PHILOSOPHICAL TRANSACTIONS.

I. *The Integration of the Equations of Propagation of Electric Waves.*

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1. In the older forms of the Undulatory Theory of Light, the propagation of the waves was traced by means of Huygens' principle; each element of a wave front was regarded as becoming a source of disturbance from which secondary waves are emitted. The principle is indefinite, inasmuch as the nature and intensity of the sources of secondary waves are unrestricted, save by the conditions that the secondary waves must combine in advance so as to give rise to the disturbance actually propagated, and must interfere in rear so as to give rise to no disturbance. That these conditions are insufficient, for the complete determination of the nature and intensity of the sources in question, is proved by observing that different writers, proceeding by different methods, have arrived at different expressions for "the law of disturbance in secondary waves," all these expressions satisfying the imposed conditions.*

In the more modern forms of the theory, the propagation of the waves is traced by means of a system of partial differential equations. This system has the same form, whether we regard the luminiferous medium as similar in its mode of action to an elastic solid, transmitting transverse waves, or regard light as an electromagnetic disturbance obeying the fundamental equations of the electric field. In both cases it appears that all the components of the vector quantities which represent the disturbance satisfy a partial differential equation of the form $\partial^2\phi/\partial t^2 = c^2\nabla^2\phi$.

2. This equation is the same as occurs in the Theory of Sound. It has been integrated in two ways. Poisson† expressed the value of ϕ, at any point, at time t, in terms of the initial values of ϕ and $\partial\phi/\partial t$ on a sphere of radius ct, with its centre at the point. Kirchhoff‡ obtained a more general integral, in which the value of ϕ at any point is expressed in terms of the values of ϕ, $\partial\phi/\partial\nu$ and $\partial\phi/\partial t$ on a closed surface S, separating the point from the singularities of the function ϕ, $d\nu$ being the

* Lord Rayleigh, "Wave Theory of Light," 'Encycl. Brit.,' 9th ed., vol. 24, pp. 429, 453.

† 'Mém. de l'Institut,' vol. 3 (1820), p. 121. *Cf.* Lord Rayleigh, 'Theory of Sound,' vol. 2, ch. 14.

‡ 'Wied. Ann.,' vol. 18 (1883), p. 663; or 'Vorlesungen ü. math. Optik' (Leipz. 1891), pp. 23 *et seq.* The result is given explicitly in equation (2) of § 7, *infra.*

element of the normal to S. KIRCHHOFF's integral can be shown to include POISSON's by taking, for S, a sphere, of radius ct, with its centre at the point. In the case of sound, or for any scalar disturbance, KIRCHHOFF's integral is directly interpretable in terms of imagined sources of disturbance situated on the surface S; for all the quantities that occur can be interpreted in terms of condensations and velocities. It might thus be regarded as providing an exact equivalent of HUYGENS' principle,* if the disturbance involved were of a scalar character. Its application to light is open to criticism (see § 8, *infra*).

Besides satisfying the partial differential equation $\partial^2\phi/\partial t^2 = c^2\nabla^2\phi$, the components of a vector quantity, propagated by transverse waves, are also subject to the circuital condition; and the problem of integrating the system of equations is, accordingly, not the same as the problem of integrating the single equation satisfied by the several components. Sir G. STOKES† has attacked the more general problem, by extending and transforming POISSON's solution of the single equation. He has shown that the components of the disturbance at any point O, and at any time t, can be expressed as the sums of two parts, one depending on initial values on a sphere of radius ct, with its centre at the point O, and the other depending on initial values in space outside this sphere.‡ The latter part is relatively unimportant when, as in the applications made by Sir G. STOKES, the radius of the sphere is great, compared with the wave-length of the disturbance; the former part has precisely the character required for representing transverse vector disturbances, and it admits of transformation to a form in which it expresses the radiation received at the point O as due to secondary waves sent out from surfaces other than spheres with their centres at O. The transformation to a plane wave front was given in the paper above quoted, and the results, which were deduced from this form of the integrals of the system of equations, have had a very important bearing on the development of the theory.

3. The object of this paper is to present an investigation of a new system of integrals of the system of equations that govern the propagation of transverse vector disturbances, and to exemplify the use that can be made of such integrals. The components of the vectors that constitute the disturbance ought to be expressible, as in KIRCHHOFF's solution, in terms of surface values on an arbitrary surface; the elements of the integrals ought to be quantities characteristic of transverse vector disturbances, as in Sir G. STOKES's solution; and the results ought to admit of interpretation in terms of sources of disturbance of definite types, as KIRCHHOFF's result does when applied to sound waves. It is shown that the method developed by KIRCHHOFF can be adapted to the system of equations in such a way as to lead to

* It is so regarded by KIRCHHOFF (*loc. cit.*), by POINCARÉ, 'Théorie math. de la Lumière,' vol. 2 (Paris, 1892), ch. 7, and by DRUDE, 'Lehrbuch d. Optik.' (Leipz., 1900), p. 167.

† "On the Dynamical Theory of Diffraction," 'Camb. Phil. Soc. Trans.,' vol. 9 (1849), 'Papers,' vol. 2, p. 241.

‡ See equations (29) and (30) of the paper above quoted, 'Papers,' vol. 2, p. 268.

results of the kind described, and in particular that the disturbances can be regarded as due to sources of two definite types. A source of one of these types is similar to an infinitesimal Hertzian vibrator. The character of the most important part of the radiation from such a vibrator is well known; it is periodic, with a damping coefficient, and is related in a definite way to a particular axis.* The radiation from a source of the corresponding type is related in the same way to the axis, but its mode of dependence upon time is arbitrary. The assumption of infinite trains of simple harmonic radiation, with or without damping coefficients, is an unnecessary restriction of the mathematical formulæ, and is inadequate to represent many phenomena. The other type of sources is arrived at by interchanging the *rôles* of the electric and magnetic forces in the type that is similar to Hertzian vibrators. There is a theorem that disturbances, which can be represented as due to sources of both types, may also be represented as due to sources of a single type, just as acyclic irrotational motions of incompressible fluid may be regarded as due to sources and double sources, or to sources only.†

4. A very general system of integrals of the system of equations that govern the propagation of waves having been obtained, it is natural to inquire after an expression for the law of disturbance in a secondary wave that shall accord with these integrals. The expression arrived at is rather simpler than that given by Sir G. STOKES‡ as regards the intensity of the secondary waves, but rather more complicated as regards the orientation of the plane through the direction of displacement and the direction of propagation. This plane is either the plane of polarisation of the secondary wave, or else it is at right angles to that plane. At one time it might have been interesting to pursue the question further, and to determine the conclusion, as regards the relation of direction of displacement to plane of polarisation, that could be drawn from the new integrals; but the question is not now of importance, since it is certain, on many grounds, that the plane of polarisation of light contains the magnetic force, and is at right angles to the electric force.

5. [*Partly re-written March,* 1901.]—Apart from this question of the plane of polarisation of scattered waves, the chief use of a law of disturbance in secondary waves is found in the solution by elementary methods of problems of diffraction; this use is not affected§ by such differences as exist between the law here found and that obtained by Sir G. STOKES. But, in connexion with the application of any such law to problems of diffraction through apertures, there also arises the question of the distribution over the aperture of sources that would give rise to the transmitted

* The relation to the axis is the same for the forms given by HERTZ, 'Electric Waves,' p. 143, as for those given by K. PEARSON and ALICE LEE, 'Phil. Trans.' A, 193 (1899), p. 159. The forms given in § 13 include both.

† LAMB, 'Hydrodynamics,' pp. 66 and 67.

‡ 'Papers,' vol. 2, p. 286.

§ Lord RAYLEIGH, 'Wave Theory of Light,' p. 453.

radiation. When there is no screen, such sources are determined for any imagined bounding surface simply and directly by the incident radiation. [But, when there is a screen, the distribution of the sources is not determined in the same way by the portion of the incident radiation that would come to the aperture if the screen were away. It is proved below that the state of the medium on that side of the screen to which the incident radiation comes can be expressed by means of two superposed fields of electric and magnetic force. The forces of one of these fields are expressed in terms of integrals taken over the surface of the aperture; and the corresponding disturbance is a system of standing waves, the amplitudes of which diminish rapidly as the distance from the aperture increases. This disturbance can be described as the "effect of the aperture." The forces of the other field are determined by the actual sources of radiation and the boundary conditions that hold over all the unperforated portion of the screen. This disturbance can be described as the "incident radiation, as modified by the action of the screen." It is proved that, when the latter disturbance is known, the system of standing waves, described as the effect of the aperture, is also known. Further, it is proved that the distribution, over the aperture, of sources that would give rise to the transmitted radiation, is determined by the incident radiation, as modified by the action of the screen, in the same way as, if there were no screen, it would be determined by the incident radiation, unmodified. The ordinary optical rule ignores the modification of the incident radiation by the action of the screen, and the success of this rule appears to show that the effect of this modification on the transmitted radiation is practically unimportant when the wave-length is short.]

6. The results obtained, in regard to the effect of an aperture, can be applied also to the problem of the communication of electrical vibrations from a condenser to the external medium, the outer conducting sheet of the condenser being perforated by a small aperture, for, in this case, full account has been taken of the boundary-conditions at the conducting surfaces in calculating the normal modes of vibration. The communication of electrical oscillations from an electrical vibrator to the surrounding medium presents a problem, which has hitherto been solved in a few very special cases. The best known example is that of a spherical conductor, over which, at some instant, charge is distributed otherwise than according to the equilibrium law. The waves emitted have definite periods, but they decay so rapidly as to be practically dead-beat.* Such a system sends out into the medium a *pulse* of radiation, rather than a *train* of radiation. The greater permanence of the vibrations of HERTZ's "resonators," and of condensing systems, has been connected with the existence of greater electrostatic capacity† in such systems; but no

* The problem is solved by J. J. THOMSON, 'Recent Researches,' pp. 361 *et seq.* The rate of decay of the oscillations is discussed on p. 370.

† J. J. THOMSON, 'Recent Researches,' p. 396. *Cf.* J. LARMOR, 'London Math. Soc. Proc.,' vol. 26 (1895), p. 123, footnote.

problem of the decay of oscillations of a system with large capacity, through the gradual transmission of the energy to a distance, has so far been solved. For a condenser with concentric spherical conducting surfaces, the outer conducting sheet being very thin, and having a small circular aperture, the problem can be solved by means of previously known analysis and of results obtained in this paper. It appears that, so long as the outer conducting sheet, and the size of the aperture, remain the same, the rate of decay of the oscillations diminishes, as the capacity diminishes; the oscillations of a condenser with small capacity, obtained by making the radius of the inner sheet small compared with that of the outer, are much more slowly damped than those of a condenser of large capacity, obtained by making the radii nearly equal. This result applies to the oscillations of high frequency, involving a large number of nodes, as well as to those of lower frequency; and it suggests that the comparative permanence of the oscillations of condensing systems is to be traced, rather to the screening action, than to the increase of capacity.* A further result, that the oscillations of high frequency and many nodal divisions, are more rapidly damped than those of lower frequency and fewer nodal divisions, is in accordance with the conclusions arrived at by Sir G. STOKES,† for the like problems concerning sound.

[*Added, March*, 1901.—Since the paper was sent in, I have found that a similar method of integration has been employed by V. CERRUTI, 'Rome, R. Acc. Lincei, Rend.,' 1879–80, for the equations of small motion of an elastic solid. The fundamental particular solutions, there used, are the same as (17) of § 11 *infra*; and the solution of the problem of the vibrations of a solid, with a given boundary, over which the displacements, or the tractions, have assigned values, is developed on the basis of an existence-theorem, of the same kind as that assumed in § 21; no application is made of the results to problems of radiation.]

KIRCHHOFF'S *Integral*.

7. Suppose that ϕ is a function with the following properties:

(1) Outside a given closed surface S, ϕ and its first and second differential coefficients, with respect to x, y, z, are everywhere finite and continuous;

(2) ϕ vanishes at infinite distances from S;

* *March*, 1901.—Mr. LARMOR has called my attention to the fact that the work in the paper does not show that *all* methods of increasing the capacity, without altering the outer conductor or the aperture, are accompanied by increased dissipation. For instance, the capacity of the condenser might be increased by displacing the inner conductor relatively to the outer, without altering its size and shape, or by replacing part of the dielectric plate by conducting material. In such cases there are some analogies with other physical problems, which suggest a diminished rate of dissipation. Against them must be set the analogy with the problem worked out in the present paper.

† 'Phil. Trans.,' vol. 158 (1868).

(3) ϕ satisfies the equation

$$\partial^2\phi/\partial t^2 = c^2 \nabla^2\phi \qquad (1),$$

at all points outside S, c being a definite constant, and ∇^2 denoting the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The value of ϕ at any point O outside S, and at time t, can be expressed as a surface-integral taken over S; in fact this value is

$$-\frac{1}{4\pi}\iint\left\{\frac{1}{r}\left[\frac{\partial\phi}{\partial\nu}\right] + \frac{1}{cr}\frac{\partial r}{\partial\nu}\left[\frac{\partial\phi}{\partial t}\right] - \frac{\partial r^{-1}}{\partial\nu}[\phi]\right\}dS \qquad (2),$$

where $d\nu$ denotes the element of the normal to dS drawn outwards (*i.e.*, into the region of space where O is situated), and the expressions in square brackets [ ] are to be formed, for each point of the surface, at the time $t - r/c$, r being the distance of the point of the surface from the point O.

This theorem was obtained by KIRCHHOFF* by an application of GREEN's theorem† to the function ϕ and an auxiliary function V, which satisfies the equation (1) at all points except O, and has the form

$$\frac{F(r + ct)}{r}, \qquad (3),$$

where F is a function to which suitable properties are assigned.

8. [*Partly re-written March,* 1901.]—If ϕ were the velocity potential of sound waves in air, the terms of (2) that contain $[\phi]$ and $[\partial\phi/\partial\nu]$ would be interpretable in terms of velocity, and those that contain $[\partial\phi/\partial t]$ would be interpretable in terms of condensation; the expression (2) would represent the motion at any point as due to sources of definite types distributed over the surface S. But, if ϕ is one of the components of a vector quantity, propagated by transverse waves, $[\partial\phi/\partial\nu]$ has no physical significance, and the expression (2) cannot be interpreted in terms of appropriate sources of disturbance.

Again, the expression (2) may be interpreted as showing that every element of the surface S becomes the centre of diverging secondary waves. If ϕ is one of the components of a vector quantity, and the primary waves are transverse, the application of Kirchhoff's theorem is open to the criticism that the secondary waves are not always

* 'Vorlesungen ü. math. Optik,' pp. 23–27.

† The theorem referred to is the one expressed by the equation

$$\iiint(\phi\nabla^2 V - V\nabla^2\phi)\,d\tau = \iint\left(V\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial V}{\partial\nu}\right)dS.$$

transverse, although, when synthesised, the disturbance, to which they give rise, is transverse.

[The criterion of transversality of a vector disturbance, propagated by wave motion, is that the vector concerned is everywhere circuital; and this implies that, in the case of diverging waves, the direction of the vector tends, at great distances from the source, to be at right angles to the radius, drawn from the source. Now, if we take, for example, the electric radiation represented by the expressions in § 13 *infra*, and choose, as the surface S, a sphere, with its centre at the source Q, the magnetic force

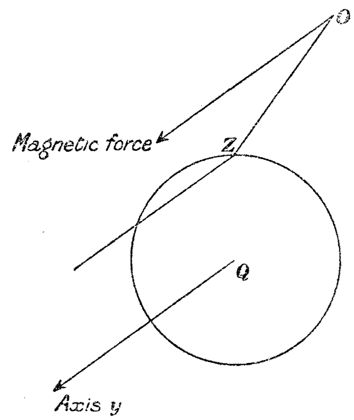

Fig. 1.

at the point Z ($x = 0$, $y = 0$) would be parallel to the axis y. KIRCHHOFF'S integral would represent the magnetic force, at any point O, as made up of components, contributed by secondary waves, diverging from the elements of S; and, in the wave diverging from Z, the magnetic force would be everywhere parallel to the axis y. It can be verified readily, by forming the expression for this force, that it is not circuital; but it can be seen at once, without forming this expression, that the secondary wave is not transverse; for, at any distance, however great, it is not at right angles to the radius vector ZO, unless O is in the plane (x, z). The particular example is sufficient to substantiate the criticism; but a little reflexion shows that there is nothing peculiar to the example. In general, let (α, β, γ) be the vector, and suppose that at some point the direction of the vector is independent of the time, we may take the surface S to pass through the point, and take the axis y parallel to the direction of the vector at the point; then α and γ vanish; the equations $\alpha = 0$ and $\gamma = 0$ will represent two surfaces passing through the point, and we may take the direction of the normal to S, at the point, to be the line of intersection of these surfaces. Then α, $\partial\alpha/\partial t$, $\partial\alpha/\partial\nu$ and γ, $\partial\gamma/\partial t$, $\partial\gamma/\partial\nu$ vanish at the point, at all times. In the secondary wave sent out from the point, the vector is everywhere parallel to the axis y; and, accordingly, the secondary wave is not a transverse wave.]

