VIII. *A Memoir on Integral Functions.*

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Received July 25,—Read November 21, 1901.

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PART I.

Introduction.

§ 1. Since the fundamental discoveries of WEIERSTRASS, much progress has been made with regard to uniform transcendental functions; but the advances of modern mathematics appear to have included no attempt formally to classify and investigate the properties of natural groups of such functions.

Consider, for instance, the case of transcendental integral functions which admit one possible essential singularity at infinity. They form the most simple class of uniform functions of a single variable, and yet of them we know, broadly speaking, the nature of but four types :—

(1) The exponential function, with which are associated circular and (rectangular) hyperbolic functions;

(2) The gamma functions;

(3) The elliptic functions and functions derived therefrom, such as the theta functions and APPELL'S generalisation of the Eulerian functions;

(4) Certain functions which arise in physical problems (such as $x^{-n} J_n(x)$) whose properties have been extensively investigated for physical purposes.

There are, of course, isolated examples of other types of functions; yet, broadly speaking, except for algebraic polynomials, the four types just mentioned comprise the extent of our knowledge.

§ 2. Take now an example of the first type of function.

We may write $\dfrac{\sinh \pi\sqrt{z}}{\pi\sqrt{z}} = \prod_{n=1}^{\infty}\left[1+\dfrac{z}{n^2}\right]$, and hence we have

$$\prod_{1}^{\infty}\left[1+\frac{z}{n^2}\right] = \frac{e^{\pi z^{\frac{1}{2}}} - e^{-\pi z^{\frac{1}{2}}}}{2\pi z^{\frac{1}{2}}};$$

so that when $|z|$ is very large, the approximate value of $\prod_{1}^{\infty}\left[1+\dfrac{z}{n^2}\right]$ is $(2\pi)^{-1} z^{-\frac{1}{2}} e^{\pi z^{\frac{1}{2}}}$, so long as $-\pi < \arg z < \pi$.

That is to say, for all points in the region of $z = \infty$ which are not at a finite distance from the zeros of $\pi^{-1} z^{-\frac{1}{2}} \sinh \pi \sqrt{z}$, this function admits what we may call the asymptotic expansion $(2\pi)^{-1} z^{-\frac{1}{4}} e^{\pi z^{\frac{1}{2}}}$, and a similar property is true of all functions of the first class.

§ 3. Consider next the second class of functions.

We have as the simplest example

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right].$$

Since the time of STIRLING it has been known that, when z is a large positive integer,

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}} e^{-z + \sum_{s=0}^{\infty} \frac{(-)^s \quad B_{s+1}}{2s+1 \,.\, 2s+2} \frac{1}{z^{2s+1}}}$$

approximately—the terms neglected involving exponentials of a lower order than those retained.

In 1889 STIELTJES* proved that this asymptotic expansion is valid for all values of z in the region of $z = \infty$, except those which are at a finite distance from the zeros of $\Gamma^{-1}(z)$.

By a different method it is possible to establish both STIELTJES' result and the analogous theorem that the double gamma function†

$$\Gamma_2^{-1}(z) = e^{\gamma_{21} \frac{z^2}{2} + z\gamma_{22}} \,.\, z \,.\, \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}} \right]$$

in which $\Omega = m_1\omega_1 + m_2\omega_2$, admits the asymptotic expansion

$$\log \frac{\Gamma_2(z)\, e^{2\pi\iota(m+m')\, {}_2S'_1(0)}}{\rho_2(\omega_1, \omega_2)} = - {}_2S'_1(z) \{ \log_{\omega_1+\omega_2} z - 2(m+m')\pi\iota \}$$
$$+ z\, {}_2S_1^{(2)}(0) + \frac{z^2}{2}\, {}_2S_1^{(3)}(0) \{ \tfrac{1}{1} + \tfrac{1}{2} \} + \sum_{m=1}^{\infty} \frac{(-)^m\, {}_2S'_{m+1}(0)}{m(m+1) z^m}.$$

This expansion was shown to be valid for all points in the plane of the complex variable z near infinity, which are not at a finite distance from the zeros of the integral function $\Gamma_2^{-1}(z)$.

A similar theorem is true for multiple gamma functions.

§ 4. As regards the elliptic functions and the integral functions associated with them which constitute the third type, there are no points in the neighbourhood of infinity which are not at a finite distance from the zeros of the function and no asymptotic approximations are known to exist.

* 'Liouville' (4), vol. 5, pp. 425–444.

† See a paper by the Author, 'Phil. Trans.,' A, vol. 196, pp. 265–387.

§ 5. The best known example of the fourth type is BESSEL's function

$$J_n(z) = \sum_{\mu=0}^{\infty} \frac{(-)^\mu \left(\frac{z}{2}\right)^{2\mu+n}}{\Gamma(\mu+1)\Gamma(\mu+n+1)}.$$

It is evident that $z^{-n}J_n(z)$ is a uniform integral function.

The investigations of POISSON,* STOKES,† LIPSCHITZ,‡ and JORDAN,§ have finally led to a rigorous demonstration by the latter that asymptotically, when n is real,

$$z^{-n}J_n(z) = \sqrt{\frac{2}{\pi}}\, z^{-n-\frac{1}{2}} \cos\left\{z - (n+\tfrac{1}{2})\frac{\pi}{2}\right\}, \text{ when } \mathfrak{R}(z) \text{ is positive,}$$

$$\text{and } z^{-n}J_n(z) = \sqrt{\frac{2}{\pi}}\, e^{\overline{2n+1}\cdot\frac{\pi\iota}{2}} z^{-n-\frac{1}{2}} \cos\left\{z + (n+\tfrac{1}{2})\frac{\pi}{2}\right\}, \text{ when } \mathfrak{R}(z) \text{ is negative.}$$

The complexity of this result is reduced by the transformation $-z^2 = t$ or $z = \iota\sqrt{t}$, which gives $z^{-n}J_n(z) = \sum_{\mu=0}^{\infty} \frac{t^\mu}{2^{2\mu+n}\Gamma(\mu+1)\Gamma(\mu+n+1)}$, an integral function of t.

And now we have for the asymptotic value of $z^{-n}J_n(z)$ the unique expression

$$(2\pi)^{-\frac{1}{2}}\, t^{-\frac{2n+1}{4}}\, e^{t^{\frac{1}{2}} + \frac{(\ldots)}{t} + \cdots},$$

which is valid for all values of arg t between $-\pi$ and π.

This shows at once that $z^{-n}J_n(z)$, quâ function of t, has no imaginary roots which are not at a finite distance from the negative part of the real axis. In point of fact, these roots are known to be real and negative when $n > -1$||. Hence the asymptotic expansion for

$$t^{-\frac{n}{2}}\, e^{\frac{-\pi\iota n}{2}} J_n(e^{\frac{\pi\iota}{2}}\sqrt{t}) = \sum_{\mu=0}^{\infty} \frac{t^\mu}{2^{2\mu+n}\Gamma(\mu+1)\Gamma(\mu+n+1)}$$

is valid for all points in the neighbourhood of $t = \infty$ except those which are at a finite distance from the zeros of the function.

§ 6. The question now forces itself upon us:—"*Do all integral functions of a single variable z admit asymptotic approximations in the domain of $z = \infty$, which are valid for all points but those which are in the immediate vicinity of the zeros of the functions?*

* POISSON, 'Journal de l'École Polyt.,' vol 19, 1823, pp. 349 *et seq.*

† STOKES, 'Camb. Phil. Trans.,' vol. 9, 1856, pp. 166 *et seq.*

‡ LIPSCHITZ, 'Crelle,' vol. 56, pp. 189 *et seq.*

§ JORDAN, 'Cours d'Analyse,' 1896, vol. 3, pp. 254–274.

|| When n is negative and between m and $m+1$ in absolute value, there may be a finite number $(2m)$ of imaginary roots of $z^{-n}J_n(z)$, but these are not associated with the essential singularity. *Cf.* MACDONALD, 'Proc Lond. Math. Soc.,' vol. 29, pp. 575–584.

The present memoir is devoted to the answer of this question; and the question is closely connected with other subjects of enquiry.

§ 7. Soon after WEIERSTRASS, in 1876, published his great theorem relating to the formation of uniform functions with assigned zeros, LAGUERRE remarked the fundamental nature of the number of terms in the exponential function which is necessary to form the "prime factor." The number was by him termed the "genre" of the function; and the questions at once arose:—

"Is the genre of a function equal to the genre of its derivative?"

"Is the sum of two functions of the same or different genre a function of genre equal to the common genre or equal to the larger genre respectively?"

§ 8. Again, by ROLLE'S Theorem it is known that the real roots of any algebraic equation, $\phi(x) = 0$, separate, and are separated by those of $\phi'(x) = 0$.

Is this true when $\phi(x)$ is an integral function?

Closely connected with this enquiry is the further one:—"If the roots of $\phi(x) = 0$ are all real, are those of $\phi'(x) = 0$ real, in the case when $\phi(x)$ is any integral function?"

Again, it is evident that the more quickly the zeros of an integral function increase, the more quickly will the TAYLOR'S series for the function converge. Can any connection be discovered between the magnitude of the coefficients of the TAYLOR'S series and the expression for the zeros of the function it represents? In other words, if we are given the general term of the TAYLOR'S series for an integral function, can we approximately determine the nature of its zeros?*

All these questions fundamentally depend on the asymptotic approximation for the function. The nature of the latter *serves to classify the nature of the integral function.*

History of the subject.

§ 9. As already remarked, WEIERSTRASS† founded the theory of transcendental integral functions by constructing functions with any assigned zeros. LAGUERRE‡ invented the term "*genre*" to denote the number of terms in the exponential associated in the prime-factor—and for functions of genre 0 and 1 proved that the real roots of the transcendental integral function $\phi(x) = 0$ are separated by those of $\phi'(x) = 0$.

He also proved, as HERMITE§ had previously proved for $\frac{1}{\Gamma(x)}$, that if the roots of $\phi(x) = 0$ are real, those of $\phi'(x) = 0$ are real, provided $\phi(x)$ is of "genre" 0 or 1.

* This question is not formally considered in the present memoir, as the expansions which are obtained, although they will give closer inequalities than any hitherto published, must be still further developed before inequality can be replaced by that asymptotic equality which alone would be a complete solution of the problem.

† WEIERSTRASS, "Zur Theorie der eindeutigen analyt. Funct.," 'Gesamm. Werke,' vol. 2.

‡ LAGUERRE, 'Compt. Rend.,' vol. 94, pp. 160-163, 635-638; vol. 95, pp. 828-831; vol. 98, pp. 79-81.

§ HERMITE, 'Crelle,' vol. 90, p. 336.

His principal proposition is, 'If, as z tends to ∞, a very great value of $|z|$ can be found for which the limit of

$$z^{-n}\frac{\phi'(z)}{\phi(z)}$$

tends uniformly to the value zero, then $\phi(z)$ is of genre n.'

Shortly afterwards, POINCARÉ* gave further criteria for the genre of a function, and made the important step of pointing out that the near connection between the genre of the function and its behaviour near infinity lead to an approximate determination of the magnitude of the general term of the TAYLOR's series for the function.

After a succession of notes by CESARO,† VIVANTI‡ (who proved that the derivative of a function is of the same genre as the function itself), and HERMITE,§ the subject remained in abeyance until HADAMARD,|| in a memoir crowned by the French Academy, gave a valuable extension of POINCARÉ's results.

The latter had proved that in the TAYLOR's series for an integral function of genre E, the coefficient of x^m multiplied by the $(\mathrm{E}+1)^{\text{th}}$ root of $m\,!$ tends to zero, as m indefinitely increases.

HADAMARD proved that, if the coefficient of x^m is less than $\left(\frac{1}{m\,!}\right)^{\frac{1}{\lambda}}$, the function is, in general, of genre less than λ. He also showed that when the coefficient of x^m is of order $\left(\frac{1}{m\,!}\right)^{\frac{1}{\lambda}}$, where λ is not an integer, the function represented by the series is of genre E, designating by $(\mathrm{E}+1)$ the integer immediately superior to λ.

Finally, BOREL,¶ continuing HADAMARD's researches, introduced a more precise notion than that of genre († § 12), and attacked the difficult problem of functions of infinite order whose convergence is very slow.

[*Note added March 20th*, 1902.] In his recent text-book, "Leçons sur les Fonctions Entières,"** BOREL has given a valuable précis of our present knowledge of integral functions. And a paper by MELLIN†† has recently come to my notice, which should be carefully read by all interested in the subjects with which the present memoir deals.

§ 10. The present contribution to this interesting theory differs from previous investigations in that it is shown to be possible to substitute actual asymptotic equalities for the inequalities which have been previously obtained.‡‡

* POINCARÉ, 'Bull. des Sciences Math.,' vol. 15, pp. 136–144.

† CESARO, 'Compt. Rend.,' vol. 99, pp. 26, 27.

‡ VIVANTI, 'Battaglini,' vol. 22, pp. 243–261, and 378–380; vol. 23, pp. 96–122; vol. 26, pp. 303–314.

§ HERMITE, 'Battaglini,' vol. 22, pp. 191–200.

|| HADAMARD, 'Liouville' (4), vol. 9, pp. 171–215.

¶ BOREL, 'Acta Mathematica,' vol. 20, pp. 357–396.

** Paris, Gauthier-Villars, 1900.

†† MELLIN, 'Acta Societatis Fennicae,' 1900, vol. 29, No. 4

[‡‡ MELLIN has obtained results of this nature.]

The memoir deals almost exclusively with simple integral functions of finite or zero order (*vide* the definitions of the succeeding paragraphs.)

I reserve the consideration of functions of infinite order, and also the results which I have obtained in connection with functions of double or multiple sequence. The latter form a self-contained theory, which is a natural extension of the investigations of the present memoir. The consideration of the asymptotic expansion of integral functions defined by a TAYLOR'S series is also postponed, although certain noteworthy extensions of HADAMARD'S results can be at once deduced from the present theory.*

My thanks are due to Professor FORSYTH for the kind way in which he has supplied me with references and criticism.

The Classification of Integral Functions.

§ 11. An *integral* function we define to be a uniform transcendental function with no poles, and a single essential singularity at infinity. [Sometimes it is convenient to include algebraic polynomials.] An integral function is thus the same as a holomorphic function, to use the translation of CAUCHY'S name; it is the equivalent of the French "*fonction entière*," and the German "*ganze Funktion*." Every meromorphic function can be expressed as the quotient of two integral functions.

The most simple integral function can be written in the form

$$e^{\mathrm{H}(z)} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) e^{\sum_{m=1}^{\rho_n - 1} \frac{1}{m} \left(\frac{z}{a_n}\right)^m} \right],$$

where $\mathrm{H}(z)$ is an integral function of z, where the zero a_n depends solely upon n and certain definite constants, and where the law of dependence of a_n upon n is the same for all zeros. Such a function we call a simple integral function with a single sequence of non-repeated zeros. The law of dependence may be broken for a finite number of arbitrary zeros in the finite part of the plane. The existence of such zeros is equivalent to the multiplication of the transcendental function by an arbitrary polynomial coupled possibly with an exponential function of the type $e^{\rho(z)}$, where $\rho(z)$ is another algebraic polynomial. Such terms do not substantially alter the character of the function.

Functions of the type $e^{\mathrm{H}(z)}$, where $\mathrm{H}(z)$ is an integral function, belong to a class apart. The integral function which we consider we shall suppose to be deprived of such extraneous factor.

* The present memoir was largely written during the summer of the year 1898. In consequence, and in spite of rigorous revision, results may sometimes appear to be tacitly claimed as new which have since been published in papers to which reference is made in connection with other investigations of the memoir.

The standard reduced simple integral function with a single simple sequence of non-repeated zeros is thus

$$\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{a_n}\right)e^{\sum_{m=1}^{\rho_n-1}\frac{1}{m}\left(\frac{z}{a_n}\right)^m}\right].$$

We shall call this briefly a *simple integral function.*

§ 12. The quantity ρ_n is the smallest integer such that the series $\sum_{n=1}^{\infty} a_n^{-\rho_n}$ is absolutely convergent. When the convergency of the series can be assured by taking for ρ_n some number p independent of n, the function is said to be of finite *genre** p. In this case, if ρ is a real positive quantity such that $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\rho+\epsilon}}$ converges and $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\rho-\epsilon}}$ diverges, however small the real positive quantity ϵ be, the function is said to be of *order*† ρ, and ρ is called the *convergence-exponent*‡ of the series $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}, \ldots$ It is sufficient that the function a_n depends uniquely upon n; if we put $a_n = \phi(n)$, the quantity $\phi(n)$ is not necessarily a uniform function: it may be a definite value of some multiform function of n.

§ 13. When there is no finite quantity ρ which will make the series $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\rho}}$ converge, the function is said to be of infinite genre and infinite order. The convergency of the series can, as WEIERSTRASS first showed, always be obtained by taking $\rho = n$. A theorem due to CAUCHY proves this at once, since

$$\underset{n=\infty}{\mathrm{L}^t}\sqrt[n]{\frac{1}{|a_n|^n}} = 0.$$

It is equally sufficient to take $\rho = \log n$, for then $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\log n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\log|a_n|}}$; and the latter series is convergent, since $|a_n|$ increases indefinitely with n.

But a smaller number still is a sufficient value for ρ, namely, the greatest integer contained in $\frac{(1+\epsilon)\log n}{\log|a_n|}$, where ϵ is any positive quantity as small as we please.§

The great difficulty in the theory of asymptotic approximations for functions of infinite order consists in finding the minimum value of ρ. I do not intend to consider such functions in the present memoir. Functions of the type $e^{\mathrm{H}(z)}$, where $\mathrm{H}(z)$ is holomorphic, are of course integral functions of infinite order.

§ 14. It is evident that if a_n does not increase more quickly than some (possibly fractional) power of n, however small, the associated integral function will be of

* LAGUERRE, 'Comptes Rendus,' vol. 94; 'Œuvres,' vol. 1, pp. 167 *et seq.*

† BOREL, 'Acta Mathematica,' vol. 20, p. 360.

‡ VON SCHAPER, 'Hadamard'schen Functionen,' p. 35; BOREL, 'Fonctions Entières,' p. 18.

§ BOREL, 'Acta Mathematica,' vol. 20, p. 360.

infinite order. On the other hand, if a_n increases faster than any algebraic power of n, however large, provided it be not actually infinite, the function is of zero order. In other words, functions whose order is "finite both ways," to use DE MORGAN's phrase, have zeros which are to a first approximation algebraic.

The zeros of the function are said to be actually *algebraic* when they are given by

$$a_n = c_0 n^\rho \left[1 + \frac{c_1}{n^{\rho_1}} + \frac{c_2}{n^{\rho_2}} + \ldots\right],$$

when ρ is of course positive and rational, the c's are constants, and $\rho_1, \rho_2, \ldots$ are in ascending order of magnitude.

It is now evident that we can form a scale of integral functions; thus, in between functions with the algebraic zeros

$$a_n = n^{\rho_1} \quad \text{and} \quad a_n = n^{\rho_2}, \quad \text{where } \rho_2 > \rho_1,$$

will come functions with zeros like

$$n^{\rho_1} \log n, \quad n^{\rho_1} \log n \,.\, \log \log n \text{ and so on.*}$$

Such functions we call simple integral functions of finite order with a single simple transcendental sequence of zeros; or, in brief, functions of transcendental sequence.

Thus
$$\prod_{n=1}^{\infty} \left[1 + \frac{z}{(n \log n)^2}\right]$$

is a function of transcendental sequence of order $\frac{1}{2}$ and genre zero.

§ 15. Functions of zero order, which must always be of transcendental sequence, can be classified in the same way. The most simple is $\prod_{n=1}^{\infty} \left[1 + \frac{z}{e^n}\right]$.

Then we consider functions whose zeros are obtained by multiplying e^n by an algebraic function of n. The next step is obviously to introduce intermediate functions by means of logarithmic terms, and so on. Then we introduce functions formed from sets of zeros of still more emphatic convergence, such as

$$\prod_{n=1}^{\infty} \left[1 + \frac{z}{e^{e^n}}\right].$$

The range is obviously limitless.

§ 16. It is worth noticing that the *density* of the zeros along the (possibly curved) line on which they lie, decreases with the increase of the convergence of the function. The zeros of the higher functions of zero order have therefore a density which becomes less as we go higher. The conception of the density of a function is perhaps the most easy way of intuitively classifying it.

* The analogy of the DE MORGAN and BERTRAND scales of convergence is almost too obvious to need mention.

The investigation of the character at infinity of the zero-lines of simple integral functions belongs to the theory of functions of real variables. I do not propose to undertake it here. It is, however, evident that such lines cannot curl round infinity when they belong to functions of finite non-zero order with algebraic zeros:* they approach this point in a line which becomes ultimately straight.

§ 17. A function with a finite number of simple sequences of zeros can evidently be built up of a number of functions, each with a single sequence of zeros.

The function will thus have a finite number of lines of zeros tending to infinity.

When the zeros of a function of order ρ are all of the same character and form m lines symmetrically ranged round the origin, the function will be equal to a function of $\zeta\ (= z^m)$ of order $\dfrac{\rho}{m}$.

Thus a function of order $\frac{1}{2}$ with the sequences

$$\left.\begin{aligned} a_n' &= n^2 \\ a_n'' &= \omega n^2 \\ a_n''' &= \omega^2 n^2 \end{aligned}\right\} \text{where } \omega^3 = 1,$$

is given by the product $\prod_{n=1}^{\infty}\left[1 - \dfrac{z^3}{n^6}\right]$ which, considered as a function of z^3, is of order $\frac{1}{6}$.

§ 18. A function, each of whose zeros is repeated a definite number of times, k (say), is substantially the k^{th} power of a function with the same sequences of non-repeated zeros.

When the n^{th} zero of a function of simple sequence is repeated a number of times dependent upon n, we call the function in brief a *simple repeated function.* We can obviously have repeated functions with any number of sequences of zeros. We may, as before, limit our consideration to a function with a single sequence of zeros; such a one may be written

$$\mathrm{F}(z) = \prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{a_n}\right)^{\mu_n} e^{\mu_n \sum\limits_{m=1}^{\rho_n - 1} \frac{1}{m}\left(-\frac{z}{a_n}\right)^m}\right].$$

The quantity μ_n must, in order that the repetition of the zero may not be meaningless, be an integral number depending upon n; but, if we take the principal values of the ensuing expressions, it is evident that we may get a generalised repeated function by regarding μ_n as a general function of n.

The quantity ρ_n must be so chosen that $\sum\limits_{n=1}^{\infty} \dfrac{\mu_n}{a_n^{\rho_n}}$ is convergent.

* This statement does not deny that they can curl a finite number of times in the finite part of the plane.

