	$-$3·6144	$-\infty$	
$\alpha = \pi/6$	·40265	·42625	...	$S_B/\mu\tau c$
$\alpha = \pi/2$	1·7276	2·0509	...	
$\alpha = \pi/6$	0	·85106	...	$S_C/\mu\tau c$
$\alpha = \pi/2$	0	2·2794	...	

We see that the greatest of the three stresses occurs at A, *i.e.*, at the vertex of the hyperbola, and it is probable that A is the true fail-point.

The following are the values of S_A/S_0, S_0 having the same meaning as in § 12 :—

TABLE of S_A/S_0.

	$\beta = \pi/2$.	$\beta = 3\pi/4$.	$\beta = \pi$.
$\alpha = \pi/6$	1·2101	1·1496	∞
$\alpha = \pi/2$	1·2192	1·6888	∞

The values of the "efficiency" are easily obtained.

TABLE of E.

	$\beta=\pi/2.$	$\beta=3\pi/4.$	$\beta=\pi.$
$\alpha=\pi/6$	·7361	·7171	0
$\alpha=\pi/2$	·6485	·4728	0

The above results do not need any detailed discussion. We see that in all cases the maximum stress is greater than the maximum stress for the circular section. Also the efficiency is always less than unity. If we compare these values of the efficiency with those in § 19, we see that, on the whole, the rule holds that the more compact the section the higher its efficiency. On the other hand, by indenting a section we render it less efficient.

§ 24. *Conclusion and Summary.*

Looking back upon the results of the paper, we see that the study of these special forms of cross-section sheds new light upon several little-explored parts of the theory of elasticity.

It confirms to a great extent DE SAINT-VENANT'S investigations concerning the behaviour under torsion of a rail, or of shafts of similar section.

Owing to the great generality of the forms treated, it enables us to correlate the results previously obtained for sections of various shapes, especially with regard to the maximum stress. It shows us what type of cross-section will give us four fail-points not at the points of the contour closest to the centre; within what limits we may expect to find this exception to the ordinary rule; and in what manner this case passes into others.

Again, with regard to the effect of keyways upon the torsional rigidity, the results of the paper tell us that, without risk of sensible error, we may in practice, in order to get the joint effect of two indefinitely thin keyways or slits, add the known effects of each keyway, taken separately.

Further, these results bring the theory of elasticity into better accordance with observed facts, by showing that the effects of keyways of moderate depth are comparatively much smaller than would have been expected from the results for the circle.