Equations of Propagation of Electric Waves.

9. The equations of propagation of electric waves in free æther are

$$\left.\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}(X, Y, Z) &= \text{curl }(\alpha, \beta, \gamma)\\ -\frac{1}{c}\frac{\partial}{\partial t}(\alpha, \beta, \gamma) &= \text{curl }(X, Y, Z)\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (4),$$

where (X, Y, Z) denotes the electric force, measured electrostatically, (α, β, γ) the magnetic force, in electromagnetic measure, and c the velocity of propagation of electrical effects. I propose to adopt, as a means of formal simplification, and without attaching to it any definite physical meaning, the view* that (α, β, γ) may be regarded as a "generalised velocity," and to introduce the corresponding "generalised displacement" (u, v, w), so that

$$\frac{\partial}{\partial t}(u, v, w) = (\alpha, \beta, \gamma) \quad \ldots \ldots \ldots \quad (5).$$

I also introduce the vector (f, g, h) by the equation

$$(f, g, h) = \text{curl }(u, v, w) \quad \ldots \ldots \ldots \quad (6),$$

so that (f, g, h) is twice the "rotation" corresponding to the displacement (u, v, w). The first of equations (4) becomes

$$(X, Y, Z) = c\,(f, g, h) \quad \ldots \ldots \ldots \quad (7),$$

and, according to the view above referred to, this equation may be regarded as expressing a purely kinematical relation, while the second of equations (4) gives the equations of motion of the æther. They are

$$\frac{\partial^2}{\partial t^2}(u, v, w) = c^2\nabla^2(u, v, w) \quad \ldots \ldots \ldots \quad (8),$$

with the circuital relation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \ldots \ldots \ldots \quad (9).$$

Further, it is convenient to derive (u, v, w) from a vector potential (F, G, H) by the equation

$$(u, v, w) = \text{curl }(F, G, H) \quad \ldots \ldots \ldots \quad (10),$$

* J. LARMOR, 'Phil. Trans.,' A, vol. 185, Part II (1894).

taking (F, G, H) to satisfy the same system of differential equations (8) and (9) as is satisfied by (u, v, w). The vectors (f, g, h) and (F, G, H) are connected by the equations

$$(f, g, h) = -\nabla^2(\mathrm{F}, \mathrm{G}, \mathrm{H}) = -\mathrm{c}^{-2}\frac{\partial^2}{\partial t^2}(\mathrm{F}, \mathrm{G}, \mathrm{H}) \quad . \quad . \quad . \quad . \quad (11).$$

It is also convenient sometimes to quote the fundamental equations in the forms

$$\left.\begin{aligned} \frac{\partial}{\partial t}(f, g, h) &= \operatorname{curl}(\alpha, \beta, \gamma) \\ -\frac{1}{\mathrm{c}^2}\frac{\partial}{\partial t}(\alpha, \beta, \gamma) &= \operatorname{curl}(f, g, h) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

The quantity (f, g, h) will sometimes be called the "electric displacement"; it is the product by 4π of the quantity so denominated by MAXWELL; the quantity (u, v, w) will sometimes be called the "magnetic displacement." MAXWELL'S vector potential would be expressed, in the above notation by $\frac{\partial}{\partial t}(\mathrm{F}, \mathrm{G}, \mathrm{H})$.

Special Types of Solution, Sources of Disturbance.

10. It has been pointed out that KIRCHHOFF'S method of integration of equation (1) depends on the application of a certain reciprocal theorem to two solutions of that equation, one of the two having the form (3). When we seek to apply a similar method to the system of equations (8) and (9), we are met at the outset by the difficulty that no simultaneous solutions of the form (3) exist, and by the necessity of devising some forms of solution, which shall become infinite at the origin, and contain arbitrary functions. If we regard the form (3) as corresponding to the solution r^{-1} of the equation $\nabla^2\phi = 0$, the appropriateness of seeking for solutions of the system (8), which correspond to spherical harmonics of order different from zero, at once suggests itself.

If in equation (1) we put

$$\phi = \phi_n \mathrm{S}_n,$$

where S_n is a spherical surface harmonic of order n, and ϕ_n is a function of r and t, we find for ϕ_n an equation which can be written

$$\left\{\frac{\partial^2}{\partial r^2} - \frac{n(n+1)}{r^2}\right\}(r\phi_n) = \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}(r\phi_n).$$

By using the relations*

* The method has been used by R. R. WEBB in the discussion of RICCATI'S equation; it is indicated by J. W. L. GLAISHER, 'Phil. Trans.,' vol. 172 (1881), p. 804.

$$\left\{\frac{\partial^2}{\partial r^2} - \frac{n(n+1)}{r^2}\right\} F = \left(\frac{\partial}{\partial r} - \frac{n}{r}\right)\left(\frac{\partial}{\partial r} + \frac{n}{r}\right) F = \left(\frac{\partial}{\partial r} + \frac{n+1}{r}\right)\left(\frac{\partial}{\partial r} - \frac{n+1}{r}\right) F$$

and $$\left(\frac{\partial}{\partial r} - \frac{n}{r}\right)\left(\frac{\partial}{\partial r} - \frac{n-1}{r}\right) \ldots \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) F = r^{n+1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left(\frac{F}{r}\right),$$

where F is any function of r, we find for ϕ the form

$$\phi = r^n S_n \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left\{\frac{F(r+ct) + f(r-ct)}{r}\right\}, \quad \ldots \ldots \quad (13)$$

where F and f are arbitrary functions.

It is now easy to write down simultaneous solutions* of the system of equations (8) and (9). Taking ω_n to represent a spherical solid harmonic of positive degree n, and writing

$$\phi_0 = F(r + ct) + f(r - ct), \quad \ldots \ldots \ldots \quad (14),$$

a set of such solutions is given by the equation

$$(u, v, w) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left(\frac{\phi_0}{r}\right) \cdot \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\omega_n; \quad . \quad (15)$$

and a second set of such solutions is obtained by taking the curl of the first set. We should find for example, after a little reduction,

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = -\frac{n+1}{2n+1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n-1}\left(\frac{1}{r}\frac{\partial^2 \phi_0}{\partial r^2}\right) \cdot \frac{\partial \omega_n}{\partial x} - \frac{n}{2n+1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n+1}\left(\frac{\phi_0}{r}\right) \cdot r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right) \quad \ldots \ldots \quad (16).$$

The solutions given by (15) may be referred to as "solutions of the first type," and those given by equations such as (16), as "solutions of the second type."

11. For our immediate purpose it will be sufficient to take $n = 1$ and $\omega_1 = x$. The components of a vector which yields a solution of the first type are

$$0, \qquad z\left(\frac{1}{r^2}\frac{\partial \phi_0}{\partial r} - \frac{1}{r^3}\phi_0\right), \qquad -y\left(\frac{1}{r^2}\frac{\partial \phi_0}{\partial r} - \frac{1}{r^3}\phi_0\right), \quad \ldots \ldots \quad (17)$$

and the components of the curl of this vector are

$$\left.\begin{aligned} &(y^2 + z^2)\left(-\frac{1}{r^3}\frac{\partial^2 \phi_0}{\partial r^2} + \frac{3}{r^4}\frac{\partial \phi_0}{\partial r} - \frac{3}{r^5}\phi_0\right) - 2\left(\frac{1}{r^2}\frac{\partial \phi_0}{\partial r} - \frac{1}{r^3}\phi_0\right), \\ &\quad - xy\left(-\frac{1}{r^3}\frac{\partial^2 \phi_0}{\partial r^2} + \frac{3}{r^4}\frac{\partial \phi_0}{\partial r} - \frac{3}{r^5}\phi_0\right), \\ &\quad - xz\left(-\frac{1}{r^3}\frac{\partial^2 \phi_0}{\partial r^2} + \frac{3}{r^4}\frac{\partial \phi_0}{\partial r} - \frac{3}{r^5}\phi_0\right). \end{aligned}\right\} \quad . \quad . \quad (18).$$

* When the functions that occur are simple harmonic functions of the time, the solution of the simplified system of equations is well known. See LAMB, 'Hydrodynamics,' pp. 487 and 555 *et seq.*

In these expressions ϕ_0 is any solution of the equation

$$\frac{\partial^2\phi_0}{\partial t^2} = \text{C}^2\frac{\partial^2\phi_0}{\partial r^2} \quad \text{(19).}$$

It is worth while to observe that the components of the curl of the vector represented by (18) are

$$0, \qquad -z\left(\frac{1}{r^2}\frac{\partial^3\phi_0}{\partial r^3} - \frac{1}{r^3}\frac{\partial^2\phi_0}{\partial r^2}\right), \qquad y\left(\frac{1}{r^2}\frac{\partial^3\phi_0}{\partial r^3} - \frac{1}{r^3}\frac{\partial^2\phi_0}{\partial r^2}\right) \quad \text{(20).}$$

12. We may use the results just obtained to describe two types of sources of electromagnetic disturbances. We shall take ϕ_0 to be a function of $\text{C}t - r$, say

$$\phi_0 = \phi(\text{C}t - r) = \phi \quad \text{(21).}$$

In the case of sources of the first type, the magnetic force is axial, and the lines of electric force are circles about the axis; when the axis is the axis of x, the vector potential has the form

$$0, \qquad \frac{z}{r^3}\left(\phi + \frac{r}{\text{C}}\dot\phi\right), \qquad -\frac{y}{r^3}\left(\phi + \frac{r}{\text{C}}\dot\phi\right); \quad \text{(22)}$$

the magnetic displacement has the form

$$\left.\begin{array}{l} -\left(\dfrac{2}{r^3} - 3\dfrac{y^2+z^2}{r^5}\right)\left(\phi + \dfrac{r}{\text{C}}\dot\phi\right) + \dfrac{y^2+z^2}{r^3\text{C}^2}\ddot\phi, \\ -\dfrac{xy}{r^5}\left(3\phi + 3\dfrac{r}{\text{C}}\dot\phi + \dfrac{r^2}{\text{C}^2}\ddot\phi\right), \\ -\dfrac{xz}{r^5}\left(3\phi + 3\dfrac{r}{\text{C}}\dot\phi + \dfrac{r^2}{\text{C}^2}\ddot\phi\right); \end{array}\right\} \quad \text{(23)}$$

and the electric displacement has the form

$$0, \qquad -\frac{z}{r^3\text{C}^2}\left(\ddot\phi + \frac{r}{\text{C}}\dddot\phi\right), \qquad \frac{y}{r^3\text{C}^2}\left(\ddot\phi + \frac{r}{\text{C}}\dddot\phi\right); \quad \text{(24).}$$

In these formulæ the dots denote partial differentiation with respect to the time. The axis of the source is the axis of x, corresponding to $\omega_1 = x$ in § 11; when the axis of the source is in the direction (l', m', n'), the result will be obtained by adding the expressions for the component vectors due to sources in the directions of the axes, and given by putting $l'\phi$, $m'\phi$ and $n'\phi$ in place of ϕ (with cyclical interchanges of the letters x, y, z). If the source is at the point (x', y', z'), instead of the origin, we have to write $x - x'$, $y - y'$, $z - z'$ in place of x, y, z, and take r to be the distance of (x, y, z) from (x', y', z').

13. In the case of sources of the second type, the electric force is axial, and the lines of magnetic force are circles about the axis; when the axis is the axis of x, the vector potential has the form

$$\left.\begin{array}{l}\left(\frac{2}{r^3}-3\frac{y^2+z^2}{r^5}\right)\left(\phi+\frac{r}{\mathrm{C}}\dot{\phi}\right)-\frac{y^2+z^2}{r^3\mathrm{C}^2}\ddot{\phi}\,,\\ \frac{xy}{r^5}\left(3\phi+3\frac{r}{\mathrm{C}}\dot{\phi}+\frac{r^2}{\mathrm{C}^2}\ddot{\phi}\right),\\ \frac{xz}{r^5}\left(3\phi+3\frac{r}{\mathrm{C}}\dot{\phi}+\frac{r^2}{\mathrm{C}^2}\ddot{\phi}\right);\end{array}\right\} \quad \ldots \ldots \text{(25)},$$

the magnetic displacement has the form

$$0,\quad \frac{z}{r^3}\left(\chi+\frac{r}{\mathrm{C}}\dot{\chi}\right),\quad -\frac{y}{r^3}\left(\chi+\frac{r}{\mathrm{C}}\dot{\chi}\right), \quad \ldots \ldots \text{(26)},$$

where

$$\chi=\frac{1}{\mathrm{C}^2}\ddot{\phi}\,; \quad \ldots \ldots \text{(27)},$$

and the electric displacement has the form

$$\left.\begin{array}{l}-\left(\frac{2}{r^3}-3\frac{y^2+z^2}{r^5}\right)\left(\chi+\frac{r}{\mathrm{C}}\dot{\chi}\right)+\frac{y^2+z^2}{r^3\mathrm{C}^2}\chi\,,\\ -\frac{xy}{r^5}\left(3\chi+3\frac{r}{\mathrm{C}}\dot{\chi}+\frac{r^2}{\mathrm{C}^2}\ddot{\chi}\right),\\ -\frac{xz}{r^5}\left(3\chi+3\frac{r}{\mathrm{C}}\dot{\chi}+\frac{r^2}{\mathrm{C}^2}\ddot{\chi}\right).\end{array}\right\} \quad \ldots \ldots \text{(28)}.$$

The most important part of the radiation due to a Hertzian vibrator appears to be of this type.*

The functions ϕ and χ, which figure in the expressions for the electric and magnetic displacements due to sources of the two types, will be referred to as the "radiation functions" for the sources.

In the expressions here obtained the source is at the origin, and its axis is the axis of x; the expressions for the displacements due to a source, of arbitrary position and direction, can be deduced as before.

The Reciprocal Theorem.

14. Let (u, v, w) be a possible system of magnetic displacements, and (f, g, h) the corresponding electric displacements, which are free from singularities in space bounded by one or more closed surfaces, denoted collectively by S. Then $u, \ldots$ are functions of x, y, z, t, which, with their first and second differential coefficients, are finite and continuous throughout this space. Denoting differentiation with respect to t by a dot, we observe that the equations of motion might be obtained by transforming the variation of the Action function†

$$\int dt \iiint \{\dot{u}^2+\dot{v}^2+\dot{w}^2-\mathrm{C}^2(f^2+g^2+h^2)\}\,d\tau$$

* HERTZ, 'Electric Waves,' p. 143.

† A factor $1/8\pi$ is omitted.

according to the rules of the Calculus of Variations. If then, in the variation

$$2\int dt \iiint \{\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w} - \mathrm{c}^2(f\delta f + g\delta g + h\delta h)\}\,d\tau,$$

we replace δu, . . . by a second system* of possible displacements u', . . . we shall obtain a symmetrical expression

$$\iiint \{\dot{u}\dot{u}' + \dot{v}\dot{v}' + \dot{w}\dot{w}' - \mathrm{c}^2(ff' + gg' + hh')\}\,d\tau \quad . \quad . \quad . \quad . \quad . \quad (29),$$

which admits of a similar transformation; and the result obtained, when simplified by means of the equations of motion, will consist of the volume integral of a perfect differential coefficient with respect to t, and a surface integral. The symmetry of the expression (29) then leads to the reciprocal theorem.

We have

$$\iiint (\dot{u}\dot{u}' + \dot{v}\dot{v}' + \dot{w}\dot{w}')\,d\tau = \iiint \left\{\frac{\partial}{\partial t}(\dot{u}u' + \dot{v}v' + \dot{w}w') - (\ddot{u}u' + \ddot{v}v' + \ddot{w}w')\right\}d\tau,$$

the volume integrations being taken through the space bounded by the surfaces S. Also, denoting by l, m, n the direction cosines of the normal to S drawn into this space, we have

$$\iiint (ff' + gg' + hh')\,d\tau = -\iint \{(gn - hm)\,u' + (hl - fn)\,v' + (fm - gl)\,w'\}\,d\mathrm{S}$$
$$-\iiint \left\{\left(\frac{\partial g}{\partial z} - \frac{\partial h}{\partial y}\right)u' + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)v' + \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)w'\right\}d\tau.$$

Hence the expression (29) becomes

$$-\iiint \left[u'\left\{\ddot{u} + \mathrm{c}^2\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\right\} + v'\left\{\ddot{v} + \mathrm{c}^2\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\right\} + w'\left\{\ddot{w} + \mathrm{c}^2\left(\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y}\right)\right\}\right]d\tau$$
$$+\iiint \frac{\partial}{\partial t}(\dot{u}u' + \dot{v}v' + \dot{w}w')\,d\tau + \mathrm{c}^2\iint \{(gn - hm)\,u' + (hl - fn)\,v' + (fm - gl)\,w'\}\,d\mathrm{S}.$$

The first line vanishes identically; and, from the symmetry of the expression (29), we deduce the reciprocal theorem

$$\iiint \frac{\partial}{\partial t}(\dot{u}u' + \dot{v}v' + \dot{w}w')\,d\tau + \mathrm{c}^2\iint \{(gn - hm)\,u' + (hl - fn)\,v' + (fm - gl)\,w'\}\,d\mathrm{S}$$
$$= \iiint \frac{\partial}{\partial t}(\dot{u}'u + \dot{v}'v + \dot{w}'w)\,d\tau + \mathrm{c}^2\iint \{(g'n - h'm)\,u + (h'l - f'n)\,v + (f'm - g'l)\,w\}\,d\mathrm{S}$$
$$. \quad . \quad . \quad (30).$$

* The second system, as well as the first, satisfies the fundamental equations.