We must then, in general, have $\underset{n=\infty}{\mathrm{Lt}} \left[\frac{\mu_n^{\frac{1}{n}}}{a_n^{\frac{\rho_n}{n}}} \right] = 0$; that is to say, $a_n^{\frac{\rho_n}{n}} \mu_n^{-\frac{1}{n}}$ must increase indefinitely with n. We can no longer assign $\log n$ as a value for ρ_n, which is always sufficient to ensure convergence, as was the case with simple non-repeated functions.

§ 19. It is evident that we may regard the value of ρ for which, ϵ being a small real positive quantity,

$$\sum_{n=1}^{\infty} \frac{\mu_n}{a_n^{\rho+\epsilon}} \text{ is convergent and } \sum_{n=1}^{\infty} \frac{\mu_n}{a_n^{\rho-\epsilon}} \text{ is divergent}$$

as the *order* of the repeated function. When ρ is an integer, the order is equal to the genre: in other cases the genre is the integer next greater than ρ.

If the order is not infinite, and the sequence of zeros to a first approximation algebraic, μ_n must be algebraic also.

Suppose that

$$\underset{u=\infty}{\mathrm{Lt}} \frac{a_n}{n^\rho} = 1 \text{ and } \underset{u=\infty}{\mathrm{Lt}} \frac{\mu_n}{n^\sigma} = 1,$$

then, we shall have for the determination of ρ_n, $\rho\rho_n - \sigma > 1$, or $\rho_n > (\sigma + 1)/\rho$.

Repeated functions of infinite order will not be considered in the present memoir.

§ 20. Hitherto we have limited ourselves to integral functions which possess a finite number of simple sequences of zeros. But we have not thus exhausted the category of integral functions. Instead of the typical zero being a definite function of the single number necessary to define its position in the series to which it belongs, it may be a function of two or more numbers and belong to a doubly or multiply infinite sequence. In such case we say that the function is a double or multiple integral function.

Thus the Weierstrassian σ function is a double integral function, and another function of the same category is the double-gamma function to which reference has been made in § 3.

The multiple integral functions always have ultimately a lacunary space* in the region near infinity. In the case of WEIERSTRASS' σ function, this lacunary space covers the whole region near infinity; for the double-gamma function this space lies between the negative directions of the axes of ω_1 and ω_2.

By a well-known theorem due to JACOBI,† functions of treble or higher sequence with periodic zeros cannot exist. This theorem may be extended, and we may prove that there must, in functions whose sequence is greater than double, be such relations

* The zeros will, of course, only crowd together indefinitely on the equivalent NEUMANN sphere. The possibility, or otherwise, of summable divergent expansions is the reason for the nomenclature.

† 'Ges. Werke,' vol. 2, pp. 27–32.

among the parameters that the region near infinity is not ultimately a lacunary space. The parameters are, of course, the constants which enter into the expression of the general zero in the form

$$a_{n_1, n_2} \ldots = \phi(n_1, n_2, \ldots).$$

Such functions have been scarcely considered in analysis. The n^{ple} gamma function is the most simple example which it is possible to give. The theory requires development, since from quotients of multiple integral functions can be built up the general solution of a linear difference equation.

It is to be noticed that, by the coalescence of the parameters, multiple integral functions give rise to functions of lower sequence with repeated zeros. Thus the function*

$$G(z) = \frac{1}{\Gamma(z)} (2\pi)^{\frac{z}{2}} e^{-\frac{z}{2} - \frac{1+\gamma}{2} z^2} \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k}\right)^k e^{-z + \frac{z^2}{2k}} \right]$$

arises from the double gamma function when the parameters ω_1 and ω_2 each become equal to unity.

The separation of multiple functions into functions with repeated and non-repeated zeros and their classification would be carried out on parallel lines to the process adopted for simple functions. As, however, detailed developments of the asymptotic expansions of such functions are not investigated in the present memoir, I do not intend to consider such functions further.

It has been already observed that by the substitution of z^m (m integral) for z, we derive from any simple integral function a function with m times as many sequences of zeros. The substitution of e^z for z will transform a simple function into one of double sequence. [An example of this is given subsequently (§ 62), where LAMBERT'S function is derived from one of simple sequence.] By transformations of greater complexity we may evidently construct functions of limitless range.

§ 21. We are still far from exhausting the category of integral functions. For instance, we may have *ring* functions, that is to say, functions whose zeros are situated on concentric circles: the number of zeros on the n^{th} circle depending upon n.

We can readily see that such a function is given by the product $\prod_{n=1}^{\infty} \left[1 - \left\{ \frac{z}{\phi(n)} \right\}^{\chi(n)} \right]$, where $\chi(n)$ is a function of n which is equal to an integer for all values of n, and where, if $\chi(n) = r$, inversely $n = \psi(r)$, and

$$\underset{r=\infty}{\mathrm{L}t} \left| \sqrt[r]{\phi[\psi(r)]} \right| = \infty.$$

For, the product will converge with

$$\prod_{n=k}^{\infty} \left[1 - \left\{ \frac{z}{\phi(n)} \right\}^{\chi(n)} \right]$$

* See a paper by the author, 'Quart. Journ. Math.,' vol. 31, pp. 264 *et seq.*

the first $k-1$ terms for which $|z| \geqq \phi(n)$ being omitted. Thus it converges with

$$\text{Exp.}\left[-\sum_{m=1}^{\infty}\sum_{n=k}^{\infty}\frac{1}{m}\left\{\frac{z}{\phi(n)}\right\}^{m\chi(n)}\right].$$

The modulus of the term inside the bracket is less than

$$\sum_{m=1}^{\infty}\sum_{n=k}^{\infty}\left[\frac{|z|}{|\phi(n)|}\right]^{m\chi(n)} < \sum_{n=k}^{\infty}\frac{\left|\dfrac{z}{\phi(n)}\right|^{\chi(n)}}{1-\left|\dfrac{z}{\phi(n)}\right|^{\chi(n)}}.$$

Now $1-\left|\dfrac{z}{\phi(n)}\right|^{\chi(n)}$ $(n = k, k+1 \ldots \infty)$ has for its greatest value a finite positive quantity A (say). The product then converges if $\sum\limits_{n=k}^{\infty}\left|\dfrac{z}{\phi(n)}\right|^{\chi(n)}$ converges, which is ensured by the condition assigned at the outset.

The function whose existence has thus been established has $\chi(n)$ zeros on a circle of radius $|\phi(n)|$. If, since the assigned condition makes the order of $\phi(n)$ greater than that of $\chi(n)$, the zeros will ultimately be separated by arcs of infinite length.

§ 22. A little ingenuity will enable us to construct other functions of types innumerable, among them what BOREL has called functions "*à croissance irrégulière.*"* The survey gradually forces upon us the conclusion that we cannot expect to find any general law as to the behaviour of all integral functions near their essential singularity which is not a disguised truism.† MM. HADAMARD‡ and BOREL have given laws relating to the increase of all integral functions. It seems to me that such laws must be limited to particular classes of functions, and that such delimitation cannot be stated too explicitly. Consequently in this memoir I have taken the most simple functions and have endeavoured to study in detail their behaviour near the essential singularity, for I believe that by such means the progress made will be sure, if slow.

* BOREL, ['Fonctions Entières,' Note III.], gives an example of such a function in the form of a TAYLOR'S series.

† Such a term I should apply to M. BOREL'S law "the maximum value of a function is equal to the inverse of its minimum value on an infinite number of circles at infinity." For this law is an immediate consequence of the possibility of asymptotic expansions (*see* Part II. of this memoir).

‡ OSGOOD ('Bulletin of the American Math. Soc.,' Nov., 1898, note, p. 80) states that the analysis used to prove HADAMARD'S most general law requires revision. And it is to be noted that HADAMARD ('Liouville,' 4 ser., t. 9, p. 173) assumes that $\phi(m)$ is continuous, increasing, and such that $L\phi(m) + \dfrac{k}{m}$ constantly increases ultimately.

Part II.

The Theory of Divergent Series.

§ 23. The development of the theory of divergent series is an interesting instance of the progress of mathematical thought. The beginning was purely arithmetic: to find some approximation to the value of $n!$, where n is a very large integer.* In the result it appeared that the value of $\log n!$ could be more and more nearly calculated by adding on successive terms of a series proceeding by powers of $\frac{1}{n}$. The error is of the order of magnitude of the term of the series next after the one at which we stop. And, most important fact of all, the series is divergent.

If $n!$ be replaced by $\Gamma(n+1)$, a similar result can be obtained, which holds for all real positive values of n.

Finally, there comes the enquiry as to what meaning, if any, can be attached to the equality in the case in which n is any complex quantity.

Other approximations undergo the same process of development, so that it becomes necessary to try and construct a formal theory.

What we may call the arithmetic theory has been given by Poincaré,† for the case in which all the quantities involved are real:—a restriction which the author subsequently assumes to be unnecessary.

For the more extended case, when z is any complex quantity, we may say that the divergent series $a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} + \ldots$ of which the sum of the first $(n+1)$ terms in S_n, will, when $|z|$ is very large, be an asymptotic expansion for a function $J(z)$ if the expression $|z^n(J - S_n)|$ tends to zero, as z tends to infinity.

Thus, if z be sufficiently large, $|z^n(J - S_n)| < \epsilon$ where ϵ is very small.

The error $J - S_n = e/z^n$ committed in taking for the function J the first $\overline{n+1}$ terms of the series has a modulus which is infinitely smaller than the modulus of the error $J - S_{n-1} = a_n + e/z^n$ obtained by taking only the first n terms, for $|a_n|$ is in general finite, and $|e|$ is very small.

In view of subsequent results, it proves necessary to define the equality of the function and divergent series for values of z which lie along some definite line tending to infinity. We do not then assume that the expansion is possible all round the point $z = \infty$.

It will be sufficient to recapitulate the results which Poincaré obtains.

We may multiply two asymptotic series together by the same rules as we should apply to absolutely convergent series.

* Stirling, 'Methodus Differentialis' (1730).

† 'Acta Mathematica,' 8, pp. 295–344; 'Mécanique Céleste,' vol. 2, pp. 12–14.

In particular, we may raise an asymptotic series to any finite power, and it will then represent the corresponding power of the function represented by the original series.

The term-by-term integral of an asymptotic series is equal to the integral of the function which it represents: in brief, we may integrate an asymptotic series.

In general, we may not differentiate an asymptotic equality.

[Nevertheless, we may differentiate most of the expansions which arise naturally in analysis, and are not constructed artificially.]

Similarly, if an asymptotic equality involves an arbitrary parameter, we may not in general (but we may fairly safely in practice) differentiate with respect to that parameter.

Such are the main propositions of the arithmetic theory of asymptotic expansions.

The difficulties inherent in the theory are obvious when we attempt its application. We have, in all cases, to investigate a superior limit to the remainder of the series after the first $(n+1)$ terms have been taken; and, to do this, we must have command, even for the most simple cases, of analytical processes of great complexity and power.

§ 24. We proceed then to consider these series from the function-theoretic point of view.

That is to say, on the one hand, we attempt to give a definition to a divergent series which shall harmonise with the development of WEIERSTRASS' theory, and on the other, we enter more deeply into the nature of the essential singularity of the function of which the divergent series is the expansion.

Suppose first that we have a series $a_0 + a_1 z + \ldots + a_n z^n + \ldots$ of finite radius of convergency ρ, so that by CAUCHY's rule, $\underset{n=\infty}{\mathrm{Lt}} \sqrt[n]{a_n} = \rho^{-1}$.

When $|z|$ is greater than ρ, the series is divergent and our fundamental conception of a series as a command to add in order successive terms leads to no result.

And yet, if the function which the series represents be not one which has the circle of radius ρ as a line of essential singularity, the function exists outside this circle, and admits an analytic continuation. Thus the function exists even when the series is divergent.

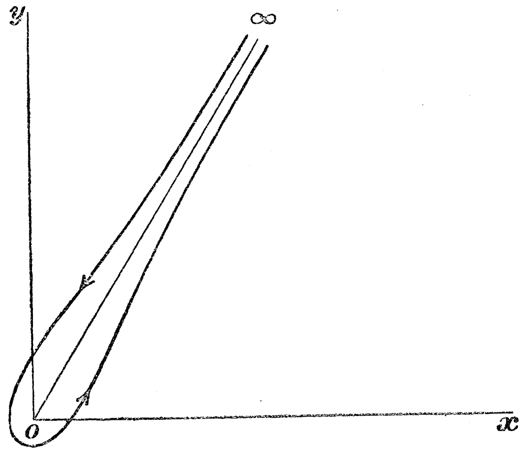
Can we not then regard the series when divergent as a command to perform certain operations which shall yield the analytic continuation of the function? We can do so, and in an infinite number of ways.

The most simple is, perhaps, given by an extension of a process developed by BOREL.*

Let the plane of the variable x be dissected by some line going from 0 to ∞ to the right of the axis of y.

* "Théorie des séries divergentes sommables," 'Liouville,' 5 sér., t. 2, pp. 103 *et seq.* "Mémoire sur les séries divergentes," Ann. de l'École Normale Supérieure, 3 sér., t. 16, pp. 1 *et seq.*

This line of section will render $(-x)^{z-1} = e^{(z-1)\log(-x)}$ uniform, and we shall take that value which is real when x is real and negative.



Then it is known that*

$$\Gamma(\theta) = \frac{\iota}{2 \sin \pi\theta} \int (-x)^{\theta-1} e^{-x} \, dx$$

where the contour of the integral embraces the line of section as in the figure.

From the original series

$$a_0 + a_1 z + \ldots + a_n z^n + \ldots$$

and an auxiliary function

$$\chi(z) = c_0 + c_1 z + \ldots + c_n z^n + \ldots$$

in which $c_n = \dfrac{1}{\Gamma(n+\theta)}$—$\theta$ being any arbitrary quantity—construct the function

$$G(x) = a_0 c_0 + a_1 c_1 x + \ldots + a_n c_n x^n + \ldots$$

This function will be an integral function, for

$$\operatorname*{Lt}_{n=\infty} \sqrt[n]{a_n c_n} = \operatorname*{Lt}_{n=\infty} \frac{\sqrt[n]{a_n}}{\sqrt[n]{\Gamma(n+\theta)}} = \operatorname*{Lt}_{n=\infty} \frac{e}{\rho n} = 0.$$

Consider now the integral

$$\frac{\iota}{2 \sin \pi\theta} \int G(xz) \, e^{-x} (-x)^{\theta-1} \, dx.$$

This integral is equal to

$$\frac{\iota}{2 \sin \pi\theta} \int \sum_{n=0}^{\infty} \left[a_n c_n x^n z^n\right] e^{-x} (-x)^{\theta-1} \, dx, \quad \text{or} \quad \sum_{n=0}^{\infty} a_n z^n.$$

That is to say, when $|z| < \rho$, the integral represents the same function as the original series. For *all* values of $|z|$, the integral, provided it has a meaning, represents the analytic continuation of the series. And if, when the series is divergent, we regard it as a command to perform the processes which lead to the integral

$$\frac{\iota}{2 \sin \pi\theta} \int G(xz) \, e^{-x} (-x)^{\theta-1} \, dx,$$

we shall obtain a conception of such a divergent series which is in harmony with WEIERSTRASS' theory of functions.

* See a paper by the author, 'Messenger of Mathematics,' vol. 29, p. 105.

§ 25. We now enquire whether the domain of existence of the integral is coextensive with the domain of existence of the analytic function defined by the original series. Just as the series ceases to define the function by becoming divergent, so the integral may cease to be an adequate expression by becoming infinite.

Consider the series $1 + z + z^2 + \ldots + z^n + \ldots$

The "sum" of this series, when divergent, is represented by the integral

$$\frac{\iota}{2 \sin \pi\theta} \int \mathrm{G}(xz)\, e^{-x} (-x)^{\theta-1}\, dx, \text{ in which } \mathrm{G}(xz) = \sum_{n=0}^{\infty} \frac{(xz)^n}{\Gamma(n+\theta)},$$

and the integral is taken round some contour embracing an axis in the positive half of the z-plane.

Make now θ tend to unity. Then $\mathrm{G}(xz)$ becomes e^{xz}, and the integral becomes $\int_0^\infty e^{-x(1-z)}\, dx$, taken along some line in the positive half of the z-plane.

Suppose now that $x = \rho e^{\iota\theta}$, $z = 1 + re^{i\phi}$ where θ and ϕ are both in absolute value not greater than π. Since the axis of the integral lies in the positive half of the z-plane, $\frac{\pi}{2} - \epsilon \geqq \theta \geqq -\frac{\pi}{2} + \epsilon$, where ϵ is a positive quantity as small as we please. The amplitude of $x(z-1)$ is $\theta + \phi$, and that the integral may be finite this quantity must be such that $\mathfrak{R}(z-1)$ is negative. Therefore $\frac{3\pi}{2} > \theta + \phi > \frac{\pi}{2}$ or $-\frac{\pi}{2} > \theta + \phi > -\frac{3\pi}{2}$. These conditions can always be satisfied by values of θ within the assigned range, if ϕ does not lie between or at the limits of the range bounded by ϵ and $-\epsilon$.

We thus see that the function $1/(1-z)$ is represented by the series

$$1 + z + z^2 + \ldots + z^n + \ldots$$

within a circle of radius unity; and by the integral

$$\frac{\iota}{2 \sin \pi\theta} \int \left[\sum_{n=0}^{\infty} \frac{(xz)^n}{\Gamma(n+\theta)} \right] e^{-x} (-x)^{\theta-1}\, dx$$

for all values of z except those which lie on that part of the real axis between the points 1 and ∞.

§ 26. Similarly the series $\sum_{n=0}^{\infty} \frac{z^n}{n+1}$, or its integral equivalent when it is divergent, will represent $-z^{-1} \log(1-z)$, provided z does not lie on that part of the real axis between 1 and ∞. And the same is true of $(1-z)^{-m}$ and its equivalent series, when m is not necessarily an integer. These statements form easy examples which the reader can at once work out for himself.

It is interesting to notice that the lines from the singularities to infinity intervene to give uniformity to the non-uniform functions to which divergent series may

"sum." Thus the divergent series $\sum_{n=0}^{\infty} \frac{z^n}{n+1}$ represents the non-uniform function $z^{-1} \log(1-z)$, which becomes uniform when a cross-cut is made along the real axis from 1 to $+\infty$.

§ 27. Suppose now that we have any function with singularities lying outside a circle of radius ρ, within which the function is represented by the convergent series

$$a_0 + a_1 z + \ldots + a_n z^n + \ldots$$

We may join the singularities by straight lines to infinity, each line being the continuation of the direction from the origin to its initial point. Then within the simply connected area thus formed we may replace the function by a set of integrals of the type

$$\frac{\iota}{2 \sin \iota\theta} \int \mathrm{G}(xz)\, e^{-x} (-x)^{\theta-1}\, dx.$$

Which we can therefore regard as the "sum" of the divergent series within the region in question, whenever this set of integrals has a meaning.

Although in general this will not be the case, we can nevertheless, if the function represented by the series has only a finite number of poles outside its circle of convergence and within a circle of finite radius σ, greater than the radius of convergence ρ, split up the given series into a sum of others each of which, except the last, will be divergent, but capable of being represented by an integral of the foregoing type, while the last series is convergent within this circle of radius σ. The circumstances under which the whole series can be represented by a definite integral over the region of its existence I hope to discuss elsewhere. The problem is bound up with the determination of the number and nature of the singularities of a TAYLOR's series and is, therefore, connected naturally with the researches of DARBOUX,* HADAMARD,† BOREL,‡ FABRY,§ LE ROY,‖ LINDELOF,¶ and LEAU.**

§ 28. So far we have been concerned with the summation of divergent series of ascending powers of z which are convergent for sufficiently small values of $|z|$. We will now *define* asymptotic series as those which are divergent, however small $|z|$ may be, and we proceed to consider their summation.

At the outset we can see that the problem is essentially different from the one

* DARBOUX, 'Liouville' (1878), 3 sér., t. 4, pp. 5–56, 377–416.

† HADAMARD, 'Liouville' (1892), 4 sér., t. 8, pp. 101–186.

‡ BOREL, 'Comptes Rendus,' October 5 and December 14, 1896; December 12, 1898; 'Acta Mathematica,' 21; 'Liouville' (1896), 5 sér., t. 2.

§ FABRY, 'Ann. de l'Éc. Nor. Sup.' (1896), 3 sér., t. 13, pp. 367–399; 'Acta Mathematica' (1899), t. 22, pp. 65–87; 'Liouville' (1898), 5 sér., t. 4, pp. 317–358.

‖ LE ROY, 'Comptes Rendus,' October 21, 1898, and February 20, 1899.

¶ LINDELOF, 'Acta Societatis Fennicæ,' 1898.

** LEAU 'Liouville' (1899), 5 sér., t. 5, pp. 365–425.

just considered. Instead of the process of summation leading to the same result, whatever the nature of the integral process chosen, we can obtain an infinite number of results, each associated function leading to a different function, of which the given series may be regarded as the asymptotic expansion. For when the divergent series is convergent for sufficiently small values of $|z|$, it defines a function over that area of convergence, and any summation process can only lead to the analytic continuation of a definite branch of that function. But a true asymptotic series has no area of convergence, and any meaning which we care to attach to it will harmonise with WEIERSTRASS' theory of functions.

The essential nature of the difference between the two kinds of series may be brought out in another way. A series convergent for sufficiently small values of $|z|$ represents a function regular in the neighbourhood of the origin. But any function which a true asymptotic series can represent will have the origin as an essential singularity. And, therefore, not only can many functions with an essential singularity at the origin have the same asymptotic expansion, but also the same function may have different asymptotic expansions in different areas having the origin as apex. It is almost impossible to imagine a vagary which an essential singularity will not possess, and this fact we cannot, throughout the whole of the investigation, too carefully bear in mind.

Inasmuch as any means of regarding an asymptotic series leads to a result peculiar to that means, we must choose our process with care so as to obtain the most simple result, and, if possible, so as to ensure that our conception of such series agrees with the arithmetic point of view by which historically they were generated.

§ 29. Suppose, in the first place, that we have given the asymptotic series

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots,$$

in which, by CAUCHY's rule, $\underset{n=\infty}{\mathrm{Lt}}\ \sqrt[n]{a_n} = \infty$. And suppose further that $\underset{n=\infty}{\mathrm{Lt}}\ \frac{\sqrt[n]{a_n}}{n} = 0$.

Then the associated function

$$G(z) = a_0 c_0 + a_1 c_1 z + \ldots + a_n c_n z^n + \ldots,$$

in which $c_n = \frac{1}{\Gamma(n+\theta)}$, will be an integral function.

It is a natural extension, then, of our previous ideas to regard the asymptotic series as the expansion of the integral

$$\frac{\iota}{2\sin\pi\theta}\int G(xz)\, e^{-x}(-x)^{\theta-1}\, dx,$$

and, conversely, to regard the integral as the "sum" of the asymptotic series.

The point $z = 0$ will be an essential singularity of the function of which the integral is the formal expression. For certain values of z near $z = 0$ the integral can probably take values which differ infinitely for the smallest change in the value of z. This will happen when z lies on a line of zeros or poles crowding to $z = 0$. Along such lines, or quite possibly within areas of the same nature, the asymptotic series will cease to represent the function.

Further, there must always be such lines or areas of non-representation, for the only functions to the essential singularities of which poles or zeros or other singularities do not crowd are of the types $e^{\frac{1}{z}}$, $e^{e^{\frac{1}{z}}}$, . . . , which cannot admit of asymptotic expansion.

We have then the fundamental result that the integral cannot represent the "sum" of the series right round $z = 0$. There will be certain lines or areas with $z=0$ as extremities or vertices along which the asymptotic series cannot be "summed" by any process which we may employ: these lines or areas will differ with the different processes, but will never be absent altogether.

There are, of course, asymptotic series of the prescribed type which can never be "summed" by any process which we may employ. Such a one is $\sum_{n=0}^{\infty} a_{c_n} z^{c_n}$, in which $\underset{n=\infty}{\mathrm{Lt}} (c_{n+1} - c_n) = \infty$ and $\underset{n=\infty}{\mathrm{Lt}} \sqrt[n]{a_n} = \infty$. But such series will never arise naturally in analysis, and we do not, therefore, need to trouble about them.

§ 30. We have now to consider whether, when a series of the prescribed type is "summed" by means of the process indicated, the function which results admits the series as an arithmetically asymptotic expansion according to POINCARÉ's definition.

Denote by $f(z)$ the integral which is the result of summing the series

$$a_0 + a_1 z + \ldots + a_n z^n + \ldots ,$$

in which $\underset{n=\infty}{\mathrm{Lt}} \sqrt[n]{a_n} = \infty$ and $\underset{n=\infty}{\mathrm{Lt}} \dfrac{\sqrt[n]{a_n}}{n} = 0$.

The associated function for the series is

$$\mathrm{G}(z) = a_0 c_0 + a_1 c_1 z + \ldots + a_n c_n z^n + \ldots ,$$

and for s_n, the sum of the first n terms of the series, is

$$a_0 c_0 + a_1 c_1 z + \ldots + a_n c_n z^n.$$

Hence

$$\frac{f(z) - s_n}{z^n} = \frac{\iota}{2 \sin \pi\theta} \int \frac{\mathrm{G}_{n+1}(z, x)}{z^n} e^{-x} (-x)^{\theta-n-1} \, dx,$$

where

$$z^{-n} \mathrm{G}_{n+1}(z, x) = zx [a_n c_n + a_{n+1} c_{n+1} (xz) + \ldots].$$

Now $z^{-n} G_{n+1}(z, x)$ is an absolutely convergent series, and $|z^{-n} G_{n+1}(zx)|$ tends to zero as n tends to infinity.

Moreover $z^{-n} G_{n+1}(zx)$ and $G(xz)$ are functions which, while n has any finite value, have the same character near $|xz| = \infty$, for they only differ by the polynomial $a_0c_0 + a_1c_1zx + \ldots + a_nc_n(zx)^n$.

Therefore the integral $\frac{\iota}{2\sin\pi\theta}\int\frac{G_{n+1}(zx)}{z^n} e^{-x}(-x)^{\theta-n-1}\,dx$ will represent an analytic function of z whenever $\frac{\iota}{2\sin\pi\theta}\int G(xz)\,e^{-x}(-x)^{\theta-1}\,dx$ does so; and the two functions will have the same character near $z = 0$.

Therefore, within those areas for which the second integral represents the "sum" of the given asymptotic series, the first integral is finite, and $|z^{-n}\{f(z) - s_n\}|$ tends to zero as $|z|$ tends to zero.

Thus the asymptotic equality satisfies POINCARÉ's arithmetic definition. The reader must note very carefully that this theorem does *not* apply to divergent series which have a finite radius of convergence. It is necessary that $|z|$ should tend to zero. No computer, for instance, could make $1 - 2 + 2^2 - 2^3 + \ldots$ tend to $\frac{1}{1+2}$.

§ 31. Suppose now that we differentiate the series

$$a_0 + a_1z + \ldots \qquad + a_nz^n + \ldots$$

in which

$$\underset{n=\infty}{\mathrm{Lt}}\ \sqrt[n]{a_n} = \infty, \text{ and } \underset{n=\infty}{\mathrm{Lt}}\ \frac{\sqrt[n]{a_n}}{n} = 0.$$

We shall obtain the series

$$a_1 + 2a_2z + \ldots + na_nz^{n-1}\ldots$$

If we "sum" this series by the exponential process (the name which it is convenient to give to the process employed in the preceding paragraphs) we obtain the integral

$$\frac{\iota}{2\sin\pi\theta}\int G_1(xz)\,e^{-x}(-x)^{\theta-1}\,dx, \text{ in which}$$

$$G_1(xz) = a_1c_1 + 2a_2c_2xz + \ldots + na_nc_n(xz)^{n-1} + \ldots$$

We thus see that, since this series is an integral function,

$$G_1(xz) = \frac{\partial}{\partial z}G(xz).$$

Therefore the "sum" of the series $a_1 + 2a_2z + \ldots + na_nz^{n-1} + \ldots$ is

$$\frac{\iota}{2\sin\pi\theta}\int\frac{\partial}{\partial z}G(xz)\,e^{-x}(-x)^{\theta-1}\,dx$$

Now so long as the integral

$$\frac{\iota}{2 \sin \pi\theta} \int G(xz)\, e^{-x} (-x)^{\theta-1}\, dx$$

is within the regions surrounding $z = 0$, for which the original series can be summed, the differential coefficient of the function which it represents is the function represented by the integral

$$\frac{\iota}{2 \sin \pi\theta} \int \frac{\partial}{\partial z} G(xz)\, e^{-x} (-x)^{\theta-1}\, dx,$$

for we do not transgress the rules which govern differentiation under the sign of integration.*

Therefore, within the region for which an asymptotic equality is valid, such equality may be differentiated.

Similarly such equality may be integrated. And the process of differentiation or integration may be repeated any number of times.

§ 32. We have hitherto limited ourselves to the consideration of asymptotic series of the type

$$a_0 + a_1 z + \ldots \qquad + a_n z^n + \ldots$$

in which $\underset{n=\infty}{\mathrm{L}t} \sqrt[n]{a_n} = \infty$ and $\underset{n=\infty}{\mathrm{L}t} \frac{\sqrt[n]{a_n}}{n} = 0.$

The first condition is necessary that the series may have zero radius of convergency, that is to say, that it may be asymptotic.

The second condition was requisite in order to ensure the applicability of the exponential process.

It is convenient to call an asymptotic series for which $\underset{n=\infty}{\mathrm{L}t} \frac{\sqrt[n]{a_n}}{n} = 0$ an asymptotic series of the first order; one for which this limit is greater than zero, but $\underset{n=\infty}{\mathrm{L}t} \frac{\sqrt[n]{a_n}}{n^2} = 0$, a series of the second order, and so on.

We have given in the preceding paragraphs the theory of summation of series of the first order. But suppose that we wish to sum one of the most simple asymptotic series, that for $\log \frac{\Gamma(z+a)}{\Gamma(z)\, z^a}$, namely $\sum_{n=1}^{\infty} \frac{(-)^{n-1} S_n(a)}{n z^n}$, where $S_n(a)$ is HERMITE's Bernoullian function.

By CAUCHY's theorem, re-discovered by HADAMARD, we know that

$$\underset{n=\infty}{\mathrm{L}t} \sqrt[n]{\frac{Sn(a)}{n!}} = 2\pi;$$

* JORDAN, 'Cours d'Analyse,' 2nd edition, vol. 2, pp. 154–157.

for the expansion

$$\frac{e^{ax}-1}{e^x-1} - a = \sum_{n=1}^{\infty} \frac{S_n(a)}{n!} x^n$$

is only valid within a circle of radius 2π.

We see then that the asymptotic series is such that, if we denote the coefficient of $\frac{1}{z^n}$ by a_n,

$$\underset{n=\infty}{\mathrm{Lt}} \frac{\sqrt[n]{a_n}}{n} > 0, \quad \text{and} \quad \underset{n=\infty}{\mathrm{Lt}} \frac{\sqrt[n]{a_n}}{n^2} = 0.$$

The series is thus of the second order. And the associated function formed as in the preceding paragraphs will be

$$G(n) = \sum_{n=1}^{\infty} \frac{(-)^{n-1} S_n(a)}{n \, . \, n!} n^n$$

which is not an integral function.

Our analytical machinery therefore breaks down, and we must attempt to extend it.

Just as the original problem admitted of an infinite number of solutions, so we may now proceed in an infinite number of ways to give an analytical meaning to asymptotic series of the second or higher orders.

Of these two would appear to be most natural. We may either use some more powerful associated function than we used in the exponential process, or we may repeat the exponential process until we arrive at a finite analytical function.

§ 33. Let us consider in the first place the second of these alternatives.

If we have the asymptotic series

$$a_0 + a_1 z + \ldots, + a_n z^n + \ldots,$$

we have agreed to say that this series is the expression of the analytic function $\frac{\iota}{2 \sin \pi\theta} \int G_1(xz) \, e^{-x} (-x)^{\theta-1} \, dx$, whenever this integral has a meaning, that is, whenever $G_1(xz)$ is an integral function, and the integral is not infinite.

Now

$$G_1(z) = \frac{a_0}{\Gamma(\theta)} + \frac{a_1}{\Gamma(1+\theta)} z + \ldots + \frac{a_n}{\Gamma(n+\theta)} z^n + \ldots$$

$$= a_0' + a_1' z + \ldots + a_n' z^n + \ldots \text{(say)},$$

and, if the series is not absolutely convergent over the whole plane, we shall be consistent with our former generalised point of view, if we regard it as determining an analytic function

$$\frac{\iota}{2 \sin \pi\theta} \int G_2(xz) \, e^{-x} (-x)^{\theta-1} \, dx,$$

whenever this integral has a meaning.

Now

$$G_2(z) = \frac{a_0'}{\Gamma(\theta)} + \ldots + \frac{a_n'}{\Gamma(n+\theta)} z^n + \ldots$$

$$= a_0'' + \ldots + a_n'' z^n + \ldots \text{(say)}.$$

If the series $G_1(z)$ had a finite radius of convergence, or zero radius of the first order, the function $G_2(z)$ will be an integral function, and by the process just sketched, a definite meaning has been assigned to $G_1(z)$ and the original series.

When, however, $G_1(z)$ has zero radius of convergency of the second or higher order, $G_2(z)$ will not be an integral function, but we must regard the series which it denotes as determining an analytic function

$$\frac{\iota}{2 \sin \pi\theta} \int G_3(xz) e^{-x} (-x)^{\theta-1} dx,$$

whenever this integral has a meaning, that is, as a preliminary condition, whenever

$$G_3(z) = \frac{a_0''}{\Gamma(\theta)} + \ldots + \frac{a_n''}{\Gamma(n+\theta)} z^n + \ldots$$

is convergent over the whole plane.

The procedure may be repeated indefinitely. If we have started with an asymptotic series which does not ultimately give rise to a function $G_n(z)$ whose finite radius of convergency is a line of essential singularity, we shall ultimately get an analytic function of which the original series is the asymptotic expansion in the vicinity of its essential singularity $z = 0$.

§ 34. The extension which we have just indicated is in harmony with the general theory, but we have still to determine the important point as to whether the asymptotic equality of series and functions satisfies POINCARÉ's arithmetic definition.

Take for simplicity the series of the second order $a_0 + a_1 z + \ldots + a_n z^n + \ldots$, for which the associated series

$$\frac{a_0}{\Gamma(\theta)} + \frac{a_1}{\Gamma(1+\theta)} z + \ldots + \frac{a_n}{\Gamma(n+\theta)} z^n + \ldots$$

has finite radius of convergency and represents the function

$${}_1G(z).$$

The given series gives rise to the function

$$G(z) = \frac{\iota}{2 \sin \pi\theta} \int {}_1G(xz) e^{-x} (-x)^{\theta-1} dx.$$

Let

$$G_n(z) = a_0 + \ldots + a_n z^n,$$

then

$$G(z) - G_n(z) = \frac{\iota}{2\sin\pi\theta}\int \{{}_1G(xz) - {}_1G_n(xz)\} e^{-x}(-x)^{\theta-1}\,dx,$$

where

$${}_1G_n(z) = \sum_{r=0}^{n} \frac{a_r}{\Gamma(r+\theta)} z^n.$$

Now $z^{-n-1}\{{}_1G(xz) - {}_1G_n(xz)\}$ is an analytic function of x of the same character as ${}_1G(xz)$: hence the natures of the two functions $G(z)$ and $z^{-n-1}\{G(z) - G_n(z)\}$ near $z = 0$ are substantially the same. And therefore, in general, if $G(z)$ tends uniformly to a finite limit as z tends to zero in any direction, $z^{-n-1}\{G(z) - {}_nG(z)\}$ also tends uniformly to a finite limit as z tends to zero in the same direction. That is to say, $|z^{-n}\{G(z) - {}_nG(z)\}|$ tends to zero as z tends to zero, so that the divergent series is arithmetically asymptotic for the function $G(z)$.

It is evident that a repetition of the same argument will prove the arithmetic nature of the asymptotic dependence of a series of any order and the function to which it gives rise by successive applications of the exponential process. But one case of exception must be noticed. At each step the equivalence of the asymptotic series and the derived function fails along certain lines or within certain areas radiating from $z = 0$. And, since the effect of such failure is cumulative, it may happen that before the process is finished the equivalence has failed over the whole area around $z = 0$. Either the series is hopeless—an artificial monstrosity that cannot arise in practice—or we need some other process by means of which it can be interpreted.

§ 35. As an example of the process just sketched, consider the asymptotic expansion

$$-\frac{1}{1-s} z^{s-1} + \sum_{n=1}^{\infty} \binom{-s}{n} \frac{S_n'(a)\, z^{n+s-1}}{s+n-1},$$

which, *quâ* function of z, is an asymptotic series of the second order and wherein s and a are any complex or real parameters.

Applying the integral process associated with the exponential function to the series, we obtain the integral

$$\frac{\iota}{2\pi}\int G_1(zx)\, e^{-x} z\, dx,$$

where

$$G_1(u) = -\Gamma(1-s)(-u)^{s-2}\left\{1 + \sum_{n=1}^{\infty} \frac{(-)^n S_n'(a)}{n!} u^n\right\},$$

and we have, for convenience, taken the auxiliary function to be

$$\sum_{n=0}^{\infty} (-)^n \Gamma(2-n-s)\, x^n,$$

so that θ is absorbed in s.

Now $G_1(u)$ is a series of finite radius of convergency, and the analytic function

which we take the series to represent is the function of which the series is the expansion within the circle of convergency. The series will therefore denote, with our present conceptions, the function

$$- \Gamma(1 - s) \frac{e^{-au}}{1 - e^{-u}} (- u)^{s-1}.$$

The series with which we commenced may therefore be regarded as giving rise to the analytic function

$$- \frac{\iota}{2\pi} \Gamma(1 - s) \int \frac{e^{-axz}}{1 - e^{-xz}} (- xz)^{s-1} e^{-x} z \, dx = - \frac{\iota}{2\pi} \Gamma(1 - s) \int \frac{e^{-\left(a+\frac{1}{z}\right)t}}{1 - e^{-t}} (- t)^{s-1} dt$$

on making the substitution $t = xz$.

This function admits when $|z|$ is small, the arithmetically asymptotic expansion from which we started.

When z^{-1} is a large real positive integer, the series and integral become fundamental in the asymptotic definition of the extended Riemann ζ function.

But there can be obtained by other processes an indefinite number of analytic functions, each of which has an essential singularity at $z = 0$, near which point it admits the given series as an arithmetically asymptotic expansion. We proceed to indicate one alternative process by which such an analytic function can be obtained at a single step.

§ 36. For this purpose we use certain results of the theory of the connection between linear difference and differential equations.

Consider the function

$$\mathrm{F}(\alpha, \rho_1, \ldots, \rho_m, -x) = 1 - \frac{\alpha}{1 . \rho_1 \ldots \rho_m} x + \ldots + \frac{\alpha \, \overline{\alpha + 1} \ldots \overline{\alpha + r - 1}}{r! \rho_1 \ldots \rho_m \ldots \overline{\rho_1 + r - 1} \ldots \overline{\rho_m + r - 1}} (-x)^r + \ldots$$

It is evidently a transcendental integral function which is a solution of the differential equation

$$\left[(\vartheta + \alpha) + \frac{1}{x} \vartheta (\vartheta + \rho_1 - 1) \ldots (\vartheta + \rho_m - 1)\right] y = 0,$$

wherein the operator $\vartheta = x \dfrac{d}{dx}$.

If y be any solution of this equation, form the function

$$\frac{\iota}{2 \sin \pi z} \int y (- x)^{z-1} dx,$$

where the contour of the integral and the prescription for $(- x)^{z-1}$ are exactly those employed in the definition of the integral for $\Gamma(z)$ previously employed (§ 24).

On integrating by parts, we have

$$z f(z) = -\frac{\iota}{2 \sin \pi z} \int [\vartheta y] (-x)^{z-1} \, dx,$$

a relation which may also be written

$$(z+1) f(z+1) = -\frac{\iota}{2 \sin \pi z} \int [x \vartheta (y)] (-x)^{z-1} \, dx.$$

We thus see that

$$z (z + 1 - \rho_1) \ldots (z + 1 - \rho_m) f(z)$$
$$= (-)^{m+1} \frac{\iota}{2 \sin \pi z} \int [\vartheta (\vartheta + \rho_1 - 1) \ldots (\vartheta + \rho_m - 1) y] (-x)^{z-1} \, dx,$$

and also

$$(z + 1 - \alpha) f(z+1) = \frac{-\iota}{2 \sin \pi z} \int [x (\vartheta + \alpha) y] (-x)^{z-1} \, dx.$$

Therefore, since y is a solution of the equation

$$\vartheta (\vartheta + \rho_1 - 1) \ldots (\vartheta + \rho_m - 1) y = -x (\vartheta + \alpha) y,$$

we have

$$f(z+1) = (-)^{m-1} \frac{z \, . \, (z + 1 - \rho_1) \ldots (z + 1 - \rho_m)}{(z + 1 - \alpha)} f(z).$$

The general solution of this difference-equation is

$$\Gamma(z) \frac{\Gamma(\alpha - z)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\rho_1) \ldots \Gamma(\rho_m)}{\Gamma(\rho_1 - z) \ldots \Gamma(\rho_m - z)} \varpi (z, \alpha, \rho_1, \ldots \rho_m),$$

where $\varpi (z, \alpha, \rho_1, \ldots \rho_m)$ is a simply periodic function of z of period unity.

We have then established the identity

$$\frac{\iota}{2 \sin \pi z} \int \mathrm{F}_m (\alpha, \rho_1, \ldots \rho_m, -x) (-x)^{z-1} \, dx$$
$$= \frac{\Gamma(z) \Gamma(\alpha - z)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\rho_1) \ldots \Gamma(\rho_m)}{\Gamma(\rho_1 - z) \ldots \Gamma(\rho_m - z)} \varpi (z, \alpha, \rho_1, \ldots \rho_m).$$

When $\alpha = \rho_1$, the expression on the left-hand side, and therefore that on the right-hand side must involve $\rho_2 \ldots \rho_m$ only. Thus, when $\alpha = \rho_1$, $\varpi (z, \alpha, \rho_1, \ldots \rho_m)$ involves $\rho_2 \ldots \rho_m$ only. It must therefore be a function of $\alpha - \rho_1, \ldots \alpha - \rho_m$. Not only so, but it cannot involve these quantities at all; for when $\alpha = \rho_1$, ϖ will be a function of $\rho_1 - \rho_2, \ldots \rho_1 - \rho_m$, and yet it is independent of ρ_1; and so on when $\alpha = \rho_2, \ldots \alpha = \rho_m$. Thus ϖ is a function of z, simply periodic of period unity, independent of $\alpha, \rho_1, \ldots \rho_m$ and m.

Let $m = 1$, $\alpha = \rho_1 = 1$; then $\mathrm{F}_m (\alpha, \rho_1, \rho_2, \ldots \rho_m, -x)$ becomes e^{-x}, and the integral becomes

$$\frac{\iota}{2 \sin \pi z} \int e^{-x} (-x)^{z-1} \, dx = \Gamma(z).$$

Thus $\varpi(z) = 1$, and we have finally for all values of $\alpha, \rho_1, \ldots \rho_m$ and z the identity

$$\frac{\iota}{2 \sin \pi z} \int F_m(\alpha, \rho_1, \ldots \rho_m, -x)(-x)^{z-1} dx = \Gamma(z) . \frac{\Gamma(\alpha - z)}{\Gamma(\alpha)} . \frac{\Gamma(\rho_1) \ldots \Gamma(\rho_m)}{\Gamma(\rho_1 - z) \ldots \Gamma(\rho_m - z)} .^*$$

This identity is the direct generalisation of the identity

$$\frac{\iota}{2 \sin \pi z} \int e^{-x} (-x)^{z-1} dx = \Gamma(z),$$

and we may therefore expect to be able to use it to extend our former process of "summing" asymptotic series.

§ 37. We may, in fact, show at once that we can sum any series of convergency zero $f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots$, in which $\underset{n=\infty}{\mathrm{L}t}\, a_n = (n!)^k$, where k is any finite quantity.

For this purpose we put $\alpha = \rho_1 = \ldots = \rho_m = 1$;

$$F_m(x) = 1 + \frac{x}{(1!)^m} + \ldots + \frac{x^r}{(r!)^m} + \ldots, \text{ and we have } \frac{\iota}{2\pi} \int \frac{\pi^m F_m(-x)}{[\sin \pi z]^m} (-x)^{z-1} dx = [\Gamma(z)]^m.$$

Then, with our former notation, we take the auxiliary function

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n, \text{ where } 1/c_n = [\Gamma(n + \theta)]^m = \frac{\iota}{2\pi} \int \frac{\pi^m F_m(-x)}{[\sin \pi\theta]^m} (-)^{nm} (-x)^{\theta - 1} (-x)^n \, dx.$$

And now $f(z)$ is defined by the integral

$$\frac{\iota}{2\pi} \int G(-xz) \frac{\pi^m F_m(-x)(-x)^{\theta-1}}{[\sin \pi\theta]^m} dx,$$

in which

$$G(u) = \sum_{n=0}^{\infty} (-)^{nm} a_n c_n u^n.$$

We take $m > k$, and then $G(u)$ will be an integral function.

For $\underset{n=\infty}{\mathrm{L}t}\, a_n c_n = n^{n(k-m)} e^{-n(k-m)+\ldots}$; and therefore $\underset{n=\infty}{\mathrm{L}t} \sqrt[n]{a_n c_n} = 0$.