We integrate this equation with respect to the time, between two fixed values t_0 and t_1, and we thus obtain the equation

$$\int_{t_0}^{t_1} dt \iint \{u(g'n - h'm) + v(h'l - f'n) + w(f'm - g'l) + f(v'n - w'm) + g(w'l - u'n) + h(u'm - v'l)\}\, dS$$
$$= \frac{1}{c^2}\left[\iiint (\dot{u}u' - \dot{u}'u + \dot{v}v' - \dot{v}'v + \dot{w}w' - \dot{w}'w)\, d\tau\right]_{t_0}^{t_1} \quad . \quad . \quad . \quad (31).$$

If the functions involved are such that the volume integral on the right vanishes, and that the order of integrations on the left can be interchanged, we have

$$\iint dS \int_{t_0}^{t_1} \{u(g'n - h'm) + v(h'l - f'n) + w(f'm - g'l)\}\, dt$$
$$= \iint dS \int_{t_0}^{t_1} \{u'(gn - hm) + v'(hl - fn) + w'(fm - gl)\}\, dt \quad . \quad . \quad . \quad (32).$$

This equation plays the same part in the present theory as GREEN's equation

$$\iint \phi \frac{\partial V}{\partial \nu}\, dS = \iint V \frac{\partial \phi}{\partial \nu}\, dS,$$

V and ϕ being harmonic, plays in the Theory of Potential.

Integration of the General Equations.

15. We shall now suppose the boundaries of the region of space, to which the theorem of § 14 is applied, to be (1) a closed surface σ_1 containing none of the singularities of the functions u, v, w, (2) a small sphere σ_2 with its centre at a point O, inside σ_1. Then, taking O as origin, we shall assume for u', v', w', the expressions (17) of § 11, and for f', g', h' the corresponding expressions (18). For l, m, n at σ_2 we have to put x/r, y/r, z/r, and the contribution of σ_2 to the left-hand member of (31) becomes

$$\int_{t_0}^{t_1} dt \iint (vz - wy)\left\{\frac{2}{r^3}\left(\frac{\partial \phi_0}{\partial r} - \frac{\phi_0}{r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi_0}{\partial r^2} - \frac{3}{r}\frac{\partial \phi_0}{\partial r} + \frac{3}{r^2}\phi_0\right)\right\} d\sigma_2$$
$$+ \int_{t_0}^{t_1} dt \iint \left(f\frac{y^2 + z^2}{r^4} - g\frac{xy}{r^4} - h\frac{xz}{r^4}\right)\left(r\frac{\partial \phi_0}{\partial r} - \phi_0\right) d\sigma_2 \quad . \quad . \quad . \quad (33).$$

We take ϕ_0 to be of the form $\phi(r + ct)$, so that

$$\frac{\partial \phi_0}{\partial r} = \frac{1}{c}\frac{\partial \phi}{\partial t}, \quad . \quad . \quad . \quad (34),$$

and suppose* that the function ϕ is very nearly zero for all values of its argument ζ, except such as lie in the interval between $-\zeta_0$ and ζ_1, where ζ_0 and ζ_1 are two very small positive numbers; further, we suppose that between these values $\phi(\zeta)$ becomes so great that

$$\int_{-\zeta_0}^{\zeta_1} \phi(\zeta)\, d\zeta = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

We may then choose t_0 and t_1 so that, if r_0 is the radius of the sphere σ_2,

$$r_0 + \mathrm{c}t_0 < -\zeta_0, \quad \text{and} \quad r_0 + \mathrm{c}t_1 > \zeta_1;$$

then we shall have

$$\int_{t_0}^{t_1} \phi_0 dt = \frac{1}{\mathrm{c}} \int_{-\zeta_0}^{\zeta_1} \phi(\zeta)\, d\zeta = \frac{1}{\mathrm{c}},$$

$$\int_{t_0}^{t_1} r_0 \frac{\partial \phi_0}{\partial r} dt = \frac{r_0}{\mathrm{c}} \int_{-\zeta_0}^{\zeta_1} \phi'(\zeta)\, d\zeta = 0,$$

and, provided r_0 is sufficiently small, these will hold for any negative value of t_0 and any positive value of t_1.

With this choice of ϕ_0, the second line of the expression (33) becomes $-\frac{2}{3}\, 4\pi f_0 \mathrm{c}^{-1}$ in the limit, when r_0 is indefinitely diminished, f_0 being the value of f for the point O and the time $t = 0$.

In the first line of the expression (33) we develope v and w in such forms as

$$v = (i)_0 + x\left(\frac{\partial v}{\partial x}\right)_0 + y\left(\frac{\partial v}{y}\right)_0 + z\left(\frac{\partial v}{\partial z}\right)_0 + \text{terms of higher order},$$

where $(\ )_0$ indicates that the value at O is to be taken; we observe that

$$\int_{t_0}^{t_1} \frac{\partial^2 \phi_0}{\partial r^2}\, dt = \frac{1}{\mathrm{c}} \int_{-\zeta_0}^{\zeta_1} \phi''(\zeta)\, d\zeta = 0,$$

and find that, when r_0 is diminished indefinitely, the limit of the first line is the same as that of

$$\iint d\mathrm{S} \int_{t_0}^{t_1} \left\{ z^2 \left(\frac{\partial v}{\partial z}\right)_0 - y^2 \left(\frac{\partial w}{\partial y}\right)_0 \right\} \frac{\phi_0}{r^4}\, dt,$$

or it is $-\frac{1}{3}\, 4\pi f_0 \mathrm{c}^{-1}$.

* The process is adapted from KIRCHHOFF, 'Optik,' pp. 24, 25.

Accordingly, the contribution of σ_2 to the left-hand member of equation (31) is $-4\pi f_0 \mathrm{c}^{-1}$.

In obtaining this result, we have interchanged the order of the integration with respect to t and the integration over the surface σ_2. This step is certainly permissible if the subject of integration is, in each case, a continuous function of x, y, z, t, for all the values that occur. Equation (35) and this condition can be satisfied in any number of ways, and, in particular, by taking a very large value of μ, and putting, after KIRCHHOFF,

$$\phi(\zeta) = \frac{\mu}{\sqrt{\pi}} e^{-\mu^2\zeta^2}, \qquad (36),$$

provided we suppose that r_0 is small of order μ^{-6}.

With the same choice of ϕ_0, t_0, t_1, the right-hand member of equation (31) can have the limit zero. For this it is sufficient that, for all points between σ_1 and σ_2, the quantities u', v', w' and $\dot{u}', \dot{v}', \dot{w}'$ should be ultimately zero when t is either t_0 or t_1. This is the case if $r + \mathrm{c}t_0$ is negative and $r + \mathrm{c}t_1$ is positive for all values of r that occur. Equation (31) then takes the form

$$\iint d\mathrm{S} \int_{t_0}^{t_1} dt \{ u'(mh - ng) + v'(nf - lh) + w'(lg - mf) \\ + f'(mw - nv) + g'(nu - lw) + h'(lv - mu) \} = 4\pi f_0 \mathrm{c}^{-1} \qquad (37),$$

where the surface integration is taken over σ_1 only.

The quantities u', v', w' and f', g', h' have values, which are not extremely near to zero, only when $r + \mathrm{c}t$ is very near to zero, r being the distance of a point on the surface from the point O. The integration with respect to t, in the left-hand member of (37), can accordingly be carried out by observing the rules

$$\int_{t_0}^{t_1} \chi\phi_0 dt = \frac{1}{\mathrm{c}} (\chi)_{t=-\frac{r}{\mathrm{c}}},$$

$$\int_{t_0}^{t_1} \chi \frac{\partial\phi_0}{\partial r} dt = \frac{1}{\mathrm{c}} \int_{t_0}^{t_1} \chi \frac{\partial\phi_0}{\partial t} dt = -\frac{1}{\mathrm{c}} \int_{t_0}^{t_1} \phi_0 \frac{\partial\chi}{\partial t} dt = -\frac{1}{\mathrm{c}^2} \left(\frac{\partial\chi}{\partial t}\right)_{t=-\frac{r}{\mathrm{c}}},$$

$$\int_{t_0}^{t_1} \chi \frac{\partial^2\phi_0}{\partial r^2} dt = \frac{1}{\mathrm{c}^2} \int_{t_0}^{t_1} \chi \frac{\partial^2\phi_0}{\partial t^2} dt = \frac{1}{\mathrm{c}^2} \int_{t_0}^{t_1} \phi_0 \frac{\partial^2\chi}{\partial t^2} dt = \frac{1}{\mathrm{c}^3} \left(\frac{\partial^2\chi}{\partial t^2}\right)_{t=-\frac{r}{\mathrm{c}}}.$$

We thus find for the value of f, at the point O, and at the time $t = 0$, the equation

$$
\begin{aligned}
4\pi f_0 = \iint d\mathrm{S}\Big[&\frac{y}{r^3}\Big\{(lg - mf) + \frac{r}{\mathrm{C}}(l\dot{g} - m\dot{f})\Big\} - \frac{z}{r^3}\Big\{(nf - lh) + \frac{r}{\mathrm{C}}(n\dot{f} - l\dot{h})\Big\} \\
&+ \Big\{\frac{2}{r^3} - \frac{3(y^2 + z^2)}{r^5}\Big\}\Big\{(mw - nv) + \frac{r}{\mathrm{C}}(m\dot{w} - n\dot{v})\Big\} \\
&- \frac{y^2 + z^2}{r^3\mathrm{C}^2}(m\ddot{w} - n\ddot{v}) \\
&+ \frac{xy}{r^5}\Big\{3(nu - lw) + 3\frac{r}{\mathrm{C}}(n\dot{u} - l\dot{w}) + \frac{r^2}{\mathrm{C}^2}(n\ddot{u} - l\ddot{w})\Big\} \\
&+ \frac{xz}{r^5}\Big\{3(lv - mu) + 3\frac{r}{\mathrm{C}}(l\dot{v} - m\dot{u}) + \frac{r^2}{\mathrm{C}^2}(l\ddot{v} - m\ddot{u})\Big\}\Big], \quad . \quad . \quad . \quad . \quad (38),
\end{aligned}
$$

in which the values of u, $\dot{u}$, . . . $\dot{h}$, at any point of the surface, at time $t = -r/\mathrm{C}$, are to be calculated, and the integral formed with these values.

16. In this equation, the point O is the origin, and the point (x, y, z) is on the surface. We can express the value of f, at any point (x, y, z), and at any time t, by suitable changes. We have to write $x' - x$, $y' - y$, $z' - z$ for x, y, z, and, in the expressions for u, $\dot{u}$ . . . in terms of x', y', z' and t, we have to substitute for t, $t - r/\mathrm{C}$, where r is the distance between the points (x, y, z) and (x', y', z'). Further, when the form of f has been obtained, the forms for g and h can be written down by symmetry, and the forms for u, v, w can be deduced from those for f, g, h, by writing F, G, H instead of u, v, w, and u, v, w instead of f, g, h.

It is convenient to have, for reference, the explicit expression of the results. For u, v, w we have

$$
\begin{aligned}
4\pi u = \iint d\mathrm{S}\Big[&-\frac{y - y'}{r^3}\Big\{(lv - mu) + \frac{r}{\mathrm{C}}(l\dot{v} - m\dot{u})\Big\} + \frac{z - z'}{r^3}\Big\{(nu - lw) + \frac{r}{\mathrm{C}}(n\dot{u} - l\dot{w})\Big\} \\
&+ \Big\{\frac{2}{r^3} - 3\frac{(y - y')^2 + (z - z')^2}{r^5}\Big\}\Big\{(m\mathrm{H} - n\mathrm{G}) + \frac{r}{\mathrm{C}}(m\dot{\mathrm{H}} - n\dot{\mathrm{G}})\Big\} \\
&- \frac{(y - y')^2 + (z - z')^2}{r^3\mathrm{C}^2}(m\ddot{\mathrm{H}} - n\ddot{\mathrm{G}}) \\
&+ \frac{(x - x')(y - y')}{r^5}\Big\{3(n\mathrm{F} - l\mathrm{H}) + 3\frac{r}{\mathrm{C}}(n\dot{\mathrm{F}} - l\dot{\mathrm{H}}) + \frac{r^2}{\mathrm{C}^2}(n\ddot{\mathrm{F}} - l\ddot{\mathrm{H}})\Big\} \\
&+ \frac{(x - x')(z - z')}{r^5}\Big\{3(l\mathrm{G} - m\mathrm{F}) + 3\frac{r}{\mathrm{C}}(l\dot{\mathrm{G}} - m\dot{\mathrm{F}}) + \frac{r^2}{\mathrm{C}^2}(l\ddot{\mathrm{G}} - m\ddot{\mathrm{F}})\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
4\pi v = \iint d\mathrm{S}\Big[&-\frac{z - z'}{r^3}\Big\{(mw - nv) + \frac{r}{\mathrm{C}}(m\dot{w} - n\dot{v})\Big\} + \frac{x - x'}{r^3}\Big\{(lv - mu) + \frac{r}{\mathrm{C}}(l\dot{v} - m\dot{u})\Big\} \\
&+ \frac{(x - x')(y - y')}{r^5}\Big\{3(m\mathrm{H} - n\mathrm{G}) + 3\frac{r}{\mathrm{C}}(m\dot{\mathrm{H}} - n\mathrm{G}) + \frac{r^2}{\mathrm{C}^2}(m\ddot{\mathrm{H}} - n\ddot{\mathrm{G}})\Big\} \\
&+ \Big\{\frac{2}{r^3} - 3\frac{(z - z')^2 + (x - x')^2}{r^5}\Big\}\Big\{(n\mathrm{F} - l\mathrm{H}) + \frac{r}{\mathrm{C}}(n\dot{\mathrm{F}} - l\dot{\mathrm{H}})\Big\} \\
&- \frac{(z - z')^2 + (x - x')^2}{r^3\mathrm{C}^2}(n\ddot{\mathrm{F}} - l\ddot{\mathrm{H}}) \\
&+ \frac{(y - y')(z - z')}{r^5}\Big\{3(l\mathrm{G} - m\mathrm{F}) + 3\frac{r}{\mathrm{C}}(l\dot{\mathrm{G}} - m\dot{\mathrm{F}}) + \frac{r^2}{\mathrm{C}^2}(l\ddot{\mathrm{G}} - m\ddot{\mathrm{F}})\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
4\pi w = \iint d\mathrm{S}\Big[& -\frac{x-x'}{r^3}\Big\{(nu-lw)+\frac{r}{\mathrm{C}}(n\dot{u}-l\dot{w})\Big\}+\frac{y-y'}{r^3}\Big\{(mw-nv)+\frac{r}{\mathrm{C}}(m\dot{w}-n\dot{v})\Big\} \\
& + \frac{(x-x')(z-z')}{r^5}\Big\{3(m\mathrm{H}-n\mathrm{G})+3\frac{r}{\mathrm{C}}(m\dot{\mathrm{H}}-n\mathrm{G})+\frac{r^2}{\mathrm{C}^2}(m\ddot{\mathrm{H}}-n\ddot{\mathrm{G}})\Big\} \\
& + \frac{(y-y')(z-z')}{r^5}\Big\{3(n\mathrm{F}-l\mathrm{H})+3\frac{r}{\mathrm{C}}(n\dot{\mathrm{F}}-l\dot{\mathrm{H}})+\frac{r^2}{\mathrm{C}^2}(n\ddot{\mathrm{F}}-l\ddot{\mathrm{H}})\Big\} \\
& + \Big\{\frac{2}{r^3}-3\frac{(x-x')^2+(y-y')^2}{r^5}\Big\}\Big\{(l\mathrm{G}-m\mathrm{F})+\frac{r}{\mathrm{C}}(l\dot{\mathrm{G}}-m\dot{\mathrm{F}})\Big\} \\
& - \frac{(x-x')^2+(y-y')^2}{r^3\mathrm{C}^2}(l\ddot{\mathrm{G}}-m\ddot{\mathrm{F}})\Big].
\end{aligned}
$$