* In connection with the proof of this formula, the reader may with advantage refer to:—

MELLIN, 'Acta Mathematica,' 8, pp. 37–80; 9, pp. 137–166; 15, pp. 317–384.
" 'Acta Societatis Fennicæ,' t. 20, pp. 1–115.
POINCARÉ, "American Journal," vol. 7, pp. 203–258.
PINCHERLE, 'Accad. dei Lincei,' ser. iv., t. 4, pp. 694–700.
POCHAMMER, 'Mathematische Annalen,' Bd. 38, pp. 586–597; Bd. 41, pp. 197–218.
" 'Crelle,' Bd. 71, pp. 316–352.
ORR, 'Cambridge Phil. Trans.,' vol. 17, pp. 182–199.

We can thus sum any natural series of convergency zero whose n^{th} coefficient is of the same order as a finite power of $n!$*

§ 38. But we can go further than this: we can construct inverse functions which will enable us to sum any series of convergency zero.

For we have seen that

$$\frac{\iota}{2\pi}\int \frac{\pi^{2m}\mathrm{F}_{2m}(-x)}{(\sin \pi\theta)^{2m}}(-x)^{n+\theta-1}\,dx = [\Gamma(n+\theta)]^{2m}.$$

Suppose, now, that we construct the function $f(x) = \sum\limits_{m=1}^{\infty} \dfrac{\pi^{2m}\mathrm{F}_{2m}(-x)}{(\sin \pi\theta)^{2m}}$.

This function will be an integral function of x, for we have

$$f(x) = \sum_{r=0}^{\infty}(-x)^r \sum_{m=1}^{\infty}\left[\frac{\pi}{r!\sin \pi\theta}\right]^{2m}$$

$$= \sum_{r=0}^{\infty}(-x)^r \frac{\left(\dfrac{\pi}{r!\sin \pi\theta}\right)^2}{1-\left(\dfrac{\pi}{r!\sin \pi\theta}\right)^2},$$

which is absolutely convergent for all values of $|x|$.

But, if we operate by our integral on this function, we have $\dfrac{\iota}{2\pi}\displaystyle\int f(x)(-x)^{n+\theta-1}\,dx$ $= \sum\limits_{m=1}^{\infty}[\Gamma(n+\theta)]^{2m}$; and the function $\sum\limits_{m=1}^{\infty}[\Gamma(n+\theta)]^{2m}$ is infinite if $\mathfrak{R}(n+\theta)$ be positive.

If, now, we take $f(x) = \sum\limits_{m=1}^{\infty}\dfrac{b_m\pi^{2m}\mathrm{F}_{2m}(-x)}{(\sin \pi\theta)^{2m}}$, where b_m is so chosen as to make the series $\sum\limits_{m=1}^{\infty} b_m[\Gamma(n+\theta)]^{2m}$ converge for all finite values of n, it is obvious that $f(x)$ will itself converge for all values of x, and so be an integral function.

We may now take for the associated function

$$\chi(z) = c_0 + c_1 z + \ldots + c_n z^n + \ldots, \text{ where } c_n^{-1} = \sum_{m=1}^{\infty} b_m[\Gamma(n+\theta)]^{2m};$$

and by suitable choice of the coefficients b we may make c_n vanish to an order as great as we please.

We can then sum the series $a_0 + a_1 z + \ldots + a_n z^n + \ldots$, where a_n is infinite with n to an order as high as we please. In other words, we have invented the analytical machinery necessary to sum any (natural) asymptotic series.

§ 39. As an example, suppose that we wish to sum a series $a_0 + a_1 z + \ldots + a_n z^n + \ldots$, where a_n is infinite like $e^{\alpha\{\Gamma(n)\}^2}$, where $0 < \alpha < 1$.

* This theorem corrects a mistake in my paper, 'Theory of the Gamma Function,' p. 112.

With our previous notation we take $b_m = \frac{1}{m!}$ and

$$f(x) = \sum_{m=1}^{\infty} \frac{\pi^{2m} F_{2m}(-x)}{m!\,(\sin \pi\theta)^{2m}} = \sum_{r=0}^{\infty} (-x)^r \sum_{m=1}^{\infty} \frac{1}{m!} \left[\frac{\pi}{r!\sin \pi\theta}\right]^{2m} = \sum_{r=0}^{\infty} (-x)^r \left[e^{\left(\frac{\pi}{r!\sin \pi\theta}\right)^2} - 1\right]$$

so that $f(x)$ is a transcendental integral function.

Then we have $\frac{\iota}{2\pi} \int f(x)\,(-x)^{n+\theta-1}\,dx = \sum_{m=1}^{\infty} \frac{[\Gamma(n+\theta)]^{2m}}{m!} = e^{[\Gamma(n+\theta)]^2} - 1.$

We take the associated function

$$\chi(z) = c_0 + c_1 z + \ldots + c_n z^n + \ldots \text{ where } c_n^{-1} = e^{[\Gamma(n+\theta)]^2} - 1;$$

and the integral function

$$G(z) = a_0 c_0 + \ldots + a_n c_n z^n + \ldots \text{ in which } \mathop{\mathrm{Lt}}_{n=\infty} a_n c_n = \mathop{\mathrm{Lt}}_{n=\infty} \frac{e^{\alpha[\Gamma(n)]^2}}{e^{[\Gamma(n+\theta)]^2}} = 0.$$

Then the sum of the series will be represented by*

$$\frac{\iota}{2\pi} \int G(-xz)\,f(x)\,(-x)^{\theta-1}\,dx.$$

§ 40. We have now, by means of the generalised exponential functions, given the machinery by which we may expect to be able to "sum" a natural asymptotic series of any order.

It may be proved just as for the fundamental exponential process that the series and the function derived from it have asymptotic equality of the arithmetic type.

Moreover, if we regard the series as having a finite radius of convergency, on which one or more singularities lie, which has shrunk indefinitely, we, as it were, magnify it again by means of the function $F_m(x)$ so as to obtain the associated function

$$G(u) = \sum_{n=0}^{\infty} (-)^{nm} a_n c_n u^n$$

whose radius of convergency is infinite.

The alternative process consists in successive magnifications by means of the function e^x.

These two processes will in general lead to different results: in each case we shall obtain functions with $z = 0$ as an essential singularity; both functions will have the

* When we take $b_m^{-1} = (m!)^{\frac{1}{s}}$, we have

$$\frac{\iota}{2\pi} \int f(x)\,(-x)^{n+\theta-1}\,dx = \sum_{m=1}^{\infty} \frac{[\Gamma(n+\theta)]^{2m}}{(m!)^{1/s}};$$

and, when n is large, the series is, by a theorem due to STOKES, infinite like exp. $\left\{\frac{1}{s}[\Gamma(n+\theta)]^{2s}\right\}$ to the first approximation. We thus sum any series for which a_n is of order exp. $\{[\Gamma(n)]^r\}$, by taking s greater than r.

same arithmetically asymptotic expansion. But the expansion in all probability will not be valid in the two cases along the same lines or within the same areas tending to $z = 0$. Moreover, such a result is not surprising. The original series, except from the point of view of the computer, had no meaning; it did not define an analytic function over any area of the plane of the complex variable and therefore could not uniquely represent such a function. We have, however, now given two processes (out of an infinite number) by which we may conceive the series to define an analytic function, and the functions thus defined each satisfy all that the computer can demand.

§ 41. It will, perhaps, elucidate the theory which has been developed if we give two actual examples of its application.

We will first investigate the Maclaurin sum formula, which gives an asymptotic value for $\sum_{n=1}^{m-1} \phi(n)$ when m is large, under certain restrictions as to the nature of the function $\phi(n)$.

In the first place it is evident that such restrictions must exist: the function must either be uniform or be limited to a definite branch of a multiform function; and, as z takes increasing integral values, $\phi(z)$ must increase uniformly.

We will assume, therefore, that $\phi(z)$ is an integral function, which may be represented by a TAYLOR's series, $a_0 + a_1 z + \ldots + a_r z^r + \ldots$

Then, if the integral be taken along a contour embracing an axis in the positive half of the z plane, we shall have, by the usual expression for the gamma function,

$$\sum_{n=1}^{m} \phi(n) = \sum_{n=1}^{m} \frac{\iota}{2\pi} \int \sum_{r=0}^{\infty} a_r \Gamma(1+r)\, e^{-nz} (-z)^{-r-1}\, dz = \frac{\iota}{2\pi} \int e^{-z} \frac{1 - e^{-mz}}{1 - e^{-z}} \sum_{r=0}^{\infty} \frac{a_r \Gamma(1+r)}{(-z)^{r+1}}\, dz.$$

Suppose now that the series $\sum_{r=0}^{\infty} a_r \Gamma(1+r) z^r$ has a finite radius of convergence ρ.

Then $\sum_{r=0}^{\infty} a_r \Gamma(1+r)(-z)^{-r-1}$ will be the expansion of a function convergent outside a circle of radius ρ^{-1}.

We can always make the bulb of the contour along which the fundamental integral is taken expand so as to entirely include this circle of radius ρ^{-1}, and the subsequent integral will then be finite.

Let now $Z = \frac{\iota}{2\pi} \int \frac{e^{-z}}{1 - e^{-z}} \sum_{r=0}^{\infty} \frac{a_r r!}{(-z)^{r+1}}\, dz$, so that Z is a definite finite quantity depending on the coefficients in the expansion of $\phi(z)$.

Then $\sum_{n=1}^{m-1} \phi(n) = Z - \frac{\iota}{2\pi} \int \frac{e^{-(m+1)z}}{1 - e^{-z}} \sum_{r=0}^{\infty} \frac{a_r \Gamma(1+r)}{(-z)^{r+1}}\, dz.$

The second integral may be written in the form

$$\frac{\iota}{2\pi}\int\left\{1+\ldots+(-z)^n\frac{S'_n(1)}{n!}+\ldots\right\}e^{-mz}\sum_{r=0}^{\infty}\frac{a_r\Gamma(1+r)}{(-z)^{r+2}}\,dz,$$

if we postulate that we are reversing the process by which we "sum" an asymptotic series.

The integral is equal to the asymptotic expansion

$$\sum_{n=0}^{\infty}\frac{S_n'(1)}{n!}\frac{\iota}{2\pi}\int e^{-mz}\sum_{r=0}^{\infty}\frac{a_r\Gamma(1+r)}{(-z)^{r+2-n}}\,dz=\sum_{n=0}^{\infty}\frac{S'_n(1)}{n!}\left[\sum_{r=0}^{\infty}\frac{a_r\Gamma(1+r)}{\Gamma(2+r-n)\,m^{n-r-1}}\right]$$

$$=\sum_{r=0}^{\infty}\frac{a_rm^{r+1}}{1+r}+\tfrac{1}{2}\sum_{r=0}^{\infty}a_rm^r+\sum_{n=2}^{\infty}\frac{S_n'(1)}{n!}\left\{\sum_{r=0}^{\infty}a_rm^{r+1-n}\,r\,.\,\overline{r-1}\ldots\overline{r+2-n}\right\}$$

$$=\int^{m}\phi(m)\,dm+\tfrac{1}{2}\phi(m)+\sum_{n=1}^{\infty}\frac{S'_{n+1}(0)}{\overline{n+1}\,!}\frac{d^n}{dm^n}\phi(m).$$

When n is odd $S'_{n+2}(0)=0$: the integral is therefore equal to the asymptotic expansion

$$\int^{m}\phi(m)\,dm+\tfrac{1}{2}\phi(m)+\sum_{n=0}^{\infty}\frac{S'_{2n+2}(0)}{\overline{2n+2}\,!}\frac{d^{2n+1}}{dm^{2n+1}}\phi(m)$$

$$=\int^{m}\phi(m)\,dm+\tfrac{1}{2}\phi(m)+\sum_{n=0}^{\infty}\frac{(-)^n B_{n+1}}{(2n+2)\,!}\frac{d^{2n+1}}{dm^{2n+1}}\phi(m).$$

We have finally the asymptotic equality*

$$\sum_{n=1}^{m-1}\phi(n)=Z+\int^{m}\phi(m)\,dm-\tfrac{1}{2}\phi(m)+\sum_{n=0}^{\infty}\frac{(-)^n B_{n+1}}{(2n+2)\,!}\frac{d^{2n+1}}{dm^{2n+1}}\phi(m).$$

This equality is valid when $\phi(z)$ is a uniform integral function of z such that if it be expanded in the form $a_0+a_1z+\ldots+a_rz^r+\ldots$, the series $\sum_{r=0}^{\infty}a_r\Gamma(1+r)\,z^r$ has a finite or infinite radius of convergency.

We must therefore have $\sqrt[n]{a_n n\,!}$ equal to a finite or zero quantity, so that $a_n^{-\frac{1}{n}}$ must increase as fast as or faster than n. The function $\phi(z)$ must therefore be a function whose "order" is greater than or equal to unity.

In the particular case when the series $\sum_{r=0}^{\infty}a_r\Gamma(1+r)\,z^r$ represents an integral function, we may conveniently express Z in terms of the Riemann ζ functions of negative integral argument.

* In a subsequent paper I shall show that it is better to write this formula in the form

$$\sum_{n=0}^{m-1}\phi(a+n\omega)=Z+\sum_{n=0}^{\infty}\frac{S'_n(a\,|\,\omega)}{n\,!}\frac{d^n}{dx^n}\left[\int^{x}\phi(x)\,dx\right]_{x=m\omega}$$

in order to exhibit its analogy to more general extensions.

For in this case the bulb of the contour integral which expresses Z may be taken so small as not to include the poles of $\frac{1}{e^z - 1}$, and we shall therefore have

$$Z = \frac{\iota}{2\pi} \sum_{r=0}^{\infty} a_r r! \int \frac{dz}{e^z - 1} (-z)^{-r-1} = \sum_{r=0}^{\infty} a_r \zeta(-r) = \sum_{t=0}^{\infty} a_{2t+1} \frac{(-)^{t-1} B_{t+1}}{2t+2}.$$

This series is evidently convergent if $\phi(z)$ is an integral function whose order is greater than or equal to 2, a condition which is equivalent to the convergency for all values of $|z|$ of the series

$$\sum_{r=0}^{\infty} a_r \Gamma(1+r) z^r.$$

It is evident that the Maclaurin sum formula will hold good in many cases in which $\phi(z)$ is not a uniform function. If it be a function like $z^{\frac{1}{\rho}}$ or $z^{\frac{1}{\rho}\log z}$, or either of these functions multiplied by an integral function of order greater than unity, the Maclaurin formula will be valid if we suitably specify the branch of the function considered. Instead of attempting to tabulate such cases, it is perhaps better that we should go back to the genesis of the formulae when they actually arise. Applications of the formulae which will be made subsequently in this memoir will usually be to cases in which $\phi(z)$ has very simple values; and all general formulæ will be tacitly supposed subordinate to what we may call the Maclaurin restrictions.

§ 42. As a second example of the theory of asymptotic series we propose to try and find the function of which the series

$$e^x \left[\frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \ldots + \frac{n!}{x^{n+1}} + \ldots \right]$$

is the asymptotic expansion near the essential singularity $x = \infty$.

We know that, if n be a positive integer,

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$$

where the line integral is taken along any straight line L from the origin to infinity which lies in the half of the z plane to the right hand side of the imaginary axis.

Therefore the given expansion asymptotically represents the function

$$\int_0^\infty (\mathrm{L}) \frac{e^x}{x} \mathrm{G}\left(\frac{t}{x}\right) e^{-t} dt$$

where $\mathrm{G}(u)$ is the function which is represented by the series $\sum_{n=0}^{\infty} u^n$, and the integral is taken along the straight line L.

The given series is therefore the asymptotic expansion of the function

$$f(x) = \int_0^\infty (\mathrm{L}) \frac{e^{x-t}}{x} \cdot \frac{1}{1 - \dfrac{t}{x}} \, dt.$$

Suppose first that the real part of x is positive; then, putting $t = xz$ we have

$$f(x) = \int_0^\infty \frac{e^{x(1-z)}}{1-z} \, dz,$$

the integral being taken along a line along which $\mathfrak{R}(z)$ is positive, that is, along which $\mathfrak{R}(1-z)$ is negative. [These two conditions only differ when we consider a line practically parallel to the imaginary axis and therefore initially excluded.]

Putting $1 - z = -y$, we have $f(x) = -\int_{-1}^\infty \frac{e^{-xy}}{y} \, dy$, the integral being taken along a line along which $\mathrm{R}(y)$ is positive; and therefore

$$f(x) = -\int_{-x}^\infty \frac{e^{-z} \, dz}{z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the integral being taken along a line still in the positive half of the z plane. Thus $f(x) = -\int_\epsilon^\infty \frac{e^{-z} \, dz}{z} - \int_{-x}^{+\epsilon} \frac{e^{-z} - 1}{z} \, dz - \int_{-x}^{\epsilon} \frac{dz}{z}$, where we take $|\epsilon|$ to be very small.

Hence, if γ be EULER's constant,*

$f(x) = \log \epsilon + \gamma - \int_{-x}^{\epsilon} \frac{e^{-z} - 1}{z} \, dz - \log \epsilon + \log(-x) +$ terms which vanish with $|\epsilon|$.

Finally, on making $|\epsilon| = 0$,

$$f(x) = \gamma + \log(-x) + \int_0^x \frac{e^z - 1}{z} \, dz.$$

It will be noticed that the integral (1) obtained for $f(x)$ has a pole along the line of integration so that it has an infinite number of values, all differing by $2\pi\iota$, which are implicitly involved in the logarithmic term.

We see then that, when the real part of x is positive, the given series is the asymptotic expansion of the function $\gamma + \log(-x) + \sum_{r=1}^{\infty} \frac{x^r}{r \, . \, r\,!}$.

Take next the case when the real part of x is negative. As in the first case, the series in the asymptotic expansion of $\int_0^\infty \frac{e^{x(1-z)}}{1-z} \, dz$, the integral being taken along a line for which $\mathfrak{R}(z)$ is negative.

Thus it is the expansion of

$$-\int_{-1}^\infty \frac{e^{-xy}}{y} \, dy \text{ along a line for which } \mathfrak{R}(y) \text{ is negative}$$

$$= -\int_{-x}^\infty \frac{e^{-z} \, dz}{z} \text{ along a line for which } \mathfrak{R}(z) \text{ is positive.}$$

* See the author's paper, 'Messenger of Mathematics,' vol. 29, pp. 98 and 99.

On pursuing the same course as before we find that the given series is the asymptotic expansion of the function

$$\gamma + \log(-x) + \int_0^x \frac{e^z - 1}{z}\, dz.$$

But there is the important difference that now the integral which has led to this result has no pole along the line of integration. And $\log(-x)$, instead of being allowed to take any one of an infinite number of values, has such a value that $\log(-x)$ is real when x is real and negative, and has for complex values of x whose real part is negative an amplitude which lies between $\pm\frac{\pi}{2}$.

We see then that the process employed has led, when $\mathfrak{R}(x)$ is positive to an infinite number of functions, all of which have the same asymptotic expansion; and, when $\mathfrak{R}(x)$ is negative, to but one such function.

Evidently when we seek an asymptotic expansion for the function *

$$f(x) = e^{-x} \sum_{r=1}^{\infty} \frac{x^{r+1}}{r \cdot r!}$$

we may say that we get, when $\mathfrak{R}(x)$ is positive,

$$f(x) = \left[1 + \frac{1!}{x} + \ldots + \frac{n!}{x^n} + \ldots\right]$$

for terms like $x\{\gamma + \log(-x) + 2m\pi\iota\}e^{-x}$ are negligible compared with the least term of the asymptotic series; but when $\mathfrak{R}(x)$ is negative, we get

$$f(x) = e^{-x}x\{-\log(-x) - \gamma\} + 1 + \frac{1!}{x} + \ldots + \frac{n!}{x^n} + \ldots$$

in which successive terms are of decreasing order of magnitude.

The zeros of the function $f(x)$ near the essential singularity $x = \infty$, are ultimately along the imaginary axis.

We thus have an illustration of two important propositions :—

(1.) A uniform integral function may admit of asymptotic expansions of different form in different areas with their vertices at its essential singularity.

(2.) These portions of the plane are separated by lines of zeros of the function.

§ 43. Inasmuch as in Parts III. and IV. of this paper we proceed to actually obtain asymptotic expansions satisfying these laws for all the most simple types of

* I was asked to investigate this function by Mr. G. W. WALKER, Fellow of Trinity College, who desired to compute it in certain physical researches. Originally I obtained the expansion by considering the differential equation $x^2\frac{dy}{dx} + y = x$, in a way bearing great resemblance to that employed by HORN, 'Crelle,' vol. 120, pp. 17 and 18.

integral functions, we now proceed to sketch the process which will be adopted, and, in the course of our outline, to prove at once the validity of that process and the laws which govern its results.

Suppose, in the first place, that we have the absolutely convergent expansion $F(z) = a_0 + a_1 z + \ldots + a_s z^s + \ldots$ in which the coefficients are functions of a variable t, asymptotically given for large values of $|t|$ by expansions of the type $a_n = \frac{b_{n_0}}{t^{n_0}} + \frac{b_{n_1}}{t^{n_1}} + \frac{b_{n_2}}{t^{n_2}} + \ldots$ where the quantities $b_{n_0}, b_{n_1}, \ldots$ are constants and $n_0, n_1, \ldots$ are numbers arranged in ascending orders of magnitude and tending to $+\infty$ as a limit, the first numbers of the series being possibly negative.

Suppose that we substitute these asymptotic values of the coefficients and rearrange the expression for $F(z)$ in powers of $\frac{1}{t}$.

We shall obtain, when $|t|$ is large, an asymptotic equality

$$F(z) = \frac{1}{t^{n_0}}\left[b_{00} + b_{10}z + \ldots + b_{m0}z^m + \ldots\right] + \frac{1}{t^{n_1}}\left[b_{01} + b_{11}z + \ldots + b_{m1}z^m + \ldots\right]$$
$$+ \ldots + \frac{1}{t^{n_s}}\left[b_{0s} + b_{1s}z + \ldots + b_{ms}z^m + \ldots\right] + \ldots$$

This expansion will be arithmetically asymptotic: the computer would use it to calculate $F(z)$ for given values of z and t when $|t|$ is large.

The series which enter as coefficients will be, in all probability, divergent; but, as we are looking at the whole matter from the point of view of the computer, we are at liberty to "sum" them by the methods which have been developed in the present part of this memoir.

If, as will be the case in the applications which we subsequently make of this theory, these series have a finite radius of convergence, we can "sum" them each to a definite, possibly non-uniform, analytic function; and we shall have an expansion $F(z) = \sum_{s=0}^{\infty} \frac{f_s(z)}{t^{n_s}}$, which will satisfy POINCARÉ's definition of arithmetically asymptotic dependence. We shall thus have obtained a unique asymptotic expansion for the function $F(z)$. The case in which the series of the type

$$b_{0s} + b_{1s}z + \ldots + b_{ms}z^m + \ldots$$

have zero radius of convergence does not arise. In such a case we should be able to obtain an infinite number of functions, of which these series are the asymptotic expansions, and we should have the absurdity that the asymptotic expansion of $F(z)$ in ascending powers of $\frac{1}{t}$ is not unique.