And for f, g, h we have

$$
\begin{aligned}
4\pi f = \iint d\mathrm{S}\Big[& -\frac{y-y'}{r^3}\Big\{(lg-mf)+\frac{r}{\mathrm{C}}(l\dot{g}-m\dot{f})\Big\}+\frac{z-z'}{r^3}\Big\{(nf-lh)+\frac{r}{\mathrm{C}}(n\dot{f}-l\dot{h})\Big\} \\
& + \Big\{\frac{2}{r^3}-3\frac{(y-y')^2+(z-z')^2}{r^5}\Big\}\Big\{(mw-nv)+\frac{r}{\mathrm{C}}(m\dot{w}-n\dot{v})\Big\} \\
& - \frac{(y-y')^2+(z-z')^2}{r^3\mathrm{C}^2}(m\ddot{w}-n\ddot{v}) \\
& + \frac{(x-x')(y-y')}{r^5}\Big\{3(nu-lw)+3\frac{r}{\mathrm{C}}(n\dot{u}-l\dot{w})+\frac{r^2}{\mathrm{C}^2}(n\ddot{u}-l\ddot{w})\Big\} \\
& + \frac{(x-x')(z-z')}{r^5}\Big\{3(lv-mu)+3\frac{r}{\mathrm{C}}(l\dot{v}-m\dot{u})+\frac{r^2}{\mathrm{C}^2}(l\ddot{v}-m\ddot{u})\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
4\pi g = \iint d\mathrm{S}\Big[& -\frac{z-z'}{r^3}\Big\{(mh-ng)+\frac{r}{\mathrm{C}}(m\dot{h}-n\dot{g})\Big\}+\frac{x-x'}{r^3}\Big\{(lg-mf)+\frac{r}{\mathrm{C}}(l\dot{g}-m\dot{f})\Big\} \\
& + \frac{(x-x')(y-y')}{r^5}\Big\{3(mw-nv)+3\frac{r}{\mathrm{C}}(m\dot{w}-n\dot{v})+\frac{r^2}{\mathrm{C}^2}(m\ddot{w}-n\ddot{v})\Big\} \\
& + \Big\{\frac{2}{r^3}-3\frac{(z-z')^2+(x-x')^2}{r^5}\Big\}\Big\{(nu-lw)+\frac{r}{\mathrm{C}}(n\dot{u}-l\dot{w})\Big\} \\
& - \frac{(z-z')^2+(x-x')^2}{r^3\mathrm{C}^2}(n\ddot{u}-l\ddot{w}) \\
& + \frac{(y-y')(z-z')}{r^5}\Big\{3(lv-mu)+3\frac{r}{\mathrm{C}}(l\dot{v}-m\dot{u})+\frac{r^2}{\mathrm{C}^2}(l\ddot{v}-m\ddot{u})\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
4\pi h = \iint d\mathrm{S}\Big[& -\frac{x-x'}{r^3}\Big\{(nf-lh)+\frac{r}{\mathrm{C}}(n\dot{f}-l\dot{h})\Big\}+\frac{y-y'}{r^3}\Big\{(mh-ng)+\frac{r}{\mathrm{C}}(m\dot{h}-n\dot{g})\Big\} \\
& + \frac{(x-x')(z-z')}{r^5}\Big\{3(mw-nv)+3\frac{r}{\mathrm{C}}(m\dot{w}-n\dot{v})+\frac{r^2}{\mathrm{C}^2}(m\ddot{w}-n\ddot{v})\Big\} \\
& + \frac{(y-y')(z-z')}{r^5}\Big\{3(nu-lw)+3\frac{r}{\mathrm{C}}(n\dot{u}-l\dot{w})+\frac{r^2}{\mathrm{C}^2}(n\ddot{u}-l\ddot{w})\Big\} \\
& + \Big\{\frac{2}{r^3}-3\frac{(x-x')^2+(y-y')^2}{r^5}\Big\}\Big\{(lv-mu)+\frac{r}{\mathrm{C}}(l\dot{v}-m\dot{u})\Big\} \\
& - \frac{(x-x')^2+(y-y')^2}{r^3\mathrm{C}^2}(l\ddot{v}-m\ddot{u})\Big].
\end{aligned}
$$

In all these $t - r/c$ is to be put for t before integration.

17. It may be verified without difficulty by using KIRCHHOFF's method that the integrals written down in § 16, when taken over a closed surface S, not containing the point (x, y, z), but containing all the singularities of the functions u, v, w, f, g, h, represent the values of $4\pi u, \ldots$ at the external point (x, y, z), provided the normal (l, m, n) is drawn towards the exterior of S. It may also be verified, in the same way, that, if the surface S contains the point (x, y, z) and all the singularities of the functions, the integrals in question vanish identically.

A particular case, which leads to a verification of the formulæ, is afforded by taking the surface S to be a sphere, of radius $\mathrm{c}t$, having its centre at the point (x, y, z). For this we have

$$x - x' = lr, \quad y - y' = mr, \quad z - z' = nr, \quad r = \mathrm{c}t,$$

and the values of the quantities $u, v, \ldots$, at points on S, are the initial values, $u_0, v_0, \ldots$, of these quantities. Now the terms of $4\pi u$ that contain u, v, w, explicitly are

$$r^{-2}\iint d\mathrm{S}\{u_0 - l(lu_0 + mv_0 + nw_0)\},$$

which $$= r^{-2}\iint d\mathrm{S}u_0 - r^{-3}\iint d\mathrm{S}\,(x - x')\,(lu_0 + mv_0 + nw_0)$$

$$= r^{-2}\iint d\mathrm{S}u_0 - r^{-3}\iiint u_0 d\tau + r^{-3}\iiint (x - x')\left(\frac{\partial u_0}{\partial x'} + \frac{\partial v_0}{\partial y'} + \frac{\partial w_0}{\partial z'}\right)d\tau,$$

where the volume integrations extend through the volume within S, and the last volume integral vanishes identically. Again, the terms that contain F, G, H explicitly are

$$r^{-3}\iint d\mathrm{S}\,(n\mathrm{G}_0 - m\mathrm{H}_0),$$

which $$= r^{-3}\iiint\left(\frac{\partial \mathrm{H}_0}{\partial y'} - \frac{\partial \mathrm{G}_0}{\partial z'}\right)d\tau, \; = r^{-3}\iiint u_0 d\tau.$$

Further, the terms that contain $\ddot{\mathrm{F}}, \ddot{\mathrm{G}}, \ddot{\mathrm{H}}$ explicitly can be written

$$r^{-1}\iint d\mathrm{S}\,(mh_0 - ng_0)$$

by observing that $\ddot{\mathrm{F}} = \mathrm{c}^2\nabla^2\mathrm{F} = -\mathrm{c}^2 f, \ldots$

The integral last written is

$$\begin{aligned} & r^{-1}\iint d\mathrm{S}\left\{m\left(\frac{\partial v_0}{\partial x'}-\frac{\partial u_0}{\partial y'}\right)-n\left(\frac{\partial u_0}{\partial z'}-\frac{\partial w_0}{\partial x'}\right)\right\} \\ &= r^{-1}\iint d\mathrm{S}\left\{-\left(l\frac{\partial u_0}{\partial x'}+m\frac{\partial u_0}{\partial y'}+n\frac{\partial u_0}{\partial z'}\right)+\left(l\frac{\partial u_0}{\partial x'}+m\frac{\partial v_0}{\partial x'}+n\frac{\partial w_0}{\partial x'}\right)\right\} \\ &= r^{-1}\iint d\mathrm{S}\frac{\partial u_0}{\partial r}-r^{-1}\iiint\frac{\partial}{\partial x'}\left(\frac{\partial u_0}{\partial x'}+\frac{\partial v_0}{\partial y'}+\frac{\partial w_0}{\partial z'}\right)d\tau, \end{aligned}$$

of which the volume integral vanishes identically. The terms which contain $\dot{u}$, $\dot{v}$, $\dot{w}$ explicitly, and those which contain $\dot{\mathrm{F}}$, $\dot{\mathrm{G}}$, $\dot{\mathrm{H}}$ explicitly, may be transformed in the same way, and we have finally

$$u=\frac{1}{4\pi \mathrm{c}^2t^2}\left\{t\iint \dot{u}_0 d\mathrm{S}+\iint u_0 d\mathrm{S}+\mathrm{c}t\iint\frac{\partial u_0}{\partial r}d\mathrm{S}\right\}; \quad . \quad . \quad . \quad . \quad (39);$$

and this is identical with POISSON'S integral* of the equation

$$\partial^2 u/\partial t^2=\mathrm{c}^2\nabla^2 u$$

in terms of initial conditions.

18. The results can be interpreted in terms of sources of disturbance of the two types previously investigated. Any point of the surface S must be regarded as the seat of a source of the first type, and of a source of the second type. The axis of the source of the first type is at right angles to the direction of (F, G, H), and is tangential to the surface; its radiation function is the product of dS, the resultant of (F, G, H) and the sine of the angle, which the direction of this resultant makes with the normal to the surface. The source of the first type is equivalent to three sources, with their axes parallel to the coordinate axes, and with radiation functions equal to

$$-d\mathrm{S}(m\mathrm{H}-n\mathrm{G}), \qquad -d\mathrm{S}(n\mathrm{F}-l\mathrm{H}), \qquad -d\mathrm{S}(l\mathrm{G}-m\mathrm{F});$$

these expressions, for any point on the surface, are functions of t, and they take the characteristic form of radiation functions, when $t-r/\mathrm{c}$ is substituted for t.

The axis of the source of the second type is at right angles to the direction of (u, v, w), and is tangential to the surface; its radiation function is the product of dS, the resultant of (u, v, w) and the sine of the angle, which the direction of this resultant makes with the normal to the surface. The source of the second type is equivalent to three sources, with their axes parallel to the coordinate axes, and with radiation functions equal to

$$-d\mathrm{S}(mw-nv), \qquad -d\mathrm{S}(nu-lw), \qquad -d\mathrm{S}(lv-mu);$$

* The form of POISSON'S integral usually given requires the performance of differentiation, with respect to t, upon an integral taken over a sphere of radius $\mathrm{c}t$, and thus, when the differentiation is carried out, there will be three terms in the complete expression; it is easy to verify that these terms are precisely those given in equation (39).

these expressions are to be formed for any time t, and then $t - r/c$ is to be substituted for t in them.

Reduction to a Single Type of Sources.

19. We may seek to express our results in terms of sources of a single type, instead of using two types of sources. The method to be followed is analogous to that used by GREEN for the Theory of Potential.* If V is a function, which is harmonic at all points outside a closed surface S, the value of V at an external point O is

$$-\frac{1}{4\pi}\iint\left(\frac{1}{r}\frac{\partial \mathrm{V}}{\partial \nu} - \mathrm{V}\frac{\partial r^{-1}}{\partial \nu}\right)d\mathrm{S}, \quad \dots \dots \dots \quad (40),$$

where $d\nu$ is the element of the normal to S drawn outwards, and r is distance from O. If now V′ is harmonic within S, and equal to V on the surface, this becomes

$$-\frac{1}{4\pi}\iint\frac{1}{r}\left(\frac{\partial \mathrm{V}}{\partial \nu} + \frac{\partial \mathrm{V}'}{\partial \nu'}\right)d\mathrm{S}, \quad \dots \dots \dots \quad (41),$$

where $d\nu'$ is the element of the normal to S drawn inwards. This result is obtained from the reciprocal theorem

$$\iint\frac{1}{r}\frac{\partial \mathrm{V}'}{\partial \nu'}d\mathrm{S} = \iint \mathrm{V}'\frac{\partial r^{-1}}{\partial \nu'}d\mathrm{S},$$

both V′ and r^{-1} being harmonic at all points within S. Further, we know that there cannot be two functions satisfying the conditions satisfied by V′; the theorem that there is one such function is the fundamental existence-theorem of the Theory of Potential. The expression (40) may be interpreted in terms of sources and doublets on S; the expression (41) admits of a similar interpretation in terms of sources only.†

20. In adapting this process to the present theory, we begin by proving that there cannot be two sets of related vectors which

(1) are free from singularities at all points within a closed surface S;
(2) satisfy the system of equations (12) of § 9 at all points within S;
(3) yield the same tangential components, for either of the two vectors, at all points on S;
(4) vanish throughout the space within S for some value t_0 of t.

If there were two such sets, their differences (α, β, γ) and (f, g, h) would satisfy

* GREEN, 'Math. Papers,' p. 29.
† LAMB, 'Hydrodynamics,' pp. 66, 67.

the conditions (1), (2) and (4), and either (α, β, γ) or (f, g, h) would be normal to S, at every point on S; so that we should have either

$$\alpha : \beta : \gamma = l : m : n,$$

or

$$f : g : h = l : m : n,$$

(l, m, n) being the direction cosines of the normal to S drawn inwards.

In both cases

$$\int_{t_0}^{t} dt \iint d\mathrm{S} \{ l(\beta h - \gamma g) + m(\gamma f - \alpha h) + n(\alpha g - \beta f)\} = 0.$$

Now this integral is

$$-\int_{t_0}^{t} dt \iiint d\tau \left[\left\{ f\left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}\right) + . + . \right\} - \left\{ \alpha \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + . + . \right\} \right],$$

the integral being taken through the volume within S, and this is

$$-\frac{1}{2} \iiint d\tau \left[(f^2 + g^2 + h^2) + \frac{1}{\mathrm{C}^2}(\alpha^2 + \beta^2 + \gamma^2) \right]_t,$$

since α, β, γ and f, g, h vanish when $t = t_0$. The expression last obtained cannot vanish, unless α, β, γ and f, g, h vanish, at all times, and at all points within S. This proves the theorem.

It follows from this theorem that, if either the tangential components of (u, v, w), or those of (f, g, h), are given at all points of S, the solution of equations (8) and (9) of § 9 is unique.

21. Now let (u_1, v_1, w_1) and (u_2, v_2, w_2) be two sets of possible magnetic displacements, for which there are no singularities within a closed surface S; and let F_1 . . . be the corresponding vector potentials, and f_1 . . . the corresponding electric displacements. Suppose further that, at a certain time t_0, all these vanish at all points within S. We shall apply the theorem of § 14 to these vectors. We identify the set with suffix 2 with the set previously accented (§ 15), so that $u_2 = u'$, . . . , the point O, which is the sole singularity of the accented set, being outside S. Then the right-hand member of equation (31) vanishes, and we have

$$\int_{t_0}^{t_1} dt \iint d\mathrm{S} \{ u_1(g'n - h'm) + v_1(h'l - f'n) + w_1(f'm - g'l)\}$$

$$= \int_{t_0}^{t_1} dt \iint d\mathrm{S} \{ u'(g_1 n - h_1 m) + v'(h_1 l - f_1 n) + w'(f_1 m - g_1 l)\}.$$

If there is a set of magnetic and electric displacements, free from singularities within S, and making the tangential components of the vector potential on S the

same as those in the solution $(u, v, \ldots h)$ for external space, this set also makes the tangential components of electric displacement the same as those in the solution for external space; we take this to be the set with suffix 1, so that, on S,

$$\left.\begin{array}{ll} g_1 n - h_1 m = gn - hm, & G_1 n - H_1 m = Gn - Hm, \\ \ldots & \ldots \\ \ldots & \ldots \end{array}\right\}$$

Equation (37) of § 15 now becomes

$$4\pi C^{-1} f_0 = \iint dS \int_{t_0}^{t_1} dt \left[f'\{m(w - w_1) - n(v - v_1)\} + g'\{n(u - u_1) - l(w - w_1)\} + h'\{l(v - v_1) - m(u - u_1)\}\right] \quad . \quad . \quad . \quad (42).$$

The corresponding equation for u_0 would be

$$4\pi C^{-1} u_0 = \iint dS \int_{t_0}^{t_1} dt \left[u'\{m(w - w_1) - n(v - v_1)\} + v'\{n(u - u_1) - l(w - w_1)\} + w'\{l(v - v_1) - m(u - u_1)\}\right] \quad . \quad . \quad . \quad (43).$$

The results would thus be interpretable in terms of sources of the second type only, the radiation functions of the sources depending upon the surface values of $u - u_1$, $v - v_1$, $w - w_1$, in the same way as those in § 18 depended upon the surface values of u, v, w. We might, in a similar way, show how the general forms of the displacements could be expressed in terms of sources of the first type only.

It is to be noted that this reduction of the number of types of sources has depended upon the possibility of choosing a time, before which there is no disturbance at any point within the given closed surface, and also that it involves an existence-theorem, which has not been proved. For a sphere, the existence-theorem could be proved by help of the formulæ in § 10.

Law of Disturbance in a Secondary Wave.