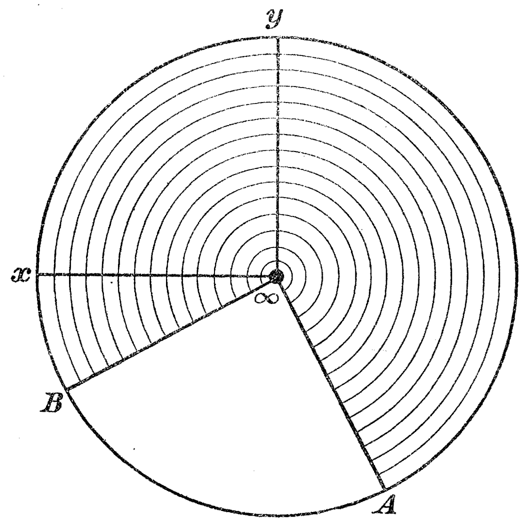
§ 44. A function cannot, as has already been stated, be represented by the same asymptotic expansion for all values of z in the neighbourhood of $z = \infty$, unless the function is an integral function of z^{-1}, and the series absolutely convergent.

For unless J (z) and the series $a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} + \ldots$ have such a character, it is impossible that $\mathrm{L}t\, z^n \left[\mathrm{J}(z) - a_0 - \frac{a_1}{z} - \ldots - \frac{a_n}{z^n}\right]$ should, as z approaches infinity, always tend uniformly to zero, whatever be the argument of z.

But, as we proceed to show, a uniform function of finite "genre," with an essential singularity at infinity can, in general, be represented by one or more asymptotic expansions valid for all points near infinity except those in the immediate vicinity of the zeros and poles of the function.

Two different asymptotic expansions cannot exist within the same region, and the regions are separated by the lines or areas of zeros or poles of the function. The theorem is true whether the function be a quotient of repeated or non-repeated integral functions, with zeros of simple or multiple sequence.

We need only consider the case of integral functions—the general theorem will follow, since every function of the type just mentioned can be represented as a quotient of two integral functions of finite genre.



The zeros of the function must proceed according to fixed laws, and therefore, in our diagram of the region near infinity, they will mass themselves infinitely close together as we approach infinity itself. They will therefore form certain lines (not necessarily straight) or areas of ultimate singularity. If the areas entirely surround $z = \infty$ there will be no asymptotic expansion possible. We thus assume that there exists an area such as ∞ AB, non-shaded in the figure, within which, if the radius ∞ A is sufficiently small, there are no zeros of $f(z)$.

Suppose first, that the zeros of the function form a single simple sequence, and are non-repeated; then it may be written

$$\mathrm{F}(z) = e^{\mathrm{H}(z)} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) e^{\sum_{m=1}^{\rho-1} \frac{1}{m}\left(\frac{z}{a_n}\right)^m}\right] = e^{\mathrm{H}(z)}\, \phi(z), \text{ (say)},$$

where ρ is the "genre" (independent of n), and H (z) is a holomorphic function.

Suppose that z lies between circles of radii $|a_n|$ and $|a_{n+1}|$ where n is very large, then those terms of the product $\phi(z)$ for which $|z| < a_{n+1}$ may be written, as in the proof of WEIERSTRASS' fundamental theorem,

$$e^{\mathrm{P}_1(z)}$$

where $\mathrm{P}_1(z)$ is a function represented by a series of positive powers of z. For those

terms which correspond to the first n zeros, we can expand $\log\left(1-\frac{z}{a_s}\right)$ in the form

$$\log z - \log(-a_s) - \frac{a_s}{z} - \tfrac{1}{2}\left(\frac{a_s}{z}\right)^2 - \dots$$

We thus obtain $F(z) = e^{H(z)}\phi(z) = e^{P(z)}$, where, when $|a_n| < |z| < |a_{n+1}|$, $P(z)$ is an absolutely convergent double series of positive and negative powers of z, together with logarithmic terms.

Now, unless z be in the immediate vicinity of the zeros of the function, this expression, considered from the point of view of divergent summable series, will be valid for *all* values of $|z|$. For, when $|z| > a_n$, the expression of $\log\left(1-\frac{z}{a_n}\right)$ in the form

$$-\left[\frac{z}{a_n} + \dots + \frac{z^s}{sa_n^s} + \dots\right]$$

still exists as a divergent summable series outside the circle of radius $|a_n|$ for all points except those near a_n. Therefore the form of $P(z)$ exists continuously as $|z|$ increases, provided we do not cross the line of, or come within the immediate vicinity of, the zeros of the function. And thus, if we treat the series entering into the expression of $P(z)$ as series which are summable though divergent, *the expansion will be independent of n.*

Now the expansion may be written

$$\sum_{r=-\infty}^{\infty}\left[\sum_{m=1}^{n}\phi_r(m)\right]z^r,$$

where $\phi_r(m)$ is a function of m which depends also on r. Expand $\sum\limits_{m=1}^{n}\phi_r(m)$ asymptotically in a series of successive differentials of $\phi_r(n)$ by the Maclaurin sum formula, and rearrange the series.

We shall get

(A) a certain series of positive and negative powers of z, each multiplying terms like $\int^n \phi_r(m)\,dm$; and

(B) an expansion consisting of a finite number of positive and an infinite number of negative powers of z, each associated with a constant arising from a corresponding Maclaurin expansion.

The other terms depend upon n and vanish identically; the coefficient of each Bernouillian number is zero.

When we apply the processes of divergent summation which have been previously developed, the series which forms the group (A) of terms will reduce to a definite

(possibly non-uniform) function $\psi(z)$ (say). The remaining (B) terms form the summable divergent series

$$\sum_{s=-p}^{\infty} \frac{A_p}{z^s}.$$

[There will only be a finite number of positive powers of z since the *genre* of the function is finite.]

We have then

$$F(z) = e^{\psi(z) + \sum_{s=-p}^{\infty} \frac{A_s}{z^s}}.$$

In other words

$$\log F(z) - \psi(z)$$

admits the asymptotic expansion $\sum_{s=-p}^{\infty} \frac{A_s}{z^s}$ valid for all points but those in the vicinity of the zeros of $F(z)$.

§ 45. The process just sketched will become much more clear when it is applied to various particular cases as in the following pages. The proof may, by mere verbal alterations, be extended so as to include functions of simple sequence with repeated zeros.

A function with a finite number of simple sequences of zeros can be expressed as a product of functions, each with a single simple sequence. The logarithm of each of these functions will admit an asymptotic expansion, and the sum of such expansions will be the asymptotic expansion for the logarithm of the function. But terms of the category $\psi(z)$ may be of different weight in different regions, separated by bands of zeros, and thus the asymptotic expansions may differ in such regions, as has previously been seen in the case of the integral function

$$e^{-x} \sum_{r=1}^{\infty} \frac{x^{r+1}}{r \, . \, r!}.$$

§ 46. The general theorem which has just been given may be proved *pari passu* for integral or meromorphic functions with multiple sequence. We refrain from formal proof, as the consideration of such functions is omitted from the subsequent development of this paper.

Neither do I make any attempt to consider functions of infinite order, or expansions near isolated essential singularities of uniform functions. The difficulties which arise are all subordinate to the main necessity of limiting the type of function under consideration; it seems doubtful whether it is possible to give any general theorem concerning integral functions and their behaviour near infinity, which will apply to every function which can be constructed. For exceptional classes must always be infinite in number compared with those which can be formally defined.

PART III.

The Asymptotic Expansion of Simple Integral Functions.

§ 47. We now proceed to consider in detail simple integral functions. After the discussion given in Part I., we may confine ourselves to functions with a single sequence of zeros.

We shall find that such functions divide themselves naturally into three groups :—

(1) Functions whose order is less than unity,

(2) Functions of non-integral order greater than unity,

(3) Functions of integral order greater than unity.

In connection with each group of functions with algebraic sequence of zeros we first consider a standard type with which all functions of the group may be compared.

These standard functions are

$$P_\rho(z) = \prod_{n=1}^{\infty} \left[1 + \frac{z}{n^\rho}\right], \text{ where } \rho > 1.$$

$$Q_\rho(z) = \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^{1/\rho}}\right) e^{-\frac{z}{n^{1/\rho}} + \dots + \frac{(-)^p z^p}{pn^{p/\rho}}}\right], \text{ where } \rho \text{ is} > 1 \text{ and not integral, and } p$$

is an integer such that $p + 1 > \rho > p$.

$$R_\rho(z) = \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^{1/\rho}}\right) e^{-\frac{z}{n^{1/\rho}} + \dots + \frac{(-z)^\rho}{\rho n}}\right], \text{ where } \rho \text{ is an integer} \gtrless 1.$$

For the logarithms of each of these functions we obtain in turn the complete asymptotic expansion near $z = \infty$. We then show how all functions of the same order with algebraic sequence of zeros yield by the same method similar asymptotic expansions. And we indicate how it is possible to apply the same methods to wide classes of simple functions with a transcendental sequence of zeros.

§ 48. The constants which enter into the analysis arise from the Maclaurin sum formula (§ 41), which may for our present purpose be written

$$\sum_{n=1}^{m-1} \phi^s(n) = \int_{\gamma_s}^{m} \phi^s(n)\, dn - \tfrac{1}{2}\phi^s(m) + \frac{B_1}{2!}\frac{d}{dm}\phi^s(m) + \dots$$
$$+ (-)^t \frac{B_{t+1}}{(2t+2)!}\frac{d^{2t+1}}{dm^{2t+1}}\phi^s(m) + \dots,$$

s being any integer, positive or negative.

What we have called the Maclaurin restrictions for the function $\phi(z)$ are always

supposed to apply. We shall call γ_s the Maclaurin integral-limit for $\phi^s(n)$. We shall also put $F_s = -\int^{\gamma_s}\phi^s(n)\,dn$, and call F_s the s^{th} Maclaurin constant for $\phi(n)$.

When $s = 0$, we have the formula

$$\sum_{n=1}^{m-1}\log\phi(n) = \int_{\gamma_0}^{m}\log\phi(n)\,dn - \tfrac{1}{2}\log\phi(m)\ldots + (-)^t\frac{B_{t+1}}{(2t+2)!}\frac{d^{2t+1}}{dm^{2t+1}}\log\phi(m)\ldots,$$

where γ_0 is the Maclaurin integral-limit for $\log\phi(n)$.

We put $\log F_0 = \int^{\gamma_0}\log\phi(n)\,dn$, and call F_0 the absolute Maclaurin constant for $\phi(n)$.

When s is a positive integer and $\underset{n=\infty}{Lt}\,[\phi^{-s}(n)] = 0$, it is evident that $\gamma_{-s} = -\infty$ and $F_{-s} = 0$.

§ 49. In the particular case when $\phi(n) = n^\rho$, ρ being real or complex, the Maclaurin constants are particular cases of RIEMANN'S ζ function.

For, for all values of s,

$$\sum_{n=1}^{m-1}\frac{1}{n^s} = \zeta(s) + \frac{1}{(1-s)\,m^{s-1}} - \frac{1}{2m^s} - \sum_{t=1}^{\infty}\binom{-s}{2t}\frac{(-)^{t-1}B_t}{(s+2t-1)\,m^{s+2t-1}}.$$

When $s = 1$, we have $\underset{\substack{s=1\\ m=\infty}}{Lt}\left[\zeta(s) + \frac{m^{1-s}}{1-s} - \log m\right] = \gamma.$

We have also the special values

$$\zeta(0) = -\tfrac{1}{2},$$

$$\zeta(s) = \frac{2^{s-1}\pi^s}{s!}B_{s+1}, \text{ when } s \text{ is an even positive integer},$$

$$\zeta(s) = 0, \text{ when } s \text{ is an even negative integer},$$

$$\zeta(s) = \frac{(-)^{t-1}B_{t+1}}{2t+2}, \text{ when } s \text{ is a negative odd integer equal to } -(2t+1).$$

We write, when s is any quantity real or complex, $\zeta(-s) = F(s)$, unless $s = 1$, in which case we put $\gamma = F(s)$.

Simple Integral Functions of Finite Order Less than Unity.

§ 50. Before we proceed to consider the general theory of the asymptotic expansion of functions typified by $P_\rho(z) = \prod_{n=1}^{\infty}\left[1 + \frac{z}{n^\rho}\right]$, where $\rho > 1$, we will

consider the function $F(z) = \prod_{n=1}^{\infty}\left[1 + \frac{z}{n^2}\right]$, which is known to be equal to $(2\pi)^{-1} z^{-\frac{1}{2}}\{e^{\pi z^{\frac{1}{2}}} - e^{-\pi z^{\frac{1}{2}}}\}$.

We have $\log F(z) = \sum_{n=1}^{m-1} \log\left(1 + \frac{z}{n^2}\right) + \sum_{n=m}^{\infty} \log\left(1 + \frac{z}{n^2}\right)$, and, if $m > |z| \gtrless m - 1$, we obtain on expanding the logarithms

$$\log F(z) = (m-1)\log z - 2\sum_{n=1}^{m-1}\log n + \sum_{n=1}^{m-1}\left[\frac{n^2}{z} - \ldots + \frac{(-)^{s-1} n^{2s}}{sz^s} + \ldots\right]$$

$$+ \sum_{n=m}^{\infty}\left[\frac{z}{n^2} - \ldots + \frac{(-)^{s-1} z^s}{sn^{2s}} + \ldots\right].$$

If now we employ the arithmetic asymptotic approximations given by the Maclaurin sum formula for $\log\{(m-1)!\}$, $\sum_{n=m}^{\infty} \frac{1}{n^{2s}}$, and $\sum_{n=1}^{m-1} n^{2s}$, we obtain, in the limit when k is infinite,

$$\log F(z) = (m-1)\log z - 2\left[(m - \tfrac{1}{2})\log m - m + \log\sqrt{2\pi} + \sum_{r=1}^{\infty}\frac{(-)^{r-1} B_r}{2r \,.\, \overline{2r-1}\, m^{2r-1}}\right]$$

$$+ \sum_{s=1}^{k}\frac{(-)^{s-1}}{sz^s}\left\{\frac{m^{2s+1}}{2s+1} - \frac{m^{2s}}{2} + \sum_{r=1}^{\infty}\frac{(-)^{r-1} B_r}{2r\,!}\frac{d^{2r}}{dm^{2r}} m^{2s}\right\}$$

$$+ \sum_{s=1}^{k}\frac{(-)^{s-1} z^s}{s}\left\{\frac{1}{(2s-1)m^{2s-1}} - \frac{1}{2m^{2s}} - \sum_{r=1}^{\infty}\frac{(-)^{r-1} B_r}{2r\,!}\frac{d^{2r}}{dm^{2r}} m^{-2s}\right\},$$

or $$\log F(z) = (m-1)\log z - (2m-1)\log m + 2m - \log 2\pi$$

$$+ \sum_{s=1}^{k}(-)^{s-1}\left\{\frac{m^{2s+1}}{s(2s+1)z^s} + \frac{z^s}{s(2s-1)m^{2s-1}}\right\} + \sum_{s=1}^{k}\frac{(-)^s}{2s}\left\{\frac{m^{2s}}{z^s} - \frac{z^s}{m^{2s}}\right\}$$

$$+ \sum_{r=1}^{\infty}\frac{(-)^{r-1} B_r}{(2r)!}\left\{-2\frac{\overline{2r-2}\,!}{m^{2r-1}} + \sum_{s=1}^{k}\frac{(-)^{s-1}}{sz^s}\frac{d^{2r}}{dm^{2r}} m^{2s} - \sum_{s=1}^{k}\frac{(-)^{s-1} z^s}{s}\frac{d^{2r}}{dm^{2r}} m^{-2s}\right\}$$

where we have re-arranged the terms of our double series in accordance with § 43.

Now by the theory of summable divergent series

$$\sum_{s=1}^{k}\frac{(-)^s}{s}\left\{\frac{m^{2s}}{z^s} - \frac{z^s}{m^{2s}}\right\} = -\log\frac{m^2}{z};$$

and $$-2\frac{\overline{2r-2}\,!}{m^{2r-1}} + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{sz^s}\frac{d^{2r}}{dm^{2r}} m^{2s} = 0.$$

Hence we have, when m is large, the approximation, asymptotic with regard to m,

$$\log F(z) = m\log\frac{z}{m^2} + 2m - \log(2\pi z^{\frac{1}{2}})$$

$$+ 2\sum_{s=1}^{k}(-)^{s-1}\left\{\left(\frac{1}{2s} - \frac{1}{2s+1}\right)\frac{m^{2s+1}}{z^s} + \left(\frac{1}{2s-1} - \frac{1}{2s}\right)\frac{z^s}{m^{2s-1}}\right\},$$

or
$$\log F(z) = -\log 2\pi z^{\frac{1}{2}} + 2m\left\{1 + \sum_{s=-k}^{k} \frac{(-)^s m^{2s}}{(2s+1) z^s}\right\}.$$

Suppose now that $|z|$ is large, and that

$$m^2 = ze^{-i\theta} \text{ where } \theta = \arg z.$$

[This assumes that $|z|$ is a large integer, a restriction which, as will be seen later, can easily be removed.]

Then, when $|z|$ is large, we have asymptotically

$$\log F(z) = -\log 2\pi z^{\frac{1}{2}} + z^{\frac{1}{2}} e^{-\frac{\iota\theta}{2}}\left[2 + \sum_{s=-k}^{k}{}' \frac{(-)^s e^{-s\iota\theta}}{s+\frac{1}{2}}\right].$$

The sum of the FOURIER's series inside the square bracket is, when $-\pi < \theta < \pi$, equal to $\pi e^{\frac{\iota\theta}{2}}$.

Therefore, when $|z|$ is large, we have asymptotically

$$\log F(z) = -\log 2\pi z^{\frac{1}{2}} + \pi z^{\frac{1}{2}}.$$

§ 51. In the preceding investigation we have assumed that the Maclaurin sum formula expresses asymptotically the values, when m is large, of the functions $\log(m!)$ and $\sum_{n=1}^{m-1} n^{2s}$ (s positive or negative).

Accurately we have of course

$$\log \overline{m-1}\,! = (m-\tfrac{1}{2})\log m - m + \log\sqrt{2\pi} + \frac{1}{\iota}\int_0^\infty \frac{dy}{e^{2\pi y}-1}\Big[\log(m+\iota y) - \log(m-\iota y)\Big];$$

$$\sum_{n=1}^{m-1} n^{2s} = \frac{m^{2s+1}}{2s+1} - \frac{m^{2s}}{2} + \frac{1}{\iota}\int_0^\infty \frac{dy}{e^{2\pi y}-1}\Big[(m+\iota y)^{2s} - (m-\iota y)^{2s}\Big], \text{ when } s \text{ is positive};$$

and

$$\sum_{n=m}^{\infty} \frac{1}{n^{2s}} = \frac{1}{(2s-1)m^{2s-1}} + \frac{1}{2m^{2s}} - \frac{1}{\iota}\int_0^\infty \frac{dy}{e^{2\pi y}-1}\Big[(m+\iota y)^{-2s} - (m-\iota y)^{-2s}\Big] \text{ when } s \text{ is positive.}$$

Hence, in the limit when k is infinite,

$$\log F(z) = -\log(2\pi z^{\frac{1}{2}}) + 2m\left\{1 + \sum_{s=-k}^{k}{}' \frac{(-)^s m^{2s}}{z^s(2s+1)}\right\}$$
$$+ \frac{1}{\iota}\int_0^\infty \frac{dy}{e^{2\pi y}-1}\left\{-2\log(m+\iota y) + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}(m+\iota y)^{2s}}{s z^s}\right\}$$
$$- \frac{1}{\iota}\int_0^\infty \frac{dy}{e^{2\pi y}-1}\left\{-2\log(m-\iota y) + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}(m-\iota y)^{2s}}{s z^s}\right\}.$$

This formula is accurate and holds whatever positive integral value m may have.

Unfortunately we may not say that the sum of an infinite series of integrals is equal to the integral whose subject of integration is the function to which the series of subjects of integration can be "summed." The two integrals last written down can then only be evaluated by reducing them to an exceeding complicated extension of the type known as DIRICHLET'S integrals. The analysis is utterly intractable.

If we make $m^2 = ze^{-i\theta}$, and expand the subjects of integration in powers of z, then we can say that the last two terms will not contribute terms whose order of magnitude when z is large is comparable with that of any positive or negative power of z. And, as we know, the sum of these two terms is equal to $\log(1 - e^{-2\pi z^{\frac{1}{2}}})$.

The formula $\log F(z) = \pi z^{\frac{1}{2}} - \log(2\pi z^{\frac{1}{4}})$ is thus asymptotic exactly as the Maclaurin series for $m\,!$ and $\sum_{n=1}^{m-1} n^{2s}$, from which it is derived, are asymptotic. That is, for large values of $|z|$, the expression $z^{-n}\{\log F(z) - \pi z^{\frac{1}{2}} + \log 2\pi z^{\frac{1}{4}}\}$ for all values of n tends to zero as $|z|$ tends to infinity. There is, in fact, POINCARÉ'S arithmetic asymptotic dependence.

The preceding example will serve to show the nature of the asymptotic expansions which we can now proceed to obtain.