22. As a first application of the general formulæ we may consider the law of disturbance in a secondary wave. We suppose that simple harmonic plane waves, of the simplest type, polarised in the plane (x, z), and propagated parallel to the axis of z, are to be resolved into secondary waves due to sources, situated on the wave front $z = z'$. Let the primary waves be given by

$$\left.\begin{array}{lll} F = 0, & G = -\dfrac{1}{\kappa}\sin\kappa(z - Ct), & H = 0, \\ u = \cos\kappa(z - Ct), & v = 0, & w = 0, \\ f = 0, & g = -\kappa\sin\kappa(z - Ct), & h = 0, \end{array}\right\} \quad . \quad . \quad (44).$$

The disturbance at any point, for which $z > z'$, is given by the following equations, in which l, m, n now denote the direction cosines of the line, of length r, drawn from the point (x', y', z') to the point (x, y, z) :—

$$\begin{aligned}4\pi u = \iint dx'dy' \Big[&\frac{n}{r^2}\{\cos\kappa(z'-\mathrm{c}t+r)+\kappa r\sin\kappa(z'-\mathrm{c}t+r)\} \\ &+\frac{2}{\kappa r^3}\{\sin\kappa(z'-\mathrm{c}t+r)-\kappa r\cos\kappa(z'-\mathrm{c}t+r)\} \\ &-\frac{(m^2+n^2)}{\kappa r^3}\{(3-\kappa^2 r^2)\sin\kappa(z'-\mathrm{c}t+r)-3\kappa r\cos\kappa(z'-\mathrm{c}t+r)\}\Big], \quad (45).\end{aligned}$$

$$4\pi v = \iint dx'dy' \frac{lm}{\kappa r^3}\{(3-\kappa^2 r^2)\sin\kappa(z'-\mathrm{c}t+r)-3\kappa r\cos\kappa(z'-\mathrm{c}t+r)\}, \quad . \ (46).$$

$$\begin{aligned}4\pi w = \iint dx'dy' \Big[&-\frac{l}{r^2}\{\cos\kappa(z'-\mathrm{c}t+r)+\kappa r\sin\kappa(z'-\mathrm{c}t+r)\} \\ &+\frac{ln}{\kappa r^3}\{(3-\kappa^2 r^2)\sin\kappa(z'-\mathrm{c}t+r)-3\kappa r\cos\kappa(z'-\mathrm{c}t+r)\}\Big] (47).\end{aligned}$$

Also

$$4\pi f = \iint dx'dy' \frac{lm}{r^3}\{(3-\kappa^2 r^2)\cos\kappa(z'+\mathrm{c}t+r)+3\kappa r\sin\kappa(z'-\mathrm{c}t+r)\}, \quad . \ (48).$$

$$\begin{aligned}4\pi g = \iint dx'dy' \Big[&\frac{-n}{r^2}\kappa\{\sin\kappa(z'-\mathrm{c}t+r)-\kappa r\cos\kappa(z'-\mathrm{c}t+r)\} \\ &+\frac{2}{r^3}\{\cos\kappa(z'-\mathrm{c}t+r)+\kappa r\sin\kappa(z'-\mathrm{c}t+r)\} \\ &-\frac{n^2+l^2}{r^3}\{(3-\kappa^2 r^2)\cos\kappa(z'-\mathrm{c}t+r)+3\kappa r\sin\kappa(z'-\mathrm{c}t+r)\}\Big], (49).\end{aligned}$$

$$\begin{aligned}4\pi h = \iint dx'dy' \Big[&\frac{m}{r^2}\kappa\{\sin\kappa(z'-\mathrm{c}t+r)-\kappa r\cos\kappa(z'-\mathrm{c}t+r)\} \\ &+\frac{mn}{r^3}\{(3-\kappa^2 r^2)\cos\kappa(z'-\mathrm{c}t+r)+3\kappa r\sin\kappa(z'-\mathrm{c}t+r)\}\Big] \quad (50).\end{aligned}$$

At a great distance, the contribution of the element of area $d\mathrm{S}$ to (u, v, w) is

$$(n+m^2+n^2\,,\ -lm\,,\ -l-ln)\frac{\kappa d\mathrm{S}}{4\pi r}\sin\kappa(z'-\mathrm{c}t+r)\,; \ . \quad . \quad . \ (51)\,;$$

the magnitude of this contribution is

$$(1+n)\frac{\kappa d\mathrm{S}}{4\pi r}\sin\kappa(z'-\mathrm{c}t+r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (52),$$

and its direction is at right angles to that of r, and makes with the plane (z, r) the same angle that this plane makes with the plane (z, x). This direction is shown by

the point P in a spherical figure (fig. 2), in which Z represents the direction of propagation of the primary waves, R the direction of r, NX is the great circle of which Z is pole, NI is the great circle of which R is pole, and P is on NI produced so that

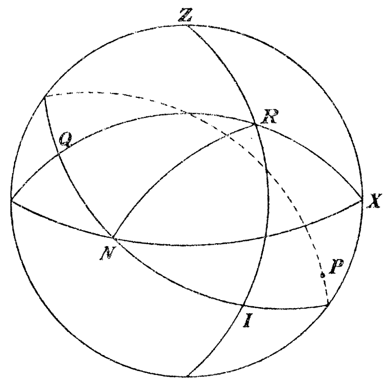

Fig. 2.

NP = NX. It is easy to verify that the direction cosines of the radius vector, drawn from the centre of the sphere to P, are

$$\frac{n + m^2 + n^2}{1 + n}, \qquad \frac{-lm}{1 + n}, \qquad -l.$$

Again, at a great distance from the plane, the contribution of the element of area dS to (f, g, h) is

$$(-lm, \; n + l^2 + n^2, \; -m - mn) \frac{\kappa d\mathrm{S}}{4\pi r} \kappa \cos \kappa (z' - \mathrm{c}t + r); \quad . \quad . \quad . \quad (53);$$

and its magnitude is

$$(1 + n) \frac{\kappa d\mathrm{S}}{4\pi r} \kappa \cos \kappa (z' - \mathrm{c}t + r); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54);$$

its direction might be shown by means of a construction similar to that used for the direction of the contribution to (u, v, w).

The result obtained by Sir G. STOKES* would be expressed, in the notation here employed, by the statements that the magnitude of the contribution of the element dS to (u, v, w) is

$$\sqrt{(m^2 + n^2)} (1 + n) \frac{\kappa d\mathrm{S}}{4\pi r} \sin \kappa (z' - \mathrm{c}t + r), \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

and that its direction is that which would be shown by the point antipodal to Q, where RX and NI intersect (fig. 2). It has been pointed out by Lord RAYLEIGH† that

* 'Papers,' 2, p. 286.

† 'Wave Theory of Light,' pp. 452, 453.

such factors as $\sqrt{(m^2 + n^2)}$ and $\frac{1}{2}(1 + n)$ are of no importance in the ordinary applications of expressions for the law of disturbance in secondary waves, and that, in fact, the enquiry after such a law involves a certain ambiguity. In the above deduction of such a law, we have used the general formulæ involving sources of two types; if we could have used formulæ involving sources of one type only, the result would probably have been different; this is the origin of the ambiguity referred to by Lord RAYLEIGH.

23. There is another difficulty attending the deduction of a law of disturbance in secondary waves from formulæ applicable to the propagation of a system of plane waves, viz.: that integrals such as (45) taken over an infinite plane are not convergent. The disturbances in the secondary waves ought to combine to give rise to the disturbance actually propagated, or the result of the integration ought to be to reproduce the displacements in the primary wave. If we form such an integral as (45) for a portion of the plane (x', y'), and afterwards extend the boundaries of this portion indefinitely, we do not arrive at a definite limit. Let O be the point at which the disturbance is to be estimated, O′ the foot of the perpendicular from O on the plane $z = z'$ (fig. 3), and let the portion of the plane be bounded by a circle, of

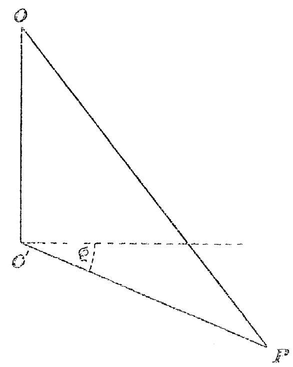

Fig. 3.

radius R′, with its centre at O′. We introduce plane polar coordinates r', ϕ, with origin at O′, and put $l = \sin\theta\cos\phi$, $m = \sin\theta\sin\phi$, $n = \cos\theta$; then the value of the expression (46) for v is

$$\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{R'} r'dr' \frac{\sin^2\theta\sin\phi\cos\phi}{\kappa r^3}\{(3 - \kappa^2 r^2)\sin\kappa(z' - ct + r) - 3\kappa r\cos\kappa(z' - ct + r)\},$$

and this vanishes identically, however great R′ may be, on account of the symmetry of the circular boundary; it would not have vanished if we had taken a boundary of a different shape, or a circular boundary with its centre away from O′. With the boundary chosen as above, we could show that w, f, h vanish. To form the expression for u, we put

$$\sin\theta = \frac{r'}{r}, \quad \cos\theta = \frac{z - z'}{r}, \quad r^2 = r'^2 + (z - z')^2, \quad R^2 = R'^2 + (z - z')^2,$$
$$\psi = \kappa(z' - ct + r)$$

and remember that z and z' are constants in the integration; we find

$$u = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{z-z'}^{R} dr \left[\frac{z-z'}{r^2}(\cos\psi + \kappa r \sin\psi) + \frac{2}{\kappa r^2}(\sin\psi - \kappa r \cos\psi) \right.$$

$$\left. - \frac{1}{\kappa r^2}\left[\frac{(z-z')^2}{r^2} + \sin^2\phi\left\{1 - \frac{(z-z')^2}{r^2}\right\}\right]\{(3-\kappa^2 r^2)\sin\psi - 3\kappa r\cos\psi\}\right]$$

$$= \tfrac{1}{2}\int_{z-z'}^{R} dr \left[\frac{z-z'}{r^2}(\cos\psi + \kappa r\sin\psi) + \frac{2}{\kappa r^2}(\sin\psi - \kappa r\cos\psi) \right.$$

$$\left. - \frac{1}{2\kappa r^2}\left\{1 + \frac{(z-z')^2}{r^2}\right\}\{(3-\kappa^2 r^2)\sin\psi - 3\kappa r\cos\psi\}\right].$$

This is immediately integrable, and we find*

$$u = -\tfrac{1}{4}\left[\left(1 + \frac{z-z'}{R}\right)^2 \cos\kappa(z' - ct + R) + \frac{1}{\kappa R}\left\{1 - \frac{(z-z')^2}{R^2}\right\}\sin\kappa(z' - ct + R)\right]$$

$$+ \cos\kappa(z - ct) \,; \quad . \quad . \quad . \quad . \quad (56);$$

and, when R is very great, this is approximately equal to

$$\cos\kappa(z - ct) - \tfrac{1}{4}\cos\kappa(z' - ct + R) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57).$$

Thus the value of u for the primary wave is reproduced by the secondary waves sent out from the parts of the plane, which are not at a very great distance. In like manner we should find for g the value

$$-\kappa\sin\kappa(z - ct)$$

$$+ \frac{\kappa}{4}\left[\left(1 + \frac{z-z'}{R}\right)^2 \sin\kappa(z' - ct + R) - \frac{1}{\kappa R}\left\{1 - \frac{(z-z')^2}{R^2}\right\}\cos\kappa(z' - ct + R)\right],$$

giving, when R is great, the approximate value

$$-\kappa\sin\kappa(z - ct) + \tfrac{1}{4}\kappa\sin\kappa(z' - ct + R) \,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (58);$$

and, as before, the value for the primary wave is reproduced by the secondary waves sent out from the parts of the plane, which are not at a very great distance. Both for u and for g, the distant parts of the plane contribute something finite to the disturbance; just as, in the ordinary elementary theory, there may remain a portion of a HUYGENS' zone uncompensated; such portions are always disregarded.†

The difficulty here considered arises entirely from our having applied to an infinite plane, formulæ, which were obtained on the express supposition that the surface, to which they are applied, lies entirely within a finite distance of the point, at which

* Equation (56) determines the intensity, at a point on the axis, of light diffracted through a circular aperture, the incident light being parallel, and the ordinary optical rule being assumed to hold (§§ 24, 30).

† *Cf.* BASSET, 'Physical Optics,' p. 46, or Lord RAYLEIGH, 'Wave Theory of Light,' p. 429.

the disturbance is estimated. The difficulty could not be evaded by adopting a different law of disturbance in secondary waves, and one aspect of it has been noticed by Sir G. STOKES* in connexion with the law obtained by him. The difficulty would not arise if we took a system of diverging *spherical* waves, and resolved the disturbance at a point O, outside some particular spherical wave front, into secondary waves due to a distribution of sources over this front. The difficulties of integration are, however, in this case considerable; when the point O is at a great distance from the sphere, the integrals can be evaluated approximately, and it can be verified that the disturbance corresponding to the primary wave is reproduced.

Passage of Waves through an Aperture.

24. The general problem of the passage of radiation across an aperture in a screen would involve a solution of the general equations (4) or (12) of § 9, subject to boundary conditions holding all over both faces of the screen; and, unless the incident radiation and the shape of the edge have very simple characters, this cannot at present be attempted.† In the theory of diffraction, it is customary to assume that the disturbance at points of the aperture, to which the disturbance on the further side is due, is that which would be found at those points if there were no screen, and also that the elements of the surface of the screen contribute nothing to the disturbance on the further side.‡ In the Theory of Sound, HELMHOLTZ§ has justified the use of a somewhat similar assumption in the problem of the open pipe. In the present theory the question may be formulated as follows:—A train of radiation is propagated on one side of a surface S towards the surface; there is an aperture in the surface, and the transmitted radiation is to be represented as due to sources situated in the aperture; how must such sources be distributed?

[25. (*Re-written March*, 1901.)—We simplify the general question by means of two suppositions:—(1.) that the incident radiation is represented by simple harmonic functions of the time, with period $2\pi/\kappa c$; (2.) that the surface S is plane. The first of these enables us to eliminate all vector potentials, by the rule

$$(F, G, H) = \kappa^{-2}(f, g, h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

It will appear later that the second supposition constitutes a practically unimportant restriction, when the aperture is small. We shall take the plane S to be given by the equation $z = z'$, and shall suppose that the incident radiation is propagated on the nearer side ($z < z'$). The transmitted radiation, on the further side ($z > z'$),

* 'Papers,' 2, p. 288. *Cf.* Lord RAYLEIGH, 'Wave Theory of Light,' p. 429.

† *Cf.* A. SOMMERFELD, 'Math. Theorie d. Diffraction,' Math. Ann., vol. 47 (1896).

‡ Lord RAYLEIGH, 'Wave Theory of Light,' p. 430; or 'Theory of Sound,' vol. 2, § 291.

§ 'J. f. Math. (Crelle),' 57 (1859); or 'Wiss. Abh.,' vol. 1, p 303.

being regarded as due to imagined sources, situated in the aperture, can be calculated directly from the formulæ of § 16, by first assigning certain functions of x', y', t as the forms of u, . . . under the sign of integration, then substituting $t - r/\mathrm{C}$ for t, and finally integrating over the aperture. We shall take the forms, that are to be substituted, for u, . . . under the sign of integration, to be given by the equations

$$\left.\begin{aligned} u &= \bar{u}_1 \cos\kappa\mathrm{C}t + \bar{u}_2 \sin\kappa\mathrm{C}t\,, \\ &\;\cdot\;\cdot\;\cdot\;\cdot \\ f &= \bar{f}_1 \cos\kappa\mathrm{C}t + \bar{f}_2 \sin\kappa\mathrm{C}t\,, \\ &\;\cdot\;\cdot\;\cdot\;\cdot \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (60),$$

where $\bar{u}_1$, . . . are functions of x', y', for which

$$\bar{h}_1 = \frac{\partial\bar{v}_1}{\partial x'} - \frac{\partial\bar{u}_1}{\partial y'}\,, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (61),$$

and similarly for $\bar{h}_2$. Further, we shall denote the values of u, . . . , resulting from the integrations, by u_+, . . . The answer to the general question of § 24, will thus lie in the determination of the functions $\bar{u}_1$, . . . These functions can be regarded as the values, at certain times, and at points within the aperture, of a certain system of magnetic and electric displacements.]