§ 52. We consider first the function $P_\rho(z) = \prod_{n=1}^{\infty}\left[1 + \frac{z}{n^\rho}\right]$, where $\rho > 1$. We have

$$\log P_\rho(z) = (m-1)\log z - \rho \sum_{n=1}^{m-1} \log n$$

$$+ \sum_{n=1}^{m-1}\left[\frac{n^\rho}{z} - \ldots + \frac{(-)^{s-1} n^{\rho s}}{s z^s} + \ldots\right]$$

$$+ \sum_{n=m}^{\infty}\left[\frac{z}{n^\rho} - \ldots + \frac{(-)^{s-1} z^s}{s n^{\rho s}} + \ldots\right].$$

Therefore, if we substitute the approximations for $\log \overline{m-1}\,!$, $\sum_{n=m}^{\infty} \frac{1}{n^{\rho s}}$, and $\sum_{n=1}^{m-1} n^{\rho s}$ given by the Maclaurin sum formula, we shall obtain the expansion, arithmetically asymptotic with regard to m,

$$\log P_\rho(z) = (m-1)\log z - \rho\left[(m - \tfrac{1}{2})\log m - m + \log\sqrt{2\pi} + \sum_{r=1}^{\infty} \frac{(-)^{r-1} B_r}{2r\,.\,\overline{2r-1}\, m^{2r-1}}\right]$$

$$+ \sum_{s=1}^{k} \frac{(-)^{s-1}}{s z^s}\left\{\frac{m^{\rho s+1}}{\rho s + 1} - \frac{m^{\rho s}}{2} + \sum_{r=1}^{\infty} \frac{(-)^{r-1} B_r}{\rho s - 2r + 1}\binom{\rho s}{2r} m^{\rho s - 2r + 1} + F(\rho s)\right\}$$

$$+ \sum_{s=1}^{k} \frac{(-)^{s-1} z^s}{s}\left\{\frac{m^{-\rho s+1}}{\rho s - 1} + \frac{m^{-\rho s}}{2} + \sum_{r=1}^{\infty} \frac{(-)^{r-1} B_r}{\rho s + 2r - 1}\binom{-\rho s}{2r} m^{-\rho s - 2r + 1}\right\},$$

or, $$\log P_\rho(z) = (m-1)\log z - \rho(m-\tfrac{1}{2})\log m + \rho m - \rho\log\sqrt{2\pi}$$

$$+ m\sum_{s=-k}^{k}{}' \left\{\frac{(-)^{s-1}}{sz^s}\cdot\frac{m^{\rho s}}{\rho s+1}\right\} + \sum_{s=-k}^{k}{}' \frac{(-)^s m^{\rho s}}{2sz^s} + \sum_{s=1}^{k}\frac{(-)^{s-1}F(\rho s)}{sz^s}$$

$$+ \sum_{r=1}^{\infty}\frac{(-)^{r-1}B_r}{m^{2r-1}}\left\{\frac{-\rho}{2r\,.\,2r-1} + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{\rho s-2r+1}\binom{\rho s}{2r}\frac{m^{\rho s}}{sz^s}\right\}.$$

This expansion is arithmetically asymptotic with regard to m, and the coefficients of various powers of $\frac{1}{m}$ are ultimately to be summable divergent series.

Let now $r = |z|$ be large, and such that $r^{\frac{1}{\rho}}$ lies between m and $m-1$. Then the modulus of $\frac{m^\rho}{r}$ is a quantity which is very nearly equal to unity. We proceed to "sum" the series $\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s\,.\,(\rho s+1)}\cdot\frac{m^{\rho s}}{zs}$, $\sum_{s=-k}^{k}{}' \frac{(-)^s}{2s}\frac{m^{\rho s}}{z^s}$, and

$$\frac{1}{2r!}\frac{d^{2r-1}}{dm^{2r-1}}\left\{-\rho\log m + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}m^{\rho s}}{sz^s}\right\}.$$

Write $t = \log\frac{m^\rho}{r}$, then the first series becomes

$$f(t) = \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s\,.\,(\rho s+1)}e^{s(t-i\theta)}.$$

Thus

$$f(t) = \sum_{s=-k}^{k}{}' (-)^{s-1}\left\{\frac{1}{s} - \frac{\rho}{\rho s+1}\right\}e^{s(t-i\theta)},$$

and $$\rho\frac{\partial f(t)}{\partial t} = \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}\rho}{\rho s+1}e^{s(t-i\theta)}.$$

So that

$$f(t) = -\rho\frac{\partial f(t)}{\partial t} + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}e^{s(t-i\theta)}}{s}.$$

If now we "sum" the last series we obtain

$$f(t) = -\rho\frac{\partial f(t)}{\partial t} + t - i\theta;\ \text{and therefore}$$

$$f(t) = Ae^{-\frac{t}{\rho}} + t - i\theta - \rho,\ \text{where A is independent of } t.$$

When $t = 0$, $$f(t) = \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s\,.\,(\rho s+1)}e^{-si\theta}.$$

Hence $$f(0) = \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s}e^{-si\theta} + \rho\sum_{s=-k}^{k}{}' \frac{(-)^s}{\rho s+1}e^{-si\theta}.$$

Now, by the usual theory of FOURIER's series, $\rho + \sum_{s=-k}^{k}{}' \dfrac{(-)^s}{s+\dfrac{1}{\rho}} e^{-si\theta} = \dfrac{\pi}{\sin\dfrac{\pi}{\rho}} e^{\frac{i\theta}{\rho}}$, provided $-\pi < \theta < \pi$.

Therefore
$$f(0) = -i\theta - \rho + \frac{\pi}{\sin\dfrac{\pi}{\rho}} e^{\frac{i\theta}{\rho}},$$

so that
$$f(t) = \frac{\pi}{\sin\dfrac{\pi}{\rho}} e^{\frac{i\theta}{\rho} - \frac{t}{\rho}} + t - i\theta - \rho.$$

Hence
$$\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s(\rho s+1)} \frac{m^{\rho s}}{z^s} = \frac{\pi}{\sin\dfrac{\pi}{\rho}} \frac{z^{\frac{1}{\rho}}}{m} + \log\frac{m^\rho}{z} - \rho.$$

The second series $\sum_{s=-k}^{k}{}' \dfrac{(-)^s}{2s} \cdot \dfrac{m^{\rho s}}{z^s}$ is at once seen to be equal to $-\frac{1}{2}\log\dfrac{m^\rho}{z}$.

And, since the term by term differential of a summable divergent series is equal to the differential of its sum, the third series vanishes identically for all positive integral values of r.

Thus, when $|z|^{\frac{1}{\rho}}$ lies between m and $m-1$, or possibly is equal to the latter quantity, we have the asymptotic expansion, while $-\pi < \arg z < \pi$,

$$\log P_\rho(z) = (m-1)\log z - \rho(m-\tfrac{1}{2})\log m + \rho m - \rho\log\sqrt{2\pi}$$
$$+ \frac{\pi}{\sin\dfrac{\pi}{\rho}} z^{\frac{1}{\rho}} + m\log\frac{m^\rho}{z} - \rho m - \tfrac{1}{2}\log\frac{m^\rho}{z} + \sum_{s=1}^{k} \frac{(-)^{s-1} F(\rho s)}{s z^s};$$

or,
$$\log P_\rho(z) = \frac{\pi}{\sin\dfrac{\pi}{\rho}} z^{\frac{1}{\rho}} - \rho\log\sqrt{2\pi} - \tfrac{1}{2}\log z + \sum_{s=1}^{k} \frac{(-)^{s-1} F(\rho s)}{s z^s}.$$

Thus, when $|z|$ has any large value, and $-\pi < \arg z < \pi$, we have the arithmetically asymptotic expansion

$$\prod_{n=1}^{\infty}\left\{1 + \frac{z}{n^\rho}\right\} = (2\pi)^{-\frac{\rho}{2}} z^{-\frac{1}{2}} e^{\frac{\pi}{\sin\pi/\rho} z^{1/\rho} + \sum_{s=1}^{\infty} \frac{(-)^{s-1}}{s z^s} F(\rho s)}.$$

§ 53. The approximation represents an arithmetic not a functional equality. It does not vary with the argument of z, and it exists everywhere in the neighbourhood of infinity except at points on or near the line of zeros of the function. Not only so, but at points on the line of zeros of $P_\rho(z)$ which are not in the immediate vicinity of one of its zeros, both the function and the asymptotic series have arithmetic continuity, and therefore the equality will hold at such points. These results accord with the general theory developed in Part II.

The series for $\log \dfrac{\mathrm{P}_\rho(z)\, z^{\frac{1}{2}}\, (2\pi)^{\frac{\rho}{2}}}{\exp\left\{\dfrac{\pi}{\sin \pi/\rho}\, z^{\frac{1}{\rho}}\right\}}$ can be "summed" by the methods of Part II.

The function just written tends to zero near its essential singularity $z = \infty$, and the same will be true of the function which we get by any process of summation. But, in general, the function derived from

$$\sum_{s=1}^{\infty} \frac{(-)^{s-1}}{sz^s} \mathrm{F}(\rho s)$$

will not be equal to the function from which the series has been obtained.

Since $\mathrm{F}(\rho s) = \zeta(-\rho s)$, the series is equal to $\sum\limits_{s=1}^{\infty} \dfrac{(-)^{s-1}}{s} z^s \dfrac{\iota}{2\pi} \Gamma(1 + s\rho) \displaystyle\int \dfrac{(-x)^{\rho s-1}}{e^x - 1}\, dx$, the integral being taken round the fundamental contour of § 24.

The series is thus equal to $\dfrac{\iota\rho}{2\pi} \displaystyle\int \mathrm{G}(x^\rho z) \dfrac{dx}{x(e^x - 1)}$, where $\mathrm{G}(z) = \sum\limits_{s=1}^{\infty} \Gamma(\rho s)\, z^s$.

The series for $\mathrm{G}(z)$ is divergent and of order ρ. The integral is interesting in that, in place of e^{-x}, we have used $(e^x - 1)^{-1}$ as our auxiliary of summation.

§ 54. We now pass on to consider the most general simple integral function with a single sequence of non-repeated zeros, whose order is any number (zero included) less than unity.

The function may be written $\mathrm{F}(z) = \prod\limits_{n=1}^{\infty} \left[1 + \dfrac{z}{\phi(n)}\right]$, where $\sum\limits_{n=1}^{\infty} \dfrac{1}{|\phi(n)|}$ is absolutely convergent. The n^{th} zero, $-\phi(n)$, is a definite function of n and any finite number of given constant quantities.

Suppose that if $r = \phi(n)$, then inversely $n = \psi(r)$.

Let $|z| = \mathrm{R}$ and suppose that m is a large integer such that $m - 1 < \psi(\mathrm{R}) \eqslantless m$.

Then $\log \mathrm{F}(z) = \sum\limits_{n=1}^{\infty} \log\left(1 + \dfrac{z}{\phi(n)}\right) = \sum\limits_{n=1}^{m-1} \log\left(1 + \dfrac{z}{\phi(n)}\right) + \sum\limits_{n=m}^{\infty} \log\left(1 + \dfrac{z}{\phi(n)}\right)$,

so that if we expand the logarithms in convergent series we shall get

$$\begin{aligned} \log \mathrm{F}(z) = (m - 1) \log z &- \sum_{n=1}^{m-1} \log \phi(n) \\ &+ \sum_{n=1}^{m-1} \left[\frac{\phi(n)}{z} - \frac{\phi^2(n)}{2z^2} + \ldots + \frac{(-)^{s-1}\phi^s(n)}{sz^s} + \ldots\right] \\ &+ \sum_{n=m}^{\infty} \left[\frac{z}{\phi(n)} - \frac{z^2}{2\phi^2(n)} + \ldots + \frac{(-)^{s-1} z^s}{s\phi^s(n)} + \ldots\right]. \end{aligned}$$

Now, by the Maclaurin sum formula, if s be positive,

$$\sum_{n=1}^{m-1} \phi^s(n) = \int_{\gamma_s}^{m} \phi^s(n)\, dn - \tfrac{1}{2}\phi^s(m) + \frac{\mathrm{B}_1}{2\,!} \frac{d}{dm} \phi^s(m) \ldots + (-)^t \frac{\mathrm{B}_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \phi^s(m) + \ldots$$

where γ_s is a constant quantity, depending on s and the form of $\phi(n)$, which we have proposed to call the Maclaurin integral limit for $\phi^s(n)$.

If s be negative, we have

$$-\sum_{n=m}^{\infty} \phi^s(n) = \int_{\gamma_s}^{m} \phi^s(n)\, dn - \tfrac{1}{2}\phi^s(n) + \ldots + (-)^t \frac{B_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \phi^s(m) + \ldots,$$

where $\gamma_s = \infty$, there being no term independent of m on the right-hand side.

And

$$\sum_{n=1}^{m-1} \log \phi(n) = \int_{\gamma_0}^{m} \log \phi(n)\, dn - \tfrac{1}{2}\log \phi(m) - \ldots + (-)^t \frac{B_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \log \phi(m) + \ldots$$

Hence, in the limit when $k = \infty$,

$$\begin{aligned}
\log F(z) = (m-1)\log z &- \int_{\gamma_0}^{m} \log \phi(n)\, dn \\
&+ \sum_{s=1}^{k} \left[\frac{(-)^{s-1}}{sz^s} \int_{\gamma_s}^{m} \phi^s(n)\, dn + \frac{(-)^s z^s}{s} \int_{\gamma_{-s}}^{m} \frac{dn}{\phi^s(n)} \right] \\
&- \tfrac{1}{2}\left\{ -\log \phi(m) + \sum_{s=1}^{k} \left[\frac{(-)^{s-1}}{sz^s} \phi^s(m) + \frac{(-)^s z^s}{s\phi^s(m)} \right] \right\} \\
&+ \sum_{t=0}^{\infty} \frac{(-)^t B_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \left\{ -\log \phi(m) + \sum_{s=1}^{k} \frac{(-)^{s-1}\phi^s(m)}{sz^s} - \sum_{s=1}^{k} \frac{(-)^{s-1} z^s}{s\phi^s(m)} \right\}.
\end{aligned}$$

Now, when the limiting values for $k = \infty$ of the summable divergent series are taken,

$$\begin{aligned}
&\sum_{s=1}^{k} \left[\frac{(-)^{s-1}}{sz^s} \phi^s(m) + \frac{(-)^s z^s}{s\phi^s(m)} \right] - \log \phi(m) \\
&\qquad = \log\left(1 + \frac{\phi(m)}{z}\right) - \log\left(1 + \frac{z}{\phi(m)}\right) - \log \phi(m) = -\log z.
\end{aligned}$$

Hence, asymptotically,

$$\begin{aligned}
\log F(z) &= (m - \tfrac{1}{2})\log z - \int_{\gamma_0}^{m} \log \phi(n)\, dn + \sum_{s=1}^{k} (-)^{s-1} \left[\int_{\gamma_s}^{m} \frac{\phi^s(n)\, dn}{s\, z^s} - \int_{\gamma_{-s}}^{m} \frac{z^s dn}{s\phi^s(n)} \right] \\
&= (m - \tfrac{1}{2})\log z - [n \log \phi(n)]_{\gamma_0}^{m} + \int_{\gamma_0}^{m} n \frac{d\phi(n)}{\phi(n)} \\
&\qquad + \sum_{s=1}^{k} (-)^{s-1} \left[\left\{ \frac{\phi^s(n)\, n}{s\, z^s} \right\}_{\gamma_s}^{m} - \left\{ \frac{z^s n}{s\phi^s(n)} \right\}_{\gamma_{-s}}^{m} \right] \\
&\qquad\qquad \sum_{s=1}^{k} (-)^s \left[\int_{\gamma_s}^{m} \frac{n\phi^{s-1}(n)}{z^s} d\phi(n) + \int_{\gamma_{-s}}^{m} \frac{z^s n}{\phi^{s+1}(n)} d\phi(n) \right] \\
&= -\tfrac{1}{2}\log z + [n \log \phi(n)]_{\gamma_0} + \sum_{s=1}^{k} \frac{(-)^s}{s\, z^s} [\phi^s(n)\, n]_{\gamma_s} \\
&\qquad + \int_{\phi(\gamma_0)}^{\phi(m)} \psi(t) \frac{dt}{t} + \sum_{s=1}^{k} (-)^s \left[\int_{\phi(\gamma_s)}^{\phi(m)} \psi(t) \frac{t^{s-1}}{z^s} dt + \int_{\phi(\gamma_{-s})}^{\phi(m)} \frac{z^s \psi(t)}{t^{s+1}} dt \right].
\end{aligned}$$

Now, where $\psi(t)$ is a given form, and such that we can integrate expressions like $\psi(t)\, t^{s-1}\, dt$ (s positive or negative, but integral) we need only carry out this process and "sum" the ensuing series of positive and negative powers of z to obtain the dominant term of the asymptotic expansion of $\log F(z)$. If, however, $\psi(t)$ is not thus formally given, we have to face the difficulty that the lower limits of the definite integrals are different quantities. The lower limits, however, corresponding to negative values of s, are such as to give rise to zero terms. If, then, we consider only indefinite integrals of the type $\int^{\phi(m)} \psi(t)\, t^s \frac{dt}{t}$, and take care that in any transformation of these we do not introduce arbitrary additive constants, we may take the asymptotic expansion in the form

$$\log F(z) = -\tfrac{1}{2}\log z + \int^{\gamma_0} \log \phi(n)\, dn - \sum_{s=1}^{k} \frac{(-)^{s-1}}{s z^s} \int^{\gamma_s} \phi^s(n)\, dn$$
$$+ \int^{\phi(m)} \frac{\psi(t)\, dt}{t} \left\{1 + \sum_{s=1}^{k} \left(\frac{(-)^s}{z^s} t^s + \frac{(-)^s z^s}{t^s}\right)\right\}.$$

§ 55. It is the integral

$$I = \underset{k=\infty}{\mathrm{Lt}} \int^{\phi(m)} \frac{\psi(t)\, dt}{t} \left[1 + \sum_{s=-k}^{k}{}' \left(\frac{-t}{z}\right)^s\right]$$

which gives rise to the dominant term of the asymptotic expansion of $\log F(z)$.

This integral is evidently equal to $\underset{k=\infty}{\mathrm{Lt}} \int^{\phi(m)} \frac{\psi(t)\, dt}{t} \left[\frac{\left(\frac{-t}{z}\right)^{-k} - \left(\frac{-t}{z}\right)^{k+1}}{1 + \frac{t}{z}}\right]$.

Suppose now that $z = re^{i\theta}$ and take $\iota\mu = \log\left(-\frac{t}{z}\right)$, the logarithm, when $t = r$, having a cross-cut along the negative half of the real axis, so that

$$\iota\mu = \log\frac{t}{r} + \pi\iota - \iota\theta, \text{ where } \log\frac{t}{r} \text{ is arithmetic.}$$

Then

$$I = \underset{k=\infty}{\mathrm{Lt}}\, \iota \int^{\frac{1}{2}\log\left\{\frac{\phi(m)}{r}\right\} + \pi - \theta} \psi\left(z\, e^{\iota(\mu-\pi)}\right) \frac{\sin(k+\frac{1}{2})\mu}{\sin\frac{1}{2}\mu}\, d\mu.$$

Now the form of the dominant term I does not depend on the quantity $\log\frac{\psi(m)}{r}$ which vanishes when r is sufficiently large. We have then

$$I = \underset{k=\infty}{\mathrm{Lt}} \int^{\pi-\theta} \iota\psi\left\{z\, e^{\iota(\mu-\pi)}\right\} \frac{\sin(k+\frac{1}{2})\mu}{\sin\frac{1}{2}\mu}\, d\mu$$

an integral of the type first considered by Dirichlet.*

* v. 'Crelle,' vol. 4, pp. 157, *et seq.*

The theory previously developed in Part II. tells us that this integral must, *quâ* function of z, be independent of θ; in other words, when $-\pi < \theta < \pi$, that

$$\underset{k=\infty}{\mathrm{Lt}} \int_{\pi-\theta}^{\pi} \iota\psi\{z e^{\iota(\mu-\pi)}\} \frac{\sin(k+\frac{1}{2})\mu}{\sin\frac{1}{2}\mu} d\mu = 0.$$

But this is precisely DIRICHLET's result: we thus have a valuable verification of our theory.

Finally, then, the dominant term of the asymptotic expansion of $\log F(z)$ is the function

$$f(z) = \underset{k=\infty}{\mathrm{Lt}} \int^{\pi} \iota\psi\{z e^{\iota(\mu-\pi)}\} \frac{\sin(k+\frac{1}{2})\mu}{\sin\frac{1}{2}\mu} d\mu.$$

Since we may evidently change the sign of ι without altering the value of $f(z)$ we have

$$f(z) = \underset{k=\infty}{\mathrm{Lt}} \int^{\pi} \frac{\psi\{z e^{\iota(\pi-\mu)}\} - \psi\{z e^{-\iota(\pi-\mu)}\}}{2\iota} \frac{\sin(k+\frac{1}{2})\mu}{\sin\frac{1}{2}\mu} d\mu.$$

Now ψ is the function inverse to ϕ. If, then, we suppose that ζ and η are determined from the relation $ze^{\iota(\pi-\mu)} = \phi(\zeta + \iota\eta)$, principal values of inverse expressions being taken, and ζ and η being functions of z and μ, we shall have finally $f(z) = \underset{k=\infty}{\mathrm{Lt}} \int^{\pi} \eta \frac{\sin(k+\frac{1}{2})\mu}{\sin\frac{1}{2}\mu} d\mu$, as the simplest form in which we may write $f(z)$.

There is no doubt that it is possible to construct functions $\phi(n)$ for which the preceding analysis will not hold good.* It would appear, however, to be applicable to most of the types of functions which would ordinarily arise, and a more accurate investigation will need the exquisite *finesse* of certain developments of the theory of functions of a real variable.

Note that, for the case in which $ze^{\iota(\pi-\mu)} = (\zeta - \iota\eta)^{\rho}$, we have established that $f(z) = \dfrac{\pi}{\sin\dfrac{\pi}{\rho}} z^{\frac{1}{\rho}}$.

§ 56. The dominant term $f(z)$ of the asymptotic expansion of $\log \prod_{n=1}^{\infty} \left[1 + \dfrac{z}{\phi(n)}\right]$ takes a very simple form for the case in which $\psi(t)$ can, when t is large, be expanded in descending powers of t in the form

$$\psi(t) = t^{\frac{1}{\rho}} \left[a_0 + \frac{a_1}{t} + \frac{a_2}{t^2} + \ldots\right], \text{ where } \rho > 1.$$

We have the asymptotic expansion

* We have assumed, for instance, that we can apply the Maclaurin sum formula to $\sum_{n=1}^{m-1} \phi(n)$, and, therefore, that the conditions of § 41 are satisfied.

$$-\tfrac{1}{2}\log z + \int^{\gamma_0} \log \phi(n)\, dn - \sum_{s=1}^{k} \frac{(-)^{s-1}}{sz^s} \int^{\gamma_s} \phi^s(n)\, dn$$
$$+ \int^{\phi(m)} \frac{\psi(t)\, dt}{t} \left\{ 1 + \sum_{s=1}^{k} \left(\frac{(-)^s t^s}{z^s} + \frac{(-)^s z^s}{t^s} \right) \right\}.$$

Since $\phi(m)$ is a very large quantity we may utilise the expansion of $\psi(t)$. The integral last written is, therefore, equal to

$$\sum_{r=0} a_r \int^{\phi(m)} \frac{dt}{t^{r+1-\frac{1}{\rho}}} \left\{ 1 + \sum_{s=1}^{k} \left(\frac{(-)^s t^s}{z^s} + \frac{(-)^s z^s}{t^s} \right) \right\}$$
$$= \sum_{r=0} \frac{a_r}{\phi^{r-\frac{1}{\rho}}(m)} \left[-\frac{1}{r-\frac{1}{\rho}} + \sum_{s=-k}^{k}{}' \frac{(-)^s}{z^s} \frac{\phi^s(m)}{s-r+\frac{1}{\rho}} \right]$$
$$= \sum_{r=0} \frac{a_r}{\phi^{r-\frac{1}{\rho}}(m)} \left[\frac{\pi}{\sin \pi \left(\frac{1}{\rho} - r \right)} \left\{ \frac{z}{\phi(m)} \right\}^{\frac{1}{\rho}-r} \right]$$
$$= \sum_{r=0} \left[(-)^r a_r \frac{\pi}{\sin \frac{\pi}{\rho}} z^{\frac{1}{\rho}-r} \right]$$
$$= \frac{\pi}{\sin \frac{\pi}{\rho}} z^{\frac{1}{\rho}} \left[\frac{\psi(-z)}{(-z)^{\frac{1}{\rho}}} \right].$$

If, then, we introduce the Maclaurin constants, $\log F_0 = \int^{\gamma_0} \log \phi(n)\, dn$, $-F_s = \int^{\gamma_s} \phi^s(n)\, dn$, we shall obtain the asymptotic expansion

$$\prod_{n=1}^{\infty} \left[1 + \frac{z}{\phi(n)} \right] = F_0 z^{-\frac{1}{2}} \exp \left\{ \frac{\pi z^{\frac{1}{\rho}}}{\sin \frac{\pi}{\rho}} \left[\frac{\psi(-z)}{(-z)^{1/\rho}} \right] + \sum_{s=1}^{\infty} \frac{(-)^{s-1} F_s}{sz^s} \right\}.$$

Such values of the many-valued functions introduced are to be taken as would be indicated by the analysis.