26. Before proceeding it will be convenient to record the forms for u_+, . . . in terms of the functions $\bar{u}_1$, . . . It will be sufficient to put down the terms that contain $\cos\kappa\mathrm{C}t$. We observe that in the formulæ of § 16

$$\left.\begin{aligned} &\frac{x-x'}{r^3} = -\frac{\partial r^{-1}}{\partial x} = \frac{\partial r^{-1}}{\partial x'}, \\ &\frac{2}{r^3} - 3\frac{(x-x')^2 + (y-y')^2}{r^5} = \frac{\partial^2 r^{-1}}{\partial z^2} = \frac{\partial^2 r^{-1}}{\partial z'^2}\,, \\ &3\frac{(y-y')(z-z')}{r^5} = \frac{\partial^2 r^{-1}}{\partial y\partial z} = \frac{\partial^2 r^{-1}}{\partial y'\partial z'}\,; \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (62)\,;$$

and we also observe that, when the surface S is a portion of a plane ($z = z'$), we must have $l = 0$, $m = 0$, $n = 1$, the point (x, y, z) being on the side $z > z'$. We can therefore write down the formulæ for u_+, . . . f_+, . . . as follows :—

$$\begin{aligned} 4\pi u_+ = \iint dx'dy' \Big[&-\frac{\partial r^{-1}}{\partial z}\,\bar{u}_1\{\cos\kappa(\mathrm{C}t - r) - \kappa r\sin\kappa(\mathrm{C}t - r)\} \\ &- \frac{\partial^2 r^{-1}}{\partial x^2}\,\kappa^{-2}\bar{g}_1\{\cos\kappa(\mathrm{C}t - r) - \kappa r\sin\kappa(\mathrm{C}t - r)\} \\ &+ \tfrac{1}{3}\,\frac{\partial^2 r^{-1}}{\partial x\partial y}\,\kappa^{-2}\bar{f}_1\{(3 - \kappa^2 r^2)\cos\kappa(\mathrm{C}t - r) - 3\kappa r\sin\kappa(\mathrm{C}t - r)\} \\ &- \frac{(y-y')^2 + (z-z'^2)}{r^3}\,\bar{g}_1\{\cos\kappa(\mathrm{C}t - r)\}\Big], \quad . \; . \; . \; . \; . \; . \quad (63), \end{aligned}$$

$$4\pi v_{+} = \iint dx'dy'\Big[-\frac{\partial r^{-1}}{\partial z}\bar{v}_1\{\cos\kappa(ct-r)-\kappa r\sin\kappa(ct-r)\}$$
$$-\tfrac{1}{3}\frac{\partial^2 r^{-1}}{\partial x\partial y}\kappa^{-2}\bar{g}_1\{(3-\kappa^2r^2)\cos\kappa(ct-r)-3\kappa r\sin\kappa(ct-r)\}$$
$$+\frac{\partial^2 r^{-1}}{\partial y^2}\kappa^{-2}\bar{f}_1\{\cos\kappa(ct-r)-\kappa r\sin\kappa(ct-r)\}$$
$$+\frac{(z-z')^2+(x-x')^2}{r^3}\bar{f}_1\{\cos\kappa(ct-r)\}\Big], \quad \ldots \ldots \quad (64),$$

$$4\pi w_{+} = \iint dx'dy'\Big[\Big(\bar{u}_1\frac{\partial r^{-1}}{\partial x}+\bar{v}_1\frac{\partial r^{-1}}{\partial y}\Big)\{\cos\kappa(ct-r)-\kappa r\sin\kappa(ct-r)\}$$
$$-\tfrac{1}{3}\Big(\frac{\partial^2 r^{-1}}{\partial x\partial z}\kappa^{-2}\bar{g}_1-\frac{\partial^2 r^{-1}}{\partial y\partial z}\kappa^{-2}\bar{f}_1\Big)\{(3-\kappa^2r^2)\cos\kappa(ct-r)$$
$$-3\kappa r\sin\kappa(ct-r)\}\Big]; \quad . \quad (65);$$

$$4\pi f_{+} = \iint dx'dy'\Big[-\frac{\partial r^{-1}}{\partial z}\bar{f}_1\{\cos\kappa(ct-r)-\kappa r\sin\kappa(ct-r)\}$$
$$-\frac{\partial^2 r^{-1}}{\partial x^2}\bar{v}_1\{\cos\kappa(ct-r)-\kappa r\sin\kappa(ct-r)\}$$
$$+\tfrac{1}{3}\frac{\partial^2 r^{-1}}{\partial x\partial y}\bar{u}_1\{(3-\kappa^2r^2)\cos\kappa(ct-r)-3\kappa r\sin\kappa(ct-r)\}$$
$$-\kappa^2\frac{(y-y')^2+(z-z')^2}{r^3}\bar{v}_1\{\cos\kappa(ct-r)\}\Big], \quad \ldots \ldots \quad (66),$$

$$4\pi g_{+} = \iint dx'dy'\Big[-\frac{\partial r^{-1}}{\partial z}\bar{g}_1\{\cos\kappa(ct-r)-\kappa r\sin\kappa(ct-r)\}$$
$$-\tfrac{1}{3}\frac{\partial^2 r^{-1}}{\partial x\partial y}\bar{v}_1\{(3-\kappa^2r^2)\cos\kappa(ct-r)-3\kappa r\sin\kappa(ct-r)\}$$
$$+\frac{\partial^2 r^{-1}}{\partial y^2}\bar{u}_1\{\cos\kappa(ct-r)-\kappa r\sin\kappa(ct-r)\}$$
$$+\kappa^2\frac{(z-z')^2+(x-x')^2}{r^3}\bar{u}_1\{\cos\kappa(ct-r)\}\Big], \quad \ldots \ldots \quad (67),$$

$$4\pi h_{+} = \iint dx'dy'\Big[\Big(\bar{f}_1\frac{\partial r^{-1}}{\partial x}+\bar{g}_1\frac{\partial r^{-1}}{\partial y}\Big)\{\cos\kappa(ct-r)-\kappa r\sin\kappa(ct-r)\}$$
$$-\tfrac{1}{3}\Big(\frac{\partial^2 r^{-1}}{\partial x\partial z}\bar{v}_1-\frac{\partial^2 r^{-1}}{\partial y\partial z}\bar{u}_1\Big)\{(3-\kappa^2r^2)\cos\kappa(ct-r)$$
$$-3\kappa r\sin\kappa(ct-r)\}\Big]. \quad . \quad (68).$$

Here the parts of $u_{+}, \ldots$ that arise from $\bar{u}_1\cos\kappa ct, \ldots$ are written down, the terms in $\sin\kappa ct$ being omitted.

[27. (*Re-written March*, 1901.)—The forms of the expressions for $u_{+}, \ldots$ have an important bearing on the determination of the functions $\bar{u}_1, \ldots$ The integrals,

which occur in these expressions, represent functions, which are continuous, and have definite values, at all points, that do not lie in the plane $z = z'$, and within the aperture. At points within the aperture, the functions, defined by these integrals, present discontinuities of one or other of three following kinds* :—(a.) The integral, obtained by replacing (x, y, z) by (x', y', z'), is convergent, and is different from the limit obtained by bringing (x, y, z) up to coincidence with (x', y', z') through values, for which $z > z'$, or through values, for which $z < z'$; these two limits are finite and definite, and they are not the same. The term of (63), containing $\partial r^{-1}/\partial z$, is an example of this peculiarity. (b.) The integral, obtained by replacing (x, y, z) by (x', y', z'), is not convergent; but the limit obtained by bringing (x, y, z) up to coincidence with (x', y', z'), on either side, is finite and definite, and these limits are the same. The term of (65), containing $\partial r^{-1}/\partial x$, is an example of this peculiarity. (c.) The integral obtained by replacing (x, y, z) by (x', y', z') is not convergent, nor is any definite limit obtained by bringing (x, y, z) up to coincidence with (x', y', z') on either side; but the difference of the two values, obtained by taking (x, y, z) at two points, near the aperture, and on opposite sides of it, can be made less than any assigned quantity by sufficiently diminishing the distance between the points. The term of (63), containing $\partial^2 r^{-1}/\partial x^2$, is an example of this peculiarity.

The discontinuity of the expressions for u_+, . . . arises from the representation of the disturbance on the further side $(z > z')$ as due to imagined sources in the aperture; there are not really any sources in the aperture, but the disturbance on the further side $(z > z')$ is continuous with the disturbance on the nearer side $(z < z')$. To restore continuity, it is most convenient to regard the disturbance on the nearer side as consisting of two superposed disturbances, denoted by A and B. The disturbance A is represented by functions, which are continuous in a region of space, containing all the points within the aperture, and within a finite distance on either side of it; these functions have no singularities on the nearer side, except the actual sources of the incident radiation. The disturbance B is represented by functions which have no singularities on the nearer side, but have discontinuities at the aperture, and these discontinuities may be of any of the kinds presented by the expressions for u_+, . . . We shall denote the magnetic and electric displacements, that belong to the disturbance B, by u_-, . . . , f_-, . . . , and those that belong to the disturbance A by u', . . . , f', . . . The displacements, that belong to the disturbances A and B, satisfy the general equations of § 9. We shall take them to be given by the following equations :—

for A,

$$\left.\begin{aligned} u &= u' = u_1' \cos \kappa \mathrm{c}t + u_2' \sin \kappa \mathrm{c}t, \\ &\quad \cdots \\ f &= f' = f_1' \cos \kappa \mathrm{c}t + f_2' \sin \kappa \mathrm{c}t, \\ &\quad \cdots \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; (69);$$

* *Cf.* POINCARÉ, 'Théorie du Potentiel Newtonien' (Paris 1899), ch. 3.

for B

$$\left.\begin{array}{l} u = u_{-} = u_1'' \cos \kappa \mathrm{c} t + u_2'' \sin \kappa \mathrm{c} t\,, \\ \quad . \ . \ . \ . \\ f = f_{-} = f_1'' \cos \kappa \mathrm{c} t + f_2'' \sin \kappa \mathrm{c} t\,, \\ \quad . \ . \ . \ . \end{array}\right\} \quad . \ . \ . \ . \ . \ . \quad (70).$$

We have now to express the conditions of continuity of magnetic and electric displacement at the aperture. We suppose that u_{+}, . . . are formed for a point P, ($z > z'$), and that u_{-}, . . . are formed for a point P′, ($z < z'$), and we take any

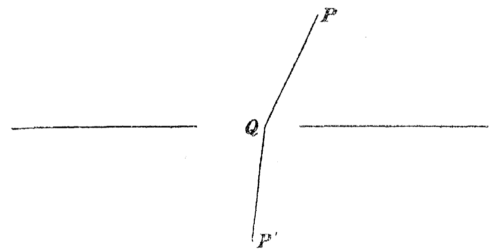

Fig. 4.

point Q in the aperture. The functions u_1', . . . are continuous in the neighbourhood of Q, and have definite values at Q. We form the difference

$$u_{+}(\mathrm{P}) - u_{-}(\mathrm{P}'),$$

and allow P and P′ to approach Q by any paths,* the tangents to which at Q do not lie in the plane of the aperture. Then the conditions of continuity are

$$\left.\begin{array}{l} \lim \{u_{+}(\mathrm{P}) - u_{-}(\mathrm{P}')\} = u_1'(\mathrm{Q}) \cos \kappa \mathrm{c} t + u_2'(\mathrm{Q}) \sin \kappa \mathrm{c} t\,, \\ \qquad . \ . \ . \ . \\ \lim \{f_{+}(\mathrm{P}) - f_{-}(\mathrm{P}')\} = f_1'(\mathrm{Q}) \cos \kappa \mathrm{c} t + f_2'(\mathrm{Q}) \sin \kappa \mathrm{c} t\,, \\ \qquad . \ . \ . \ . \end{array}\right\} \quad . \ . \quad (71).$$

The functions u_{-}, . . . satisfy the general equations of § 9 at all points on the nearer side ($z < z'$), and are free from singularities in this region; these conditions, with the conditions of continuity (71), suffice to determine the functions in question, in terms of the functions $\bar{u}_1$, . . . introduced in § 25. One such determination will be worked out presently; here it is important to observe that it is effectively unique. The conditions (71) require, in fact, that the discontinuities of u_{-}, . . . should be arranged so as to cancel exactly those of u_{+}, . . . Now let us suppose that two sets of functions u_{-}, . . ., and $u_{-} + \Delta u_{-}$, . . . have been found, both of which obey all the conditions imposed upon the functions u_{-}, . . . ; their differences Δu_{-}, . . . have no discontinuities at the aperture, and no singularities on the nearer side ($z < z'$); thus the disturbance represented by Δu_{-}, . . . belongs to the disturbance

* The path of P lies, of course, entirely in the region $z > z'$, and that of P′ in the region $z < z'$.

A, and not to the disturbance B. The relation between A and B is similar to that between the "complementary function" and a "particular integral" of a linear differential equation, with a right-hand member; the difference between two particular integrals is part of the complementary function.*

Perhaps the simplest way of building up the functions u_-, . . . is to act upon a hint, derived from a study of HELMHOLTZ's theory of acoustical resonators.† We may, in fact, attempt to satisfy the conditions, by regarding the disturbance B as consisting of a system of standing waves; and we find that the method thus suggested is successful. We shall proceed on the assumption that the displacements, represented by u_-, . . . , constitute a system of standing waves.]

28. Having regard to the proposed plan of passing to a limit when the point (x, y, z) is brought to coincidence with (x', y', z'), we see that especial importance attaches to the limiting values of such expressions as

$$\cos \kappa (\mathrm{C}t - r) - \kappa r \sin \kappa (\mathrm{C}t - r),$$

which, in § 26, have uniformly been placed in { }; and it appears that all these limiting values are numerical multiples of $\cos \kappa \mathrm{C}t$. This remark indicates that the discontinuities of the terms $\bar{u}_2 \sin \kappa \mathrm{C}t$, . . . are independent of those of the terms $\bar{u}_1 \cos \kappa \mathrm{C}t$, . . . Again, when the expressions are, as above, replaced by their limiting values, it appears that every term in u_+, . . . might be interpreted as a differential coefficient, either of the first order, or of the second order, of the potential of a distribution of surface density on the area within the aperture. Now it is known‡ that, the charged surface coinciding with the plane $z =$ const., first differential coefficients of the potential with respect to x and y are continuous in crossing the surface, and the first differential coefficient with respect to z has a definite discontinuity; further, it is known that all second differential coefficients are continuous in crossing the surface, except the two that are formed by differentiating with respect to z once and either x or y once. These considerations guide us to a proper choice of displacements in the standing waves represented by u_-, . . . ; for example, in the second line of the expression for $4\pi u_+$, the factors

$$-\frac{\partial^2 r^{-1}}{\partial x^2} \kappa^{-2} \bar{g}_1$$

must be retained, and multiplied by a function of r and t, of which the limit at $r = 0$ is $\cos \kappa \mathrm{C}t$. But we should not arrive at a proper choice by replacing the expressions in { } by their limiting values; for the system of displacements thus arrived at would not satisfy the fundamental differential equations. This consideration suggests that

* FORSYTH, 'Treatise on Differential Equations,' ch. 3.

† See the memoir already quoted, particularly equation (29c), HELMHOLTZ, 'Wiss. Abh.,' vol. 1, p. 377.

‡ *Cf.* POINCARÉ, 'Théorie du Potentiel Newtonien,' ch. 3.

the functions of r and t, that are to replace the expressions in { }, being factors in certain particular solutions of the differential equations, and even functions of t, could be arrived at by changing, in { }, t into $-t$, and taking half the sum; for example we should replace

$$\cos\kappa(\mathrm{c}t - r) - \kappa r\sin\kappa(\mathrm{c}t - r)$$

by

$$\tfrac{1}{2}\{\cos\kappa(\mathrm{c}t - r) - \kappa r\sin\kappa(\mathrm{c}t - r) + \cos\kappa(\mathrm{c}t + r) + \kappa r\sin\kappa(\mathrm{c}t + r)\},$$

or by

$$\cos\kappa\,\mathrm{c}t\,(\cos\kappa r + \kappa r\sin\kappa r).$$

This comes to the same thing as picking out from each expression in { } the terms that contain $\cos\kappa\,\mathrm{c}t$, and rejecting those that contain $\sin\kappa\,\mathrm{c}t$.

We accordingly take for u_-, . . . forms given by such equations as

$$\begin{aligned}4\pi u_- = \cos\kappa\,\mathrm{c}t\iint dx'dy'\Big[&-\frac{\partial r^{-1}}{\partial z}\bar{u}_1\{\cos\kappa r + \kappa r\sin\kappa r\}\\ &-\frac{\partial^2 r^{-1}}{\partial x^2}\kappa^{-2}\bar{g}_1\{\cos\kappa r + \kappa r\sin\kappa r\}\\ &-\tfrac{1}{3}\frac{\partial^2 r^{-1}}{\partial x\partial y}\kappa^{-2}\bar{f}_1\{(3 - \kappa^2 r^2)\cos\kappa r + 3\kappa r\sin\kappa r\}\\ &-\frac{(y-y')^2 + (z-z')^2}{r^3}\bar{g}_1\{\cos\kappa r\}\Big], \quad . \quad . \quad . \quad . \quad (72),\end{aligned}$$

$$\begin{aligned}4\pi w_- = \cos\kappa\,\mathrm{c}t\iint dx'dy'\Big[&\left(\bar{u}_1\frac{\partial r^{-1}}{\partial x} + \bar{v}_1\frac{\partial r^{-1}}{\partial y}\right)\{\cos\kappa r + \kappa r\sin\kappa r\}\\ &-\tfrac{1}{3}\kappa^{-2}\left(\bar{g}_1\frac{\partial^2 r^{-1}}{\partial x\partial z} - \bar{f}_1\frac{\partial^2 r^{-1}}{\partial y\partial z}\right)\{(3 - \kappa^2 r^2)\cos\kappa r + 3\kappa r\sin\kappa r\}\Big]. \quad . \quad (73).\end{aligned}$$

29. According to explanations already given, we shall have

$$\begin{aligned}4\pi\lim\{u_+(\mathrm{P}) - u_-(\mathrm{P}')\} &= \cos\kappa\,\mathrm{c}t\iint dx'dy'\,\bar{u}_1\left[\left(-\frac{\partial r^{-1}}{\partial z}\right)_+ + \left(\frac{\partial r^{-1}}{\partial z}\right)_-\right]\\ &= 4\pi\bar{u}_1(\mathrm{Q})\cos\kappa\,\mathrm{c}t\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).\end{aligned}$$

Again

$4\pi\lim\{w_+(\mathrm{P}) - w_-(\mathrm{P}')\}$ is the limit of

$$\begin{aligned}-\frac{1}{3\kappa^2}\iint dx'dy'\Big[&\left\{\bar{g}_1\left(\frac{\partial^2 r^{-1}}{\partial x\partial z}\right)_+ - \bar{f}_1\left(\frac{\partial^2 r^{-1}}{\partial y\partial z}\right)_+\right\}\{(3-\kappa^2 r^2)\cos\kappa(\mathrm{c}t - r) - 3\kappa r\sin\kappa(\mathrm{c}t - r)\}\\ &-\left\{\bar{g}_1\left(\frac{\partial^2 r^{-1}}{\partial x\partial z}\right)_- - \bar{f}_1\left(\frac{\partial^2 r^{-1}}{\partial y\partial z}\right)_-\right\}\{(3 - \kappa^2 r^2)\cos\kappa r + 3\kappa r\sin\kappa r\}\cos\kappa\,\mathrm{c}t\Big].\end{aligned}$$

We write $\dfrac{\partial^2 r^{-1}}{\partial x \partial z} = -\dfrac{\partial}{\partial x'}\dfrac{\partial r^{-1}}{\partial z}$, and similarly for y, and integrate by parts; the result contains a line integral round the boundary of the aperture and a surface integral, and the former contributes nothing to the limit we are seeking, unless the point Q is indefinitely close to the boundary of the aperture. Thus the limit we are seeking is that of

$$
\begin{aligned}
-\frac{1}{3\kappa^2}\iint dx'dy' \Bigg[& \left(\frac{\partial r^{-1}}{\partial z}\right)_+ \left(\frac{\partial \bar{g}_1}{\partial x'} - \frac{\partial \bar{f}_1}{\partial y'}\right)\{(3-\kappa^2 r^2)\cos\kappa(\mathrm{c}t - r) - 3\kappa r \sin\kappa(\mathrm{c}t - r)\} \\
& - \left(\frac{\partial r^{-1}}{\partial z}\right)_- \left(\frac{\partial \bar{g}_1}{\partial x'} - \frac{\partial \bar{f}_1}{\partial y'}\right)\{(3-\kappa^2 r^2)\cos\kappa r + 3\kappa r\sin\kappa r\}\cos\kappa \mathrm{c}t \\
& - \left(\frac{\partial r^{-1}}{\partial z}\right)_+ \left(\bar{g}_1\frac{x'-x}{r} - \bar{f}_1\frac{y'-y}{r}\right)\{-\kappa^3 r^2\sin\kappa(\mathrm{c}t - r) + \kappa^2 r\cos\kappa(\mathrm{c}t - r)\} \\
& + \left(\frac{\partial r^{-1}}{\partial z}\right)_- \left(\bar{g}_1\frac{x'-x}{r} - \bar{f}_1\frac{y'-y}{r}\right)\{\kappa^3 r^2\sin\kappa r + \kappa^2 r\cos\kappa r\}\cos\kappa\,\mathrm{c}t\,;
\end{aligned}
$$

in this expression, the two last lines vanish in the limit, and the others yield the value at Q of

$$4\pi\kappa^{-2}\left(\frac{\partial \bar{g}_1}{\partial x'} - \frac{\partial \bar{f}_1}{\partial y'}\right)\cos\kappa\,\mathrm{c}t\,.$$

Thus we have

$$4\pi \lim\{w_+(\mathrm{P}) - w_-(\mathrm{P}')\} = 4\pi\,\bar{w}_1(\mathrm{Q})\cos\kappa\,\mathrm{c}t\,; \quad . \quad . \quad . \quad . \quad (75),$$

and it follows that the conditions of continuity are satisfied by putting

$$\left.\begin{array}{l} \bar{u}_1 = u_1',\ \bar{u}_2 = u_2', \\ \quad \cdot\ \cdot\ \cdot\ \cdot \\ \bar{f}_1 = f_1',\ \bar{f}_2 = f_2', \\ \quad \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

By this result, the transmitted waves on the further side are connected with the waves on the nearer side; and it is manifest that the result would not be disturbed if the surface were not plane, provided that all the linear dimensions of the aperture are small compared with the radii of curvature of the surface.

[30. (*Partly re-written March,* 1901.)—We return now to the general question propounded in § 24, and seek to estimate the character of the answer that we have found. In § 25 the question is made more precise by showing that the distribution over the aperture of sources to which the transmitted radiation can be regarded as due, depends upon the values, at certain times and at points within the aperture, of a certain system of magnetic and electric displacements; these values are the quantities denoted by $\bar{u}_1$, . . . In § 26 the transmitted radiation is expressed in terms of these

quantities; the functions by which it is expressed are those denoted by u_+, . . . These functions are defined for points on the further side by expressions which are not continuous up to and across the aperture; but they represent the transmitted radiation at any finite distance from the aperture. The actual disturbance is continuous up to and across the aperture. We seek accordingly to represent the disturbance on the nearer side by means of functions which are defined for the nearer side, but are not continuous up to and across the aperture, the discontinuities being so arranged that the displacements on the nearer side shall be continuous with those on the further side. In § 27 we separate the expressions of these functions into two parts, thus regarding the disturbance on the nearer side as consisting of two superposed disturbances, there called A and B. The functions representing the disturbance A are continuous up to and across the aperture; those representing the disturbance B are not; but their discontinuities cancel exactly those of the functions u_+, . . . The determination of B is in a certain sense unique. In § 28 we verify the supposition that B may be regarded as a system of standing waves, by actually determining, in accordance with this supposition, the functions involved in B, viz., u_-, . . . , in terms of the functions $\bar{u}_1$, . . . In § 29 we show that the displacements, of which the functions $\bar{u}_1$, . . . are the values, at certain times, and at points within the aperture, are the displacements belonging to the disturbance A. The disturbance B and the transmitted radiation are thus determined in terms of A, and the general question of § 24 is reduced to the determination of A.

The components u', . . . f', . . . of A are subject to the following conditions :—

(1.) On the nearer side they satisfy the equations of § 9 everywhere, except possibly at certain singular points.

(2.) These singular points are the actual sources of the incident radiation.

(3.) The functions u', . . . f', . . . are continuous up to, and across, the aperture.

(4.) At all points of the screen, not points of the aperture, they satisfy certain boundary conditions.

The boundary conditions depend, to some extent, on the material of the screen; and they will usually take the form that some components of electric or magnetic displacement vanish. The components, affected by the condition, are those of the displacement on the nearer side compounded of A and B, *i.e.*, such quantities as $u' + u_-$; but, as B falls off rapidly, with increasing distance from the aperture, it will generally be sufficient to impose the boundary condition on the components of A only.

We may now give the following interpretation of the analysis :—The disturbance B, consisting of a system of standing waves, which are important in the neighbourhood of the aperture only, can be described as the "effect of the aperture." The disturbance A can be described as "the incident radiation, as modified by the action of the screen." The result of § 29 can be stated in the form :—The transmitted radiation is to be calculated from the incident radiation, as modified by the action

of the screen, in the same way as if this radiation passed freely through the aperture.

This result differs from the ordinary optical rule, that the transmitted radiation is to be calculated from the incident radiation, unmodified, as if this radiation passed freely through the aperture. In the application of this rule no attention is paid to the boundary conditions at the screen. If we could assume that the disturbance at points of the *aperture* when the incident radiation is modified by the action of the screen, differs very little from the unmodified incident disturbance, then the result and the optical rule would be in practical agreement. The success of the optical rule seems to show that the modification of the incident radiation by the screen is unimportant, at points within the aperture, when the wave-length is short.]

The result obtained may be applied with greater certainty when the disturbance on the nearer side of the screen has been calculated in accordance with a known boundary condition, holding over all the unperforated portion. This is the case when, instead of an incident train of waves, we have, on the nearer side, standing vibrations, for which the boundary condition is satisfied. In such a case, the values to be assigned to the components

$$u_1' \cos \kappa ct + u_2' \sin \kappa ct\,,$$
$$\cdot\ ,\ \cdot\ \cdot$$

of the disturbance A, . . . at points of the aperture, are the values that u, . . . would have if the screen were unperforated. This remark applies to the problem of the communication of vibrations from a condensing system to the surrounding æther. We shall now take up this problem, having regard especially to the example of concentric spherical conducting surfaces, with a very thin dielectric plate between them, the outer surface being perforated by a small circular aperture.

Electrical Oscillations between Concentric Spheres.

31. It has been pointed out by LARMOR* that the most important modes of electrical oscillation in a condenser, with a thin dielectric plate, are those in which the charge surges over the conducting surfaces, the lines of electric force being always normal to these surfaces, and the lines of magnetic force tangential to them. In a condenser with concentric spherical conducting surfaces such modes of oscillation exist, whatever the thickness of the dielectric plate may be; and the analysis requisite for dealing with them has been developed by LAMB.† The required solutions of the

* 'London Math. Soc. Proc.,' vol. 26 (1895), p. 119.

† 'London Math. Soc. Proc.,' vol. 13 (1882), p. 51; or 'Hydrodynamics,' pp. 555, *et seq.* The notation here used will be that of the 'Hydrodynamics.' It is worth while to recall some of the properties of the functions ψ, defined in equations (77): they satisfy the equations

fundamental equations are included among those obtained in § 10, by proper choice of the function ϕ_0.

Taking the centre of the spheres as origin, let ω_n denote any spherical solid harmonic, and write

$$\left.\begin{aligned} \psi_n(\zeta) &= (-)^n \left(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\right)^n \frac{\sin\zeta}{\zeta}, \\ \Psi_n(\zeta) &= (-)^n \left(\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\right)^n \frac{\cos\zeta}{\zeta}, \\ \mathrm{A}\psi_n(\zeta) + \mathrm{B}\Psi_n(\zeta) &= \chi_n(\zeta), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77),$$

A and B being constants. Then, in the modes of oscillation under discussion, we have

$$(u, v, w) = \cos\kappa\mathrm{c}t \,.\, \chi_n(\kappa r) \,.\, \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\omega_n, \quad . \quad (78),$$

$$f = -\frac{1}{2n+1}\cos\kappa\mathrm{c}t\left\{(n+1)\chi_{n-1}(\kappa r)\frac{\partial\omega_n}{\partial x} - n\chi_{n+1}(\kappa r)\kappa^2 r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\}, \quad (79),$$

with similar formulæ for g and h; the value of κ and the ratio A : B are to be found from the boundary conditions, which are that the electric force is normal to the bounding surfaces.

The vector (f, g, h) has a radial component of amount

$$-n(n+1)\cos\kappa\mathrm{c}t\, r^{-1}\chi_n(\kappa r)\,\omega_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80).$$

If then we form a new vector from (f, g, h), by resolving this radial component in the directions of the axes, and subtracting its resolved parts from f, g, h, the new vector will have the same tangential components at any sphere as (f, g, h) has; the x-component of the new vector is

$$\frac{n(n+1)}{2n+1}\cos\kappa\mathrm{c}t\left[\left\{\chi_n(\kappa r) - \frac{1}{n}\chi_{n-1}(\kappa r)\right\}\frac{\partial\omega_n}{\partial x} - \left\{\chi_n(\kappa r) - \frac{\kappa^2 r^2}{n+1}\chi_{n+1}(\kappa r)\right\} r^{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right],$$

$$\left(\frac{d^2}{d\zeta^2} + 1 - \frac{n(n+1)}{\zeta^2}\right)\{\zeta^{n+1}\psi_n(\zeta)\} = 0,$$

$$\psi_n'(\zeta) = -\zeta\psi_{n+1}(\zeta),$$

$$(2n+1)\psi_n(\zeta) = \psi_{n-1}(\zeta) + \zeta^2\psi_{n+1}(\zeta),$$

and these equations are also satisfied by Ψ, and by χ, provided the constants A and B are supposed not to change with n. Reductions made by using these equations will be introduced without remark.

or, what comes to the same thing, the new vector is

$$-\frac{\cos \kappa Ct}{(2n+1)r^n}\frac{\partial}{\partial r}\{r^{n+1}\chi_n(\kappa r)\}\left[(n+1)\left(\frac{\partial \omega_n}{\partial x}, \frac{\partial \omega_n}{\partial y}, \frac{\partial \omega_n}{\partial z}\right) + nr^{2n+1}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\omega_n}{r^{2n+1}}\right)\right];$$

and the condition that (f, g, h) should be radial, at the conducting surfaces $r = r_0$ and $r = r_1$, is that

$$\frac{\partial}{\partial r}\{r^{n+1}\chi_n(nr)\} = 0 \quad \ldots \ldots \ldots \ldots (81)$$

at $r = r_0$ and $r = r_1$. These two equations determine the ratio A : B and the value of κ.

When the conducting surfaces are very near together, we have approximately

$$\frac{\partial^2}{\partial r^2}\{r^{n+1}\chi_n(\kappa r)\} = 0,$$

or, in virtue of the differential equation, satisfied by χ_n,

$$\kappa^2 r_0^2 = n(n+1), \quad \ldots \ldots \ldots \ldots (82),$$

a result otherwise obtained by LARMOR.* The ratio A : B is determined by the equation

$$\mathrm{A}\frac{\partial}{\partial r}\{r^{n+1}\psi_n(\kappa r)\} + \mathrm{B}\frac{\partial}{\partial r}\{r^{n+1}\Psi_n(\kappa r)\} = 0, \quad \ldots \ldots (83),$$

which holds for $r = r_0$.

32. We consider, in particular, modes of oscillation, for which the axis z is an axis of symmetry. We take

$$z/r = \mu, \quad \omega_n = r^n \mathrm{P}_n(\mu), \quad \ldots \ldots \ldots \ldots (84),$$

where $\mathrm{P}_n(\mu)$ is the zonal surface harmonic (LEGENDRE'S coefficient) of order n. We find

$$\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\omega_n = r^{2n+1}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\{r^{-(n+1)}\mathrm{P}_n(\mu)\},$$

$$= \frac{(-)^n}{n!}r^{2n+1}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\frac{\partial^n r^{-1}}{\partial z^n}$$

$$= \frac{(-)^n}{n!}r^{2n+1}\left\{y\frac{\partial^{n+1}r^{-1}}{\partial z^{n+1}} + yz\frac{\partial^n r^{-3}}{\partial z^n}\right\}.$$

Now it is easy to show† that

$$\frac{(-)^n}{n!}\frac{\partial^n r^{-3}}{\partial z^n} = \frac{1}{r^{n+3}}\left\{(n+1)\mathrm{P}_n + \mu\frac{d\mathrm{P}_n}{d\mu}\right\}, \quad \ldots \ldots \ldots (85),$$

* 'London Math. Soc. Proc.,' vol. 26.

† One way of doing this is as follows :—

$$\frac{(-)^n}{n!}\frac{\partial^n r^{-3}}{\partial z^n} = \text{the coefficient of } h^n \text{ in the expansion of } \{x^2 + y^2 + (z-h)^2\}^{-\frac{3}{2}};$$

also $$\{x^2 + y^2 + (z-h)^2\}^{-\frac{3}{2}} = r^{-3}\left(1 - 2\mu\frac{h}{r} + \frac{h^2}{r^2}\right)^{-\frac{3}{2}},$$

and we deduce without difficulty that, when $\omega_n = r^n \mathrm{P}_n(\mu)$,

$$(u, v, w) = \cos\kappa \mathrm{C}t\, r^{n-1}\chi_n(\kappa r)\frac{d\mathrm{P}_n}{d\mu}\,.\,(y, -x, 0) \quad . \quad . \quad . \quad . \quad . \quad (86).$$

Forms for f, g, h could be obtained by a similar analysis; but the values that they would take near the boundaries r_0 and r_1 can be written down immediately from the formula (80), viz.: we have, at the boundaries,

$$(f, g, h) = -n(n+1)\cos\kappa \mathrm{C}t\, r^{n-2}\chi_n(\kappa r)\,\mathrm{P}_n(\mu)\,.\,(x, y, z) \quad . \quad . \quad . \quad (87).$$

When r_0 and r_1 are nearly equal, this formula holds approximately for all values of r that are involved.