§ 57. It is evident that the investigation of § 54 applies to all simple integral functions whose primary factors need no exponential to ensure convergency. Thus it includes all simple functions of order $\frac{1}{\rho}$, where ρ is real positive and ≥ 1 with algebraic zeros. It includes all simple functions with non-algebraic zeros of the type given by $a_n = [an^\rho + bn^{\rho_1} + \ldots] (l^\tau n)^\sigma$, where τ and ρ are both real positive and > 1, σ is positive or negative; where $l^\tau(n)$ denotes $\log \{\log \{\ldots n\}\}\ldots\}$, these being τ repetitions of the logarithm, and where $\rho, \rho_1, \ldots$ are decreasing quantities tending to $-\infty$ as a limit.

But, because of the validity of the Maclaurin sum formula, it includes simple functions with very rapid convergence—such as those for which

$$a_n = e^{e^{\cdot^{\cdot^{\cdot^{e^n}}}}} \times \left(\text{function of } n \text{ of lower order than } e^{e^{\cdot^{\cdot^{\cdot^{e^n}}}}}\right).$$

§ 58. We can now extend the result which was obtained in § 52, and find an asymptotic approximation for a simple integral function with an algebraic sequence of zeros, that is to say, of a function of which the n^{th} zero, $-a_n$, admits, when n is large, an expansion of the form $a_n = n^\rho + b_1 n^{\rho-\epsilon_1} + b_2 n^{\rho-\epsilon_2} + \ldots$, where ρ is greater than unity and the quantities $\epsilon_1, \epsilon_2, \ldots$ are real positive and in ascending order of magnitude.

We take $a_n = \phi(n) = r$ so that, by reversion of series,

$$n = r^{\frac{1}{\rho}} - \frac{b_1}{\rho} r^{\frac{1-\epsilon_1}{\rho}} + \frac{\rho + 1 - 2\epsilon_1}{\rho^2} \cdot \frac{b_1^2}{2} r^{\frac{1-2\epsilon_1}{\rho}} - \frac{b_2}{\rho} r^{\frac{1-\epsilon_2}{\rho}} + \ldots = \psi(r) \text{ (say)}.$$

Since, when s is positive, we may expand a_n^s directly by the binomial theorem, we have, when m is large,

$$\sum_{n=1}^{m-1} a_n^s = \sum_{n=1}^{m-1}\left[n^{\rho s} + s b_1 n^{\rho s-\epsilon_1} + \frac{s \,.\, s-1}{1\,.\,2} n^{\rho s - 2\epsilon_1} + s b_2 n^{\rho s - \epsilon_2} + \ldots\right]$$

$$= \int^m \phi^s(n)\, dn + \mathrm{F}(\rho s) + s b_1 \mathrm{F}(\rho s - \epsilon_1) + \frac{s\,.\,s-1}{2} b_1^2 \mathrm{F}(\rho s - 2\epsilon_1) + s b_2 \mathrm{F}(\rho s - \epsilon_2) + \ldots$$

$$- \frac{\phi^s(m)}{2} + \sum_{t=0}^{\infty} \frac{(-)^t \mathrm{B}_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \phi^s(m)$$

$$= \int^m \phi^s(n)\, dn + \mathrm{F}(\rho s) + \mathrm{Z}\left(\rho, s\,;\, \begin{matrix}\epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots\end{matrix}\right) - \frac{\phi^s(m)}{2} + \sum_{t=0}^{\infty} \frac{(-)^t \mathrm{B}_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \phi^s(m),$$

where $\mathrm{Z}\left(\rho, s\,;\, \begin{matrix}\epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots\end{matrix}\right)$ is a definite finite quantity vanishing with the quantities b, which can be expressed in terms of a series of Riemann ζ functions.

Again, when s is positive,

$$\sum_{n=m}^{\infty} \frac{1}{\phi^s(n)} = -\int^m \phi^{-s}(n)\, dn - \frac{\phi^{-s}(m)}{2} - \sum_{t=0}^{\infty} \frac{(-)^t \mathrm{B}_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \phi^{-s}(m).$$

And
$$\sum_{n=1}^{m-1} \log \phi(n) = \sum_{n=1}^{m-1}\left[\rho \log n + \frac{b_1}{n^{\epsilon_1}} - \frac{b_1^2}{2n^{2\epsilon_1}} + \frac{b_2}{2n^{\epsilon_2}} + \ldots\right]$$

$$= \int^m \log \phi(n)\, dn + \frac{\rho}{2} \log 2\pi + b_1 \mathrm{F}(-\epsilon_1) - \frac{b_1^2}{2} \mathrm{F}(-2\epsilon_1) + b_2 \mathrm{F}(-\epsilon_2)$$

$$- \frac{\log \phi(m)}{2} + \sum_{t=0}^{\infty} \frac{(-)^t \mathrm{B}_{t+1}}{(2t+2)\,!} \frac{d^{2t+1}}{dm^{2t+1}} \log \phi(m)\,;$$

and we shall put $\mathrm{Z}\left(0\,;\, \begin{matrix}\epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots\end{matrix}\right) = b_1 \mathrm{F}(-\epsilon_1) - \dfrac{b_1^2}{2} \mathrm{F}(-2\epsilon_1) + b_2 \mathrm{F}(-\epsilon_2) + \ldots$

If now we substitute in the general formula

$$\log F(z) = -\tfrac{1}{2}\log z + \int^{\gamma_0} \log \phi(n)\, dn - \sum_{s=1}^{k} \frac{(-)^{s-1}}{sz^s} \int^{\gamma_s} \phi^s(n)\, dn$$

$$+ \int^{\phi(m)} \frac{\psi(t)\, dt}{t} \left\{1 - \sum_{s=1}^{k} (-)^s \left(\frac{t^s}{z^s} + \frac{z^s}{t^s}\right)\right\},$$

we shall obtain $\log F(z) = -\tfrac{1}{2}\log z - \frac{\rho}{2}\log 2\pi - Z\left(0\,;\, \begin{matrix}\epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots\end{matrix}\right)$

$$+ \sum_{s=1}^{\infty} \frac{(-)^{s-1}}{sz^s} F(\rho s) + \sum_{s=1}^{\infty} \frac{(-)^{s-1}}{sz^s} Z\left(\rho,\, s\,;\, \begin{matrix}\epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots\end{matrix}\right)$$

$$+ \int^{\phi(m)} \left(t^{\frac{1}{\rho}} - \frac{b_1}{\rho} t^{\frac{1-\epsilon_1}{\rho}} + \frac{\rho - \epsilon_1}{\rho^2} \frac{b_1^2}{2} t^{\frac{1-2\epsilon_1}{\rho}} - \frac{b_2}{\rho} t^{\frac{1-\epsilon_2}{\rho}} \ldots\right) \frac{dt}{t} \left\{1 + \sum_{s=-k}^{k}{}' \frac{(-t)^s}{z^s}\right\}.$$

And when we sum the Fourier series which result from the last integral, we find

$$\log F(z) = -\tfrac{1}{2}\log z - \frac{\rho}{2}\log 2\pi - Z\left(0\,;\, \begin{matrix}\epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots\end{matrix}\right)$$

$$+ \sum_{s=1}^{\infty} \frac{(-)^{s-1}}{sz^s} F(\rho s) + \sum_{s=1}^{\infty} \frac{(-)^{s-1}}{sz^s} Z\left(\rho,\, s\,;\, \begin{matrix}\epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots\end{matrix}\right)$$

$$+ \frac{\pi}{\sin\frac{\pi}{\rho}} z^{\frac{1}{\rho}} - \frac{b_1\pi}{\rho} \frac{z^{\frac{1-\epsilon_1}{\rho}}}{\sin \pi\frac{1-\epsilon_1}{\rho}} + \frac{\pi b_1^2}{2} \cdot \frac{\rho + 1 - 2\epsilon_1}{\rho^2} \cdot \frac{z^{\frac{1-2\epsilon_1}{\rho}}}{\sin \pi\frac{1-2\epsilon_1}{\rho}} - \frac{b_2\pi}{\rho} \frac{z^{\frac{1-\epsilon_2}{\rho}}}{\sin \pi\frac{1-\epsilon_2}{\rho}} - \ldots$$

§ 59. This expansion is valid for all values of any z which lie between $-\pi$ and π. It is arithmetically asymptotic in the same way as the expansion from which it is derived.

We see from the results just obtained that the asymptotic approximation for $\log \prod_{n=1}^{\infty}\left[1 + \frac{z}{a_n}\right]$, where $a_n = n^\rho + b_1 n^{\rho-\epsilon_1} + \ldots$ exceeds that for $\log \prod_{n=1}^{\infty}\left[1 + \frac{z}{n^\rho}\right]$ by a quantity whose first term is $-Z\left(0\,;\, \begin{matrix}\epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots\end{matrix}\right)$ when $\epsilon_1 > 1$, and by a quantity whose first term is

$$-\frac{b_1\pi}{\rho} \frac{z^{\frac{1-\epsilon_1}{\rho}}}{\sin \pi\frac{1-\epsilon_1}{\rho}}, \text{ when } \epsilon_1 < 1.$$

When $\epsilon_1 = 1$, the difference of the two asymptotic approximations commences with the indeterminate form $-\frac{b_1\pi}{\rho}\left[\frac{z^\theta}{\sin \pi\theta}\right]_{\theta=0}$, which arises from the integral

$$\frac{b_1}{\rho} \int^{\phi(m)} \frac{dt}{t} \left\{1 + \sum_{s=-k}^{k}{}' \frac{(-t)^s}{z^s}\right\}.$$

But this integral is equal to

$$\log \phi(m) + \sum_{s=-k}^{k}{}' \frac{(-)^s \phi^s(m)}{s z^s} = \log \phi(m) - \log\left[1 + \frac{\phi(m)}{z}\right] + \log\left[1 + \frac{z}{\phi(m)}\right] = \log z.$$

Thus, where $\epsilon_1 = 1$, the difference of the two asymptotic approximations commences with the term $-b_1 \rho^{-1} \log z$, a result which may be obtained without much difficulty by elementary algebra.

Note that $-b_1 \rho^{-1} \log z$ is the expression obtained when we reject the infinite part of the function $\frac{-b_1 \pi}{\rho} \frac{z^\theta}{\sin \pi^\theta}$ for $\theta = 0$, when expanded in powers of θ.

Note also that the constant $Z\left(0\,;\, \begin{matrix} \epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots \end{matrix}\right)$ which, when $\epsilon_1 > 1$ is the first term of the asymptotic expansion of the logarithm of the ratio of our two products is equal to $\sum_{n=1}^{\infty} \log \frac{a_n}{n^\rho}$.

By means of the formula $x > \log(1 + x) > \frac{x}{1 + x}$, where x is a real quantity lying between ± 1, we may prove that this series is absolutely convergent when $\rho > 1$.

Application to Functions of Zero Order.

§ 60. Hitherto no example has been given of a function of zero order, although the general investigation of § 36 applies equally to functions of this nature. In such cases it becomes necessary to introduce Maclaurin constants of a complexity which seems, except in special cases, beyond the reach of present analytical processes. They can no longer, as for functions of finite order, be expressed in terms of RIEMANN'S ζ function nor, I believe, in terms of any functions which have so far been introduced into analysis. An example will now be given of a very rapidly converging integral function. It obviously would serve as the starting point of a series of interesting researches dealing with the classification of simple integral functions of zero order.

61. We propose to obtain the asymptotic expansion of the function $\prod_{n=1}^{\infty}\left[1 + \frac{z}{e^n}\right]$. In the notation of the general theory we have now $\phi(n) = e^n$.

Therefore $$\log \phi(n) = n\,; \quad \sum_{n=1}^{m-1} \log \phi(n) = \frac{m^2}{2} - \frac{m}{2}.$$

By the Maclaurin sum formula, if s be positive,

$$\sum_{n=1}^{m-1} e^{ns} = \int^m e^{ns} dn + C_s - \frac{e^{ms}}{2} + \frac{B_1}{2\,!} s e^{ms} - \ldots$$

where C_s is the Maclaurin constant corresponding to e^{ns}, which may be determined as in § 84. If we put $\int^m e^{ns} dn = e^{ms}/s$, we have $C_s = (1 - e^s)^{-1}$.

If then we carry out the general process, we shall obtain the asymptotic expansion

$$\log \prod_{n=1}^{\infty} \left[1 + \frac{z}{e^n}\right] = (m-1)\log z - \left(\frac{m^2}{2} - \frac{m}{2}\right) + \sum_{s=1}^{k} \frac{(-)^{s-1}}{sz^s} \mathrm{C}_s$$

$$+ \sum_{s=-k}^{k}{}' \left\{\frac{(-)^{s-1}}{sz^s} \left[\int^m e^{ns}\, dn - \frac{e^{ms}}{2}\right]\right\}$$

$$= (m-1)\log z - \frac{m^2}{2} + \frac{m}{2} + \sum_{s=1}^{k} \frac{(-)^{s-1}\mathrm{C}_s}{sz^s} + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{sz^s}\left(\frac{e^{ms}}{s} - \frac{e^{ms}}{2}\right).$$

As before, we have to "sum" the final divergent series. We take $|z|$ to be a large quantity such that $\left|\frac{e^m}{z}\right|$ is very nearly equal to unity, and then we consider the Fourier series $\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s}\left\{\frac{e^{i\theta s}}{s} - \frac{e^{i\theta s}}{2}\right\}$.

But $\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s^2} e^{i\theta s} = 2\sum_{s=1}^{k} \frac{(-)^{s-1}\cos\theta s}{s^2} = -\frac{1}{2}\left(\theta^2 - \frac{\pi^2}{3}\right).$

And $\sum_{s=-k}^{k}{}' \frac{(-)^s e^{i\theta s}}{2s} = -\frac{1}{2} i\theta.$

Therefore we have the asymptotic expansion

$$\log \prod_{n=1}^{\infty}\left[1 + \frac{z}{e^n}\right] = (m-1)\log z - \frac{m^2}{2} + \frac{m}{2} + \sum_{s=1}^{k} \frac{(-)^{s-1}\mathrm{C}_s}{sz^s} + \frac{1}{2}\left[\log\frac{e^m}{z}\right]^2 + \frac{\pi^2}{6} - \frac{1}{2}\log\frac{e^m}{z},$$

or finally* $\log \prod_{n=1}^{\infty}\left[1 + \frac{z}{e^n}\right] = \frac{1}{2}(\log z)^2 - \frac{1}{2}\log z + \frac{\pi^2}{6} + \sum_{s=1}^{\infty} \frac{(-)^{s-1}\mathrm{C}_s}{sz^s}.$

§ 62. It should be noticed that if, in the function whose asymptotic expansion has thus been obtained, we substitute e^z for z, we shall obtain the function $\prod_{n=1}^{\infty}\left[1 + \frac{e^z}{e^n}\right]$.

This is an integral function whose zeros are of the form

$$z = n + (2m-1)\pi\iota \begin{cases} n = 1, 2, 3, \ldots \infty. \\ m = -\infty, \ldots, -1, 0, 1, \ldots \infty. \end{cases}$$

It is substantially what I propose to call LAMBERT'S function. The function has properties which are a sort of mean between those of the elliptic and double gamma functions.

We can express LAMBERT'S function as a product of two double gamma functions. It is closely connected with the well-known LAMBERT'S series, and in terms of it we can express in a very elegant form the coefficients of capacity of two spheres.

* The dominant terms of this result are equivalent to those given by MELLIN, 'Acta Societatis Fennicæ,' t. 24, p. 50.

§ 63. The reader will notice that in the preceding analysis we have used the methods and not the result of the general formula.

The reason is that with an exponential subject of integration we are unable to ensure that we do not introduce arbitrary additive constants when the indefinite integrals are transformed as formerly.

For in this case $\phi(n) = e^n$ and $\psi(n) = \log n$;

and we have to consider a series of integrals of which the first is $\int^{\phi(m)} \frac{\psi(t)\, dt}{t}$.

We are tempted to say that this integral is equal to

$$\int^1 \frac{\log[\phi(m)\, t]\, dt}{t} = \int^1 \left[\log \phi(m) + \log t\right] \frac{dt}{t} = 0,$$

whereas we only avoid introducing an additive constant by saying that

$$\int^1 \frac{\log[\phi(m)\, t]\, dt}{t} = \tfrac{1}{2}\left[\log^2(\phi(m)\, t)\right]^1 = \tfrac{1}{2}\left[\log \phi(m)\right]^2.$$

§ 64. The integral function just considered is the most simple function of zero order. In carrying out the algebraical analysis of a theory of such functions, it would be necessary to consider the types

$$\prod_{n=1}^{\infty}\left[1 + \frac{z}{e^{e^n}}\right], \qquad \prod_{n=1}^{\infty}\left[1 + \frac{z}{e^{e^{e^n}}}\right] \text{ \&c.}$$

The asymptotic expansions for these successive functions are of successively lower orders of greatness—they are never, however, of so low an order as z^n, where n is finite. This agrees with the known theorem that an algebraical polynomial is the only uniforn function of such an order. Unfortunately, unless we introduce new analytical functions defined by definite integrals, we cannot investigate formally asymptotic approximations for such types; and until the properties of such new functions are investigated, we but express one unknown form in terms of another.

Simple Integral Functions of Finite Non-integral Order Greater than Unity.

§ 65. In the investigations to which we now proceed of simple integral functions of finite non-integral order greater than unity, the theoretical considerations which have been given in detail for functions of order less than unity will for the most part be suppressed, and for brevity only the bare analysis will be written down.

We consider first the standard function $Q_\rho(z) = \prod_1^{\infty}\left[\left(1 + \frac{z}{n^{1/\rho}}\right) e^{-\frac{z}{n^{1/\rho}} + \dots + \frac{(-)^p z^p}{pn^{p/\rho}}}\right]$,

where $\rho > 1$ and p is an integer such that $p + 1 > \rho > p$.

Let $z = Re^{i\theta}$, and suppose that R is very large.

Take m an integer such that $m - 1 < R^\rho \leqq m$.

Then
$$Q_\rho(z) = \prod_{n=1}^{m-1}\left[\left(1+\frac{z}{n^{1/\rho}}\right)e^{-\frac{z}{n^{1/\rho}}+\ldots+\frac{(-)^p z^p}{pn^{p/\rho}}}\right]$$
$$\times \prod_{n=m}^{\infty}\left[\left(1+\frac{z}{n^{1/\rho}}\right)e^{-\frac{z}{n^{1/\rho}}+\ldots+\frac{(-)^p z^p}{pn^{p/\rho}}}\right],$$

so that
$$Q_\rho(z) = \prod_{n=1}^{m-1}\left[1+\frac{z}{n^{1/\rho}}\right] \times \prod_{n=1}^{m-1}\left[e^{-\frac{z}{n^{1/\rho}}+\ldots+\frac{(-z)^p}{pn^{p/\rho}}}\right]$$
$$\times \prod_{n=m}^{\infty}\left[e^{\frac{(-)^p z^{p+1}}{(p+1)n^{\frac{p+1}{\rho}}}+\frac{(-)^{p+1} z^{p+2}}{(p+2)n^{\frac{p+2}{\rho}}}+\ldots}\right];$$

and hence
$$\log Q_\rho(z) = (m-1)\log z - \frac{1}{\rho}\sum_{n=1}^{m-1}\log n$$
$$+\sum_{n=1}^{m-1}\left[\frac{n^{1/\rho}}{z}+\ldots+\frac{(-)^{s-1}n^{s/\rho}}{sz^s}+\ldots\right]$$
$$+\sum_{n=1}^{m-1}\left[-\frac{z}{n^{1/\rho}}+\ldots+\frac{(-z)^p}{pn^{p/\rho}}\right]$$
$$+\sum_{n=m}^{\infty}\left[\frac{(-)^p z^{p+1}}{(p+1)n^{\frac{p+1}{\rho}}}+\frac{(-)^{p+1}z^{p+2}}{(p+2)n^{\frac{p+2}{\rho}}}+\ldots\right].$$

Now, when m is a very large integer,

$$\sum_{n=1}^{m-1}\frac{1}{n^{s/\rho}} = \frac{m^{1-s/\rho}}{1-\frac{s}{\rho}} - \frac{m^{-s/\rho}}{2} + \ldots + \frac{(-)^t}{\frac{-s}{\rho}-2t-1}\binom{\frac{-s}{\rho}}{2t+2}\frac{B_{t+1}}{m^{2t-\frac{s}{\rho}+1}} + \ldots + F\left(\frac{-s}{\rho}\right).$$

And, when s is positive and greater than ρ,

$$-\sum_{n=m}^{\infty}\frac{1}{n^{s/\rho}} = \frac{m^{1-s/\rho}}{1-\frac{s}{\rho}} - \frac{m^{-s/\rho}}{2} + \ldots + \frac{(-)^t}{\frac{-s}{\rho}-2t-1}\binom{\frac{-s}{\rho}}{2t+2}\frac{B_{t+1}}{m^{2t-\frac{s}{\rho}+1}} + \ldots$$

We use these Maclaurin approximations and rearrange the double series which results as the arithmetically asymptotic approximation for $\log Q_\rho(z)$. We obtain, in the limit, when the limits of the summable divergent series are taken for k infinite, the asymptotic expansion

$$\log Q_\rho(z) = (m-1)\log z - \frac{1}{\rho}(m-\tfrac{1}{2})\log m - \frac{m}{\rho} - \frac{1}{2\rho}\log 2\pi$$
$$+ \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} F\left(\frac{s}{\rho}\right)$$
$$+ m\rho \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s\,.\,\rho+s}\left(\frac{m^{1/\rho}}{z}\right)^s - \tfrac{1}{2}\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s}\left(\frac{m^{1/\rho}}{z}\right)^s$$
$$+ \sum_{t=0}^{\infty} \frac{(-)^t}{2t+2\,!}\,\frac{B_{t+1}}{m^{2t+1}}\left\{\frac{-(2t)\,!}{\rho} + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s}\left(\frac{m^{1/\rho}}{z}\right)^s\left[\frac{d^{2t+1}x^{s/\rho}}{dx^{2t+1}}\right]_{x=1}\right\}.$$

We suppose now that $\frac{|z|}{m^{1/\rho}}$ is a quantity which, when m is very large, is ultimately equal to unity. Then we may "sum," as before (§ 52), the Fourier series, which are the various coefficients in the preceding expansion.