33. The kinetic energy of the mode of oscillation here discussed can be calculated without difficulty. We require the value of

$$\frac{1}{8\pi}\iiint(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,d\tau,$$

$$= \frac{\kappa^2\mathrm{C}^2}{8\pi}\sin^2\kappa \mathrm{C}t\int_{r_1}^{r_0} dr\int_{-1}^{1} d\mu\, r^{2n+2}\{\chi_n(\kappa r)\}^2\, 2\pi(1-\mu^2)\left(\frac{d\mathrm{P}_n}{d\mu}\right)^2$$

$$= \tfrac{1}{2}\frac{n(n+1)}{2n+1}\kappa^2\mathrm{C}^2\sin^2\kappa \mathrm{C}t\int_{r_1}^{r_0}\{r^{n+1}\chi_n(\kappa r)\}^2\,dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88).$$

To calculate the integral in this expression, we have recourse to the differential equation

$$\left\{\frac{d^2}{dr^2} + \kappa^2 - \frac{n(n+1)}{r^2}\right\}\{r^{n+1}\chi_n(\kappa r)\} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89),$$

and the condition that

$$\frac{d}{dr}\{r^{n+1}\chi_n(\kappa r)\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

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and

$$(1 - 2\mu k + k^2)^{-\frac{3}{2}} = \frac{1}{(1 - 2\mu k + k^2)^{\frac{1}{2}}} + (2\mu - k)\frac{k}{(1 - 2\mu k + k^2)^{\frac{3}{2}}}$$

$$= \left\{1 + (2\mu - k)\frac{d}{d\mu}\right\}\frac{1}{(1 - 2\mu k + k^2)^{\frac{1}{2}}}$$

$$= \sum_0^\infty k^n\mathrm{P}_n + (2\mu - k)\sum_1^\infty k^n\frac{d\mathrm{P}_n}{d\mu}$$

$$= 1 + \sum_1^\infty k^n\left(\mathrm{P}_n + 2\mu\frac{d\mathrm{P}_n}{d\mu} - \frac{d\mathrm{P}_{n-1}}{d\mu}\right);$$

and we also have the known relation

$$\frac{d\mathrm{P}_{n-1}}{d\mu} = \mu\frac{d\mathrm{P}_n}{d\mu} - n\mathrm{P}_n.$$

at both boundaries. Taking κ and κ' to be two possible values of κ, and λ and λ' the corresponding forms of $r^{n+1}\chi_n(\kappa r)$, we find

$$(\kappa^2 - \kappa'^2)\int_{r_1}^{r_0} \lambda\lambda' dr = \left[\lambda \frac{d\lambda'}{dr} - \lambda' \frac{d\lambda}{dr}\right]_{r_1}^{r_0};$$

and then, by the usual limiting process,*

$$\int_{r_1}^{r_0} \lambda^2 dr = \frac{1}{2\kappa}\left[\frac{d\lambda}{d\kappa}\frac{d\lambda}{dr} - \lambda \frac{d}{d\kappa}\frac{d\lambda}{dr}\right]_{r_1}^{r_0},$$

or

$$\int_{r_1}^{r_0} r^{2n+2}\{\chi_n(\kappa r)\}^2 dr = \tfrac{1}{2}\left[\left\{1 - \frac{n(n+1)}{\kappa^2 r_0^2}\right\} r_0^{2n+3}\{\chi_n(\kappa r_0)\}^2 - \left\{1 - \frac{n(n+1)}{\kappa^2 r_1^2}\right\} r_1^{2n+3}\{\chi_n(\kappa r_1)\}^2\right] \quad . \quad . \quad (90).$$

When r_1 is very nearly equal to r_0, this becomes, approximately,

$$\int_{r_1}^{r_0} r^{2n+2}\{\chi_n(\kappa r)\}^2 dr = (r_0 - r_1)\,\kappa^{-2}\, n(n+1)\, r_0^{2n}\{\chi_n(\kappa r_0)\}^2 \quad . \quad . \quad . \quad (91).$$

After the appropriate expression (90) or (91) has been substituted in the expression (88), the total energy, kinetic and potential, is to be obtained by suppressing the factor $\sin^2 \kappa Ct$. Thus, when r_1 is very nearly equal to r_0, the total energy of the oscillating charges on the condenser is

$$\tfrac{1}{2}(r_0 - r_1)\frac{n^2(n+1)^2}{2n+1} C^2 r_0^{2n}\{\chi_n(\kappa r_0)\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92).$$

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34. When there is no aperture in the outer conductor, the oscillations considered in §§ 31–33 would, in the absence of dissipation due to imperfect conduction, continue indefinitely; but they would not produce any effect in any external electrical system. When there is an aperture, we may take account of it by supposing that the displacements (magnetic and electric), in space external to the condenser, are of the character corresponding to waves diverging from sources distributed over the aperture only, and that the displacements within the dielectric plate of the condenser differ from those, which would be found in a normal mode of oscillation, by the superposition of displacements corresponding to a system of standing waves, which are insensible except in the immediate neighbourhood of the aperture. We may suppose the

* *Cf.* Lord RAYLEIGH, 'Theory of Sound,' vol. 1, p. 325.

diverging waves, and the standing waves, thus introduced, to have the same period as the oscillations in the normal mode; and then the displacements corresponding to them will be determined, as in § 29, by the conditions that the electric and magnetic displacements must be continuous across the aperture. As we are concerned rather with the general features of the transmission of disturbances across an aperture than with special details, we may select any normal mode of oscillation for examination. We shall suppose the aperture to be a circle of radius a, small compared with the distance $r_0 - r_1$ between the conducting surfaces; and we shall consider particularly modes of oscillation symmetrical about the axis of the circle, taken as axis of z.

35. For the calculation of the energy dissipated we shall take, in the notation of § 25,

$$(\bar{u}_1, \bar{v}_1, \bar{w}_1) = \frac{n(n+1)}{2} r_0^{n-1} \chi_n(\kappa r_0) \,.\, (y', -x', 0)$$

$$= \xi \,.\, (y', -x', 0), \text{ say, } \quad \ldots \ldots \ldots \ldots \quad (93),$$

where $\frac{n(n+1)}{2}$ has been written for $\left(\frac{d\mathrm{P}_n}{d\mu}\right)_{\mu=1}$; and we shall take

$$(\bar{f}_1, \bar{g}_1, \bar{h}_1) = -n(n+1) r_0^{n-2} \chi_n(\kappa r_0) \,.\, (x', y', r_0)$$

$$= \eta \,.\, (x', y', r_0), \text{ say, } \quad \ldots \ldots \ldots \ldots \quad (94),$$

these being with sufficient approximation the values obtained in § 32; the normal mode of oscillation here discussed will accordingly be one for which the axis z is an axis of symmetry.

We have now to find the most important terms in $4\pi u_+, \ldots$ at a great distance from the aperture, the values above written being substituted for $\bar{u}_1, \ldots$. We shall take R for the distance of the point (x, y, z) from the centre of the aperture, and, whenever we wish to do so, we shall expand r in the form

$$r = \mathrm{R} + x' \frac{-x}{\mathrm{R}} + y' \frac{-y}{\mathrm{R}} + \ldots .$$

Now, taking $4\pi u_+$, the first line of equation (63) is

$$\xi \iint \frac{z - r_0}{\mathrm{R}^3} y' \left\{1 + 3\frac{xx' + yy'}{\mathrm{R}^2}\right\} [\sin \kappa (ct - \mathrm{R})(-\kappa \mathrm{R})$$
$$+ \cos \kappa (ct - \mathrm{R}) \{1 - \kappa^2 (xx' + yy')\}] \, dx' dy'$$

approximately, where the integration is taken over the area within the circle $x'^2 + y'^2 = a^2$, and terms of order higher than x'^2 have been neglected. The most important part of this is

$$-\frac{\pi a^4}{4} \xi \frac{z - r_0}{\mathrm{R}} \frac{\kappa^2 y}{\mathrm{R}^3} \frac{\cos \kappa (ct - \mathrm{R})}{\mathrm{R}}$$

which is of order $R^{-1} \cos \kappa (ct - R)$, the right order when R is great. We treat every term exactly as this term has been treated. The second line of $4\pi u_+$ is of order $R^{-2} \cos \kappa (ct - R)$, and is to be rejected; the third line gives

$$\frac{\pi a^4}{4} \eta \kappa \frac{x^2 y}{R^3} \frac{\sin \kappa (ct - R)}{R},$$

and the fourth line gives

$$\frac{\pi a^4}{4} \eta \kappa \frac{y}{R} \frac{y^2 + (z - r_0)^2}{R^2} \frac{\sin \kappa (ct - R)}{R}.$$

Hence the most important part of u_+ at a distance from the aperture is

$$-\frac{a^4}{16} \left\{ \xi \frac{\kappa^2 y (z - r_0)}{R^3} \cos \kappa (ct - R) - \eta \frac{\kappa y}{R^2} \sin \kappa (ct - R) \right\}.$$

In like manner the most important part of v_+ is

$$\frac{a^4}{16} \left\{ \xi \frac{\kappa^2 x (z - r_0)}{R^3} \cos \kappa (ct - R) - \eta \frac{\kappa x}{R^2} \sin \kappa (ct - R) \right\},$$

and all the terms of w_+ are of a higher order than these.

The results just obtained can be written

$$(u_+, v_+, w_+) = \frac{\kappa a^4}{16R} \left\{ \kappa \xi \frac{z - r_0}{R} \cos \kappa (ct - R) - \eta \sin \kappa (ct - R) \right\} . \left(-\frac{y}{R}, \frac{x}{R}, 0 \right) \quad (95).$$

By a similar process it may be shown that the approximate forms for f_+, g_+, h_+ at a great distance from the aperture are given by the equation

$$(f_+, g_+, h_+) = \frac{\kappa^2 a^4}{16R} \left\{ \kappa \xi \frac{z - r_0}{R} \sin \kappa (ct - R) + \eta \cos \kappa (ct - R) \right\} . \left(-\frac{x (z - r_0)}{R^2}, -\frac{y (z - r_0)}{R^2}, \frac{x^2 + y^2}{R^2} \right) \quad . \quad . \quad . \quad (96).$$

We observe that the value of the magnetic force $(\dot{u}_+, \dot{v}_+, \dot{w}_+)$ at a great distance is

$$(\dot{u}_+, \dot{v}_+, \dot{w}_+) = c \frac{\kappa^2 a^4}{16R} \left\{ \kappa \xi \frac{z - r_0}{R} \sin \kappa (ct - R) + \eta \cos \kappa (ct - R) \right\} . \left(\frac{y}{R}, -\frac{x}{R}, 0 \right) \quad . \quad . \quad . \quad (97),$$

so that the factors, that contain t, are the same for the electric and magnetic forces at a great distance from the aperture.

36. Now let l, m, n be the direction cosines of the normal to a closed surface S drawn in a specified direction (inwards or outwards); then the rate at which energy

is transmitted across the surface, in the sense of the normal (l, m, n), is, by POYNTING'S Theorem,

$$\frac{\mathrm{C}^2}{4\pi}\iint\{l(g\dot{w}-h\dot{v})+m(h\dot{u}-f\dot{w})+n(f\dot{v}-g\dot{u})\}\,d\mathrm{S} \quad . \quad . \quad . \quad (98).$$

We can, therefore, find the rate of dissipation of energy from the condenser, by forming this integral, for a sphere of large radius, R, and for the functions $\dot{u}_+, f_+, \ldots$, the normal being drawn outwards.

If we write, for brevity,

$$x = l\mathrm{R}, \qquad y = m\mathrm{R}, \qquad z - r_0 = n\mathrm{R}, \qquad \kappa(\mathrm{C}t - \mathrm{R}) = \psi,$$

the rate of dissipation is

$$\frac{\mathrm{C}^3}{4\pi}\left(\frac{a}{2}\right)^8\frac{\kappa^4}{\mathrm{R}^2}\iint(\kappa\xi n\sin\psi+\eta\cos\psi)^2\{l^2(l^2+m^2)+m^2(l^2+m^2)+n^2(l^2+m^2)\}\,d\mathrm{S} \quad . \quad . \quad . \quad (99),$$

taken over the sphere; and the amount of energy transferred across the sphere in a period, $2\pi/(\kappa\,\mathrm{C})$, is given by the expression*

$$\frac{\mathrm{C}^3}{4\pi}\left(\frac{a}{2}\right)^8\frac{\kappa^4}{\mathrm{R}^2}\frac{\pi}{\kappa\mathrm{C}}\iint(\kappa^2\xi^2n^2+\eta^2)(l^2+m^2)\,d\mathrm{S}, \quad . \quad . \quad . \quad . \quad . \quad (100),$$

or it is

$$\frac{\mathrm{C}^2}{4}\left(\frac{a}{2}\right)^8 2\pi\int_0^\pi(\kappa^5\xi^2\cos^2\theta+\kappa^3\eta^2)\sin^3\theta\,d\theta\,,$$

which is

$$2\pi\mathrm{C}^2\left(\frac{a}{2}\right)^8\left(\frac{\kappa^5\xi^2}{15}+\frac{\kappa^3\eta^2}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101).$$

Restoring the values of ξ and η, this expression becomes

$$\tfrac{2}{3}\pi\mathrm{C}^2\left(\frac{a}{2}\right)^8\kappa^3n^2(n+1)^2r_0^{2n-4}\{\chi_n(\kappa r_0)\}^2(1+\tfrac{1}{20}\kappa^2r_0^2), \quad . \quad . \quad . \quad . \quad (102),$$

where n is now the order of the spherical harmonic involved in the oscillations.

When $r_0 - r_1$ is small compared with r_0, the fraction of the total energy, which is dissipated in one period, is obtained by dividing the expression here written by the expression (92); it is

$$\tfrac{4}{3}\pi(2n+1)(\kappa r_0)^3(1+\tfrac{1}{20}\kappa^2r_0^2)\frac{a^8}{2^8r_0^7(r_0-r_1)}; \quad . \quad . \quad . \quad . \quad . \quad (103);$$

* The expression shows that equal amounts of energy are transmitted across the hemisphere in front of the aperture and that behind. This arises from the circumstance that the wave-length is of the same order of magnitude as the radius of the outer conducting surface, so that the waves bend completely round that surface.

and, at any rate when n is not too great, this is a very small fraction, if $r_0 - r_1$ is large compared with a, as has been supposed throughout the investigation.

37. The form of the result shows that the number of vibrations of the higher modes, that are executed before the disturbance sinks into insignificance, is much less than that of the lower ones. The occurrence of $r_0 - r_1$ in the denominator of (103) suggests that the principal factor in securing permanence of the vibrations is not the capacity of the vibrating system, but the screening action of the external conductor. The latter point might be illustrated further by considering the example of a spherical condenser, in the case where r_1 is small compared with r_0. The boundary condition at the inner surface can be satisfied approximately by putting, in equation (83), $B = 0$; and the frequency is determined by the equation

$$\frac{d}{dr}\{r^{n+1}\psi_n(\kappa r)\} = 0 \qquad (104),$$

when $r = r_0$. The total energy, for a mode of oscillation given by $\omega_n = r^n P_n(\mu)$, is

$$\tfrac{1}{4}\frac{n(n+1)}{2n+1}\kappa^2 C^2 \left\{1 - \frac{n(n+1)}{\kappa^2 r_0^2}\right\} r_0^{2n+3}\{\psi_n(\kappa r_0)\}^2, \qquad (105),$$

and the energy dissipated in a period is

$$\tfrac{2}{3}\pi C^2 \left(\frac{a}{2}\right)^8 \kappa^3 n^2 (n+1)^2 r_0^{2n-4}\{\psi_n(\kappa r_0)\}^2 \left(1 + \tfrac{1}{20}\kappa^2 r_0^2\right); \qquad (106),$$

and it is clear that the fraction of the total energy dissipated in one period is of the same order of magnitude as before, except that $(a/r_0)^8$ is substituted for the product of $(a/r_0)^7$ and $\{a/(r_0 - r_1)\}$. For the mode of least frequency $n = 1$, and we have $\kappa r_0 = 2\cdot75$ nearly,* instead of $1\cdot41$, its value when r_1 is nearly equal to r_0; and thus the fraction in question becomes approximately

$$82\pi\left(\frac{a}{2r_0}\right)^8, \quad \text{instead of being approximately} \quad 12\pi\left(\frac{a}{2r_0}\right)^8 \frac{r_0}{r_0 - r_1};$$

or the rate of dissipation of energy, for the spherical condenser, is less when the capacity is very small than when it is very large.

* See J. J. THOMSON, 'Recent Researches,' p. 373.