We have $\rho \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s\,.\,\rho+s}\left(\frac{m^{1/\rho}}{z}\right)^s = \frac{\pi}{\sin\pi\rho}\,\frac{z^\rho}{m} + \log\frac{m^{1/\rho}}{z} - \frac{1}{\rho}$, provided ρ is not integral, and provided $-\pi < \theta < \pi$. And $-\tfrac{1}{2}\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s}\left(\frac{m^{1/\rho}}{z}\right)^s = -\tfrac{1}{2}\log\frac{m^{1/\rho}}{z}$.

Also, exactly as before, the coefficient of B_{t+1} in the asymptotic approximation for $\log Q_\rho(z)$ vanishes identically.

Therefore we have, provided $-\pi < \theta < \pi$, the asymptotic equality

$$\log Q_\rho(z) = (m-\tfrac{1}{2})\log\frac{z}{m^{1/\rho}} - m - \tfrac{1}{2}\log z - \frac{1}{2\rho}\log 2\pi + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{sz^s}F\left(\frac{s}{\rho}\right)$$
$$+ \frac{\pi z^\rho}{\sin\pi\rho} + m\log\frac{m^{1/\rho}}{z} - \frac{m}{\rho} - \tfrac{1}{2}\log\frac{m^{1/\rho}}{z}.$$

Thus, provided $-\pi < \arg z < \pi$, we have finally

$$\log Q_\rho(z) = \frac{\pi z^\rho}{\sin\pi\rho} - \frac{1}{2\rho}\log 2\pi - \tfrac{1}{2}\log z + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{sz^s}F\left(\frac{s}{\rho}\right).$$

This expansion is exactly analogous to the one previously obtained for $\log P_\rho(z)$ and is to be regarded in the same way. It must be borne in mind that $n^{1/\rho}$ has been assumed to be the arithmetic ρ^{th} root of n. Had any other root been taken—say the arithmetic root multiplied by $\omega = e^{\frac{2\pi\iota r}{\rho}}$, where r is an integer, we should have obtained the asymptotic expansion

$$Q_\rho(z) = e^{\frac{\pi\iota r}{\rho}}\frac{(2\pi)^{-\frac{1}{2}\rho}}{z^{1/2}}\exp\left\{\frac{\pi z^\rho}{\sin\pi\rho} + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}\omega^s}{sz^s}F\left(\frac{s}{\rho}\right)\right\}$$

valid, when $-\pi < \theta - \frac{2\pi r}{\rho} < \pi$, *i.e.*, when $-\pi + \frac{2\pi r}{\rho} < \theta < \pi + \frac{2\pi r}{\rho}$.

The expansion is thus valid everywhere except along the new line of zeros.

§ 66. We proceed now to investigate the asymptotic expansion for

$$\mathrm{F}(z) = \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{a_n}\right) e^{\frac{-z}{a_n} + \ldots + \frac{(-z)^p}{p a_n^p}} \right],$$

where a_n is such a function of n, $\phi(n)$, let us say, that $\sum_{n=1}^{\infty} \frac{1}{a_n^{\rho+\epsilon}}$ converges, and $\sum_{n=1}^{\infty} \frac{1}{a_n^{\rho-\epsilon}}$ diverges, however small ϵ may be, ρ being real, finite, positive, non-integral, and greater than unity, while p is an integer such that $p + 1 > \rho > p$.

Suppose that the result of reversing the equality $r = \phi(n)$ is to give $n = \psi(r)$.

Let m be a very large integer, such that $m - 1 < \psi(|z|) \eqslantless m$.

As previously, we have

$$\begin{aligned}\log \mathrm{F}(z) = (m-1)\log z &- \sum_{n=1}^{m-1} \log \phi(n) \\ &+ \sum_{n=1}^{m-1}\left[\frac{\phi(n)}{z} + \ldots + \frac{(-)^{s-1}\phi^s(n)}{s z^s} + \ldots\right] \\ &+ \sum_{n=1}^{m-1}\left[-\frac{z}{\phi(n)} + \ldots + \frac{(-)^p z^p}{p\phi^p(n)}\right] \\ &+ \sum_{n=m}^{\infty}\left[\frac{(-)^p z^{p+1}}{(p+1)\phi^{p+1}(n)} + \frac{(-)^{p+1} z^{p+2}}{\overline{p+2}\,\phi^{p+2}(n)} + \ldots\right].\end{aligned}$$

Substitute now the arithmetically asymptotic approximations given by the Maclaurin sum formula, and we have

$$\begin{aligned}\log \mathrm{F}(z) = (m-1)\log z - \int_{\gamma_0}^{m} \log \phi(n)\,dn + \tfrac{1}{2}\log\phi(m) \\ + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s z^s}\int_{\gamma_s}^{m}\phi^s(n)\,dn - \tfrac{1}{2}\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}\phi^s(m)}{s z^s} \\ + \sum_{t=0}^{\infty}\frac{(-)^t \mathrm{B}_{t+1}}{(2t+2)!}\left\{-\frac{d^{2t+1}}{dm^{2t+1}}\log\phi(m) + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{s z^s}\frac{d^{2t+1}}{dm^{2t+1}}\phi^s(m)\right\}.\end{aligned}$$

In this expansion γ_{-s} is infinite, and there is no corresponding Maclaurin constant if, and only if, $s > \rho$.

Use indefinite integrals and transform by integrating by parts in the same way and under the same restrictions as in § 54, and we get

$$\begin{aligned}\log \mathrm{F}(z) = -\tfrac{1}{2}\log z + \int^{\gamma_0}\log\phi(n)\,dn + \sum_{s=-p}^{\infty}{}' \frac{(-)^s}{s z^s}\int^{\gamma_s}\phi^s(n)\,dn \\ + \int^{\phi(m)}\frac{\psi(t)\,dt}{t}\left\{1 + \sum_{s=-k}^{k}{}' \frac{(-t)^s}{z^s}\right\}.\end{aligned}$$

The final integral gives rise to the dominant term of the asymptotic expansion. As formerly, it may be written

$$\underset{k=\infty}{\mathrm{Lt}} \int^{\pi} \iota\psi\left\{z\, e^{\iota(\mu-\pi)}\right\} \frac{\sin\left(k+\frac{1}{2}\right)\mu}{\sin\frac{1}{2}\mu}\, d\mu.$$

If we denote the value of this integral by $f(z)$, and if we put

$$\int^{\gamma_0} \log\phi(n)\, dn = \log \mathrm{F}_0, \int^{\gamma_s} \phi^s(n)\, dn = \mathrm{F}_s,$$

we have the final asymptotic equality

$$\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{a_n}\right) e^{\frac{-z}{a_n}+\ldots+\frac{(-z)^p}{pa_n{}^p}}\right\} = \mathrm{F}_0\, z^{-\frac{1}{2}} \exp\left\{f(z) + \sum_{s=-p}^{\infty}{}' \frac{(-)^s \mathrm{F}_s}{s\, z^s}\right\}.$$

§ 67. We notice the exact analogy between this expansion and the one previously obtained in the case when the order of the function is less than unity. The only difference arises from the Maclaurin constants. In the former case, all the constants corresponding to negative values of s were zero; in the present case, the first p of them are formed from asymptotic expansions like $\sum_{n=1}^{m-1} \frac{1}{a_n{}^s} = \int_{\gamma-s}^{m} \phi^{-s}(n)\, dn + \ldots$, and give rise consequently to finite constants; while only the remaining ones, formed from expansions like $-\sum_{n=m}^{\infty} \frac{1}{a_n{}^s} = \int_{\gamma-s}^{m} \phi^{-s}(n)\, dn + \ldots$ are such that $\gamma_{-s} = \infty$.

We notice also the great elegance with which WEIERSTRASS' exponential factor enters to ensure the finiteness of the expressions obtained in the course of the analysis. Could we conceive an attempt to investigate, for functions of order greater than unity, the theory which we carried out for functions of order less than unity in the first paragraphs of this part of the present paper, we should at the outset be forced to invent again WEIERSTRASS' great theorem.

Application to Functions with Algebraic Sequence of Zeros.

§ 68. We will now evaluate the first few terms of the asymptotic expansion for $\mathrm{P}(z) = \prod_{1}^{\infty}\left[\left(1+\frac{z}{a_n}\right) e^{-\frac{z}{a_n}+\ldots+\frac{(-)^p z^p}{pa_n{}^p}}\right]$, where $a_n = n^{\frac{1}{\rho}}\left[1+\frac{b_1}{n^{\epsilon_1}}+\frac{b_2}{n^{\epsilon_2}}+\ldots\right]$, and the ϵ's are positive real quantities arranged in ascending order of magnitude.

Let $r = a_n = \phi(n)$, then on reversion of series we find

$$n = \psi(r) = r^{\rho} - \rho b_1 r^{\rho(1-\epsilon_1)} + \left\{\frac{\rho(\rho+1-2\rho\epsilon_1)}{2}\, b_1{}^2\right\} r^{\rho(1-2\epsilon_1)} - \rho b_2 r^{\rho(1-\epsilon_2)} + \ldots$$

When s is positive,

$$\sum_{n=1}^{m-1} a_n^s = \sum_{n=1}^{m-1}\left[n^{\frac{s}{\rho}} + sb_1 n^{\frac{s}{\rho}-\epsilon_1} + \frac{s(s-1)}{2} b_1^2 n^{\frac{s}{\rho}-2\epsilon_1} + sb_2 n^{\frac{s}{\rho}-\epsilon_2} + \ldots\right]$$

$$= \int^m \phi^s(n)\,dn + \mathrm{F}\left(\frac{s}{\rho}\right) + sb_1\mathrm{F}\left(\frac{s}{\rho}-\epsilon_1\right) + \frac{s(s-1)}{2} b_1^2\mathrm{F}\left(\frac{s}{\rho}-2\epsilon_1\right)$$

$$+ sb_2\mathrm{F}\left(\frac{s}{\rho}-\epsilon_2\right) + \ldots$$

$$-\frac{\phi^s(m)}{2} + \sum_{t=0}^{\infty} \frac{(-)^t \mathrm{B}_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \phi^s(m);$$

and
$$\sum_{n=1}^{m-1}\frac{1}{a_n^s} = \sum_{n=1}^{m-1}\left[n^{-\frac{s}{\rho}} - sb_1 n^{-\frac{s}{\rho}-\epsilon_1} + \frac{s(s+1)}{2} b_1^2 n^{-\frac{s}{\rho}-2\epsilon_1} - sb_2 n^{-\frac{s}{\rho}-\epsilon_2} - \ldots\right]$$

$$= \int^m \phi^{-s}(n)\,dn + \mathrm{Z}\left(\rho, -s;\, \begin{matrix}\epsilon_1\,\epsilon_2\ldots\\ b_1\,b_2\ldots\end{matrix}\right) + \mathrm{F}\left(-\frac{s}{\rho}\right) - \frac{\phi^{-s}(m)}{2}$$

$$+ \sum_{t=0}^{\infty} \frac{(-)^t \mathrm{B}_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \phi^{-s}(m),$$

where $\mathrm{Z}\left(\rho, -s;\, \begin{matrix}\epsilon_1\,\epsilon_2\ldots\\ b_1\,b_2\ldots\end{matrix}\right)$ can be expressed in terms of Riemann ζ functions, or the equivalent Maclaurin constants F by the formula

$$\mathrm{Z}\left(\rho, -s;\, \begin{matrix}\epsilon_1\,\epsilon_2\ldots\\ b_1\,b_2\ldots\end{matrix}\right) = -sb_1\mathrm{F}\left(-\frac{s}{\rho}-\epsilon_1\right) + \frac{s(s+1)}{2} b_1^2\mathrm{F}\left(-\frac{s}{\rho}-2\epsilon_1\right)$$

$$- sb_2\mathrm{F}\left(-\frac{s}{\rho}-\epsilon_2\right) + \ldots$$

As formerly, we put

$$\mathrm{Z}\left(0;\, \begin{matrix}\epsilon_1,\,\epsilon_2\ldots\\ b_1,\,b_2\ldots\end{matrix}\right) = b_1\mathrm{F}(-\epsilon_1) - \frac{b_1^2}{2}\mathrm{F}(-2\epsilon_1) + b_2\mathrm{F}(-\epsilon_2) + \ldots,$$

so that $\frac{\rho}{2}\log 2\pi + \mathrm{Z}(0)$ arises as the Maclaurin constant corresponding to the asymptotic expansion for $\sum_{n=1}^{m-1}\log\phi(n)$.

Proceeding exactly as for the case when the order of the function is equal to unity we see that the asymptotic expansion of $\log \mathrm{P}(z)$ is

$$\frac{\pi}{\sin\pi\rho} z^\rho - \tfrac{1}{2}\log z - \frac{1}{2\rho}\log 2\pi + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{sz^s}\mathrm{F}\left(\frac{s}{\rho}\right)$$

$$\frac{-\pi\rho b_1}{\sin\pi\rho(1-\epsilon_1)} z^{\rho(1-\epsilon_1)} + \frac{\rho(\rho+1-2\rho\epsilon_1)}{2} b_1^2 \frac{\pi}{\sin\pi\rho(1-2\epsilon_1)} z^{\rho(1-2\epsilon_1)}$$

$$\frac{-\pi\rho b_2}{\sin\pi\rho(1-\epsilon_2)} z^{\rho(1-\epsilon_2)} + \ldots + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{sz^s}\mathrm{Z}\left(\rho, s;\, \begin{matrix}\epsilon_1,\,\epsilon_2\ldots\\ b_1,\,b_2\ldots\end{matrix}\right)$$

$$-\mathrm{Z}\left(0;\, \begin{matrix}\epsilon_1,\,\epsilon_2\ldots\\ b_1,\,b_2\ldots\end{matrix}\right),$$

this expansion being valid when $-\pi < \arg z < \pi$.

Thus, when $\rho - \epsilon_1\rho > p$, the first term of the expansion of the ratio

$$\log \frac{\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{a_n}\right)e^{-\frac{z}{a_n}+\ldots+\frac{(-z)^p}{pa_n{}^p}}\right]}{\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n^{1/\rho}}\right)e^{-\frac{z}{n^{1/\rho}}+\ldots+\frac{(-z)^p}{pn^{p/\rho}}}\right]} \quad \text{is} \quad \frac{-\pi\rho b_1}{\sin\pi\rho(1-\epsilon_1)} z^{\rho(1-\epsilon_1)}.$$

And, when $\rho - \epsilon_1\rho < p$, the first term is $\frac{(-)^p z^p}{p} Z\left(\rho, -p; \begin{matrix}\epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots\end{matrix}\right)$, which is readily seen to be equal to $\frac{(-z)^p}{p}\sum_{n=1}^{\infty}\left[\frac{1}{a_n{}^p} - \frac{1}{n^{p/\rho}}\right]$.

§ 69. The expansion which we have obtained is valid for all points except those near the line of zeros of the function and for all finite values of the quantities b and ϵ such that a_n has a finite value. It must be carefully noticed that when any term becomes infinite through the occurrence of sines of integral multiples of π, we must revert to the genesis of that term to find the true form of the expansion. Thus, when $\rho - \rho\epsilon_1 = p$, the first term of the ratio just considered is that which arises from

$$-b_1\rho\int^{\phi(m)} t^p\left\{1+\sum_{s=-k}^{k}{}' \frac{(-t)^s}{z^s}\right\}\frac{dt}{t},$$

that is, from

$$-b_1\rho\left\{\frac{\phi^p(m)}{p} + \sum_{s=-k}^{k}{}'' \frac{(-)^s\phi^{s+p}(m)}{(s+p)z^s} + (-z)^p\log\phi(m)\right\},$$

where the double accent denotes that the terms corresponding to $s = 0$ and $s = -p$ are to be omitted from the summation.

Put now $\frac{\phi(m)}{z} = e^{-i\theta}$, then, with the argument previously used, we have to sum the series

$$-b_1\rho\phi^p(m)\left\{\frac{1}{p} + \sum_{s=-k}^{k}{}'' \frac{(-e^{-i\theta})^s}{s+p} + (-e^{+i\theta})^p\log\phi(m)\right\}.$$

Now, when ρ is not integral, we have seen that

$$\frac{1}{\rho} + \sum_{s=-k}^{k}{}' \frac{(-)^s}{s+\rho}e^{-si\theta} = \frac{\pi}{\sin\pi\rho}e^{\rho i\theta}, \quad \text{provided} \quad -\pi \lesseqgtr \theta \lesseqgtr \pi.$$

Let us put $\rho = p + \epsilon$, where p is a positive integer and ϵ is very small. Then we have, retaining only first powers of ϵ,

$$\frac{1}{p} + \sum_{s=-k}^{k}{}'' \frac{(-)^s e^{-si\theta}}{s+p} + \frac{(-)^p e^{pi\theta}}{\epsilon} = \frac{\pi}{(-)^p\pi\epsilon}e^{pi\theta}[1+\epsilon i\theta],$$

so that

$$\frac{1}{p} + \sum_{s=-k}^{k}{}'' \frac{(-)^s e^{-si\theta}}{s+p} = (-)^p e^{pi\theta} i\theta,$$

The term which we seek is then the value of

$$- b_1 \rho \phi^p(m)(-)^p \left\{\frac{z}{\phi(m)}\right\}^p \log z = (-)^{p+1} b_1 \rho z^p \log z.$$

This is, of course, the term independent of θ in the expansion of $\frac{-\pi\rho b_1}{\sin\pi(p+\theta)} z^{p+\theta}$ in ascending powers of θ.

In exactly the same manner, if $\rho - n\epsilon_s$ (say) is an integer, the corresponding term of our asymptotic expansion must undergo the same process of evaluation and will give rise to a logarithmic term. If one of the ϵ's, say ϵ_k, is equal to ρ, we obtain in the asymptotic expansion a corresponding logarithmic term

$$- \rho b_k \log z.$$

Simple Integral Functions of Finite Integral Order.

§ 70. We proceed now to consider the standard function

$$R_\rho(z) = \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^{1/\rho}}\right) e^{\frac{-z}{n^{1/\rho}} + \ldots + \frac{(-z)^\rho}{\rho n^{\rho/\rho}}}\right], \text{ where } \rho \text{ is an integer } \geqq 1.$$

Let $z = re^{i\theta}$, where r is very large, and let m be a large integer such that $m - 1 < r^\rho \leqq m$.

Then, employing the same process and argument as before,

$$\log R_\rho(z) = \sum_{n=1}^{m-1} \left\{\frac{n^{1/\rho}}{z} + \ldots + \frac{(-)^{s-1} n^{s/\rho}}{sz^s} + \ldots\right\}$$
$$+ (m-1)\log z - \frac{1}{\rho}\sum_{n=1}^{m-1} \log n + \sum_{n=1}^{m-1}\left\{\frac{-z}{n^{1/\rho}} + \ldots + \frac{(-z)^\rho}{\rho n^{\rho/\rho}}\right\}$$
$$- \sum_{n=m}^{\infty}\left\{\frac{(-z)^{\rho+1}}{(\rho+1)n^{\frac{\rho+1}{\rho}}} + \frac{(-z)^{\rho+2}}{(\rho+2)n^{\frac{\rho+2}{\rho}}} + \ldots\right\}.$$

Now, when $\frac{s}{\rho} = 1$,

$$\sum_{n=1}^{m-1} \frac{m^{s/\rho}}{n^{s/\rho}} = \gamma m^{s/\rho} + m^{s/\rho}\log m - \tfrac{1}{2} + \ldots + \frac{(-)^t}{-\frac{s}{\rho} - 2t - 1}\binom{-\frac{s}{\rho}}{2t+2}\frac{B_{t+1}}{m^{2t+1}} + \ldots,$$

and in accordance with the definition of § 49 we put $\gamma = F(-1)$.

If, then, we suitably modify the analysis formerly employed we shall obtain, when

the limits of the summable divergent series as k tends to infinity are taken, the arithmetically asymptotic expansion

$$\log R_\rho(z) = (m-1)\log z - \frac{1}{\rho}\{(m-\tfrac{1}{2})\log m - m + \tfrac{1}{2}\log 2\pi\}$$
$$+ \sum_{s=-\rho}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} F\left(\frac{s}{\rho}\right) + m \sum_{s=-k}^{k}{}'' \left\{\frac{(-)^{s-1}}{sz^s}\frac{m^{s/\rho}}{\frac{s}{\rho}+1}\right\}$$
$$+ \frac{(-z)^{-\rho}}{\rho}\log m - \tfrac{1}{2}\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{sz^s} m^{s/\rho}$$
$$+ \sum_{t=0}^{\infty} \frac{(-)^t B_{t+1}}{2t+2!}\frac{1}{m^{2t+1}}\left\{\frac{-(2t)!}{\rho} + \sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{sz^s} m^{s/\rho}\left[\frac{d^{2t+1}}{dx^{2t+1}} x^{s/\rho}\right]_{x=1}\right\},$$

where the double accent denotes that in the corresponding summation the terms for which $s=0$ and $s=-\rho$ are to be omitted.

As before, the coefficient of B_{t+1} vanishes identically.

The series $\sum_{s=-k}^{k}{}' \frac{(-)^{s-1}}{sz^s} m^{s/\rho}$ is equal to $\log\frac{m^{1/\rho}}{z}$.

It is then only necessary for us to consider the series $\sum_{s=-k}^{k}{}'' \frac{(-)^{s-1}}{sz^s}\frac{m^{s/\rho}}{\frac{s}{\rho}+1}$.

If we put $t = \log\frac{m^{1/\rho}}{z}$, we may write this series in the form

$$f(t) = \sum_{s=-k}^{k}{}'' \frac{(-)^{s-1}e^{st}}{s\left(\frac{s}{\rho}+1\right)} = \sum_{s=-k}^{k}{}'' (-)^{s-1}e^{st}\left(\frac{1}{s} - \frac{1}{s+\rho}\right).$$

Remembering that a summable divergent series may be differentiated, we find

$$f(t) = -\frac{1}{\rho}f'(t) + \sum_{s=-k}^{k}{}'' \frac{(-)^{s-1}e^{st}}{s},$$

or

$$f'(t) + \rho f(t) = \rho t + (-)^{\rho-1}e^{-\rho t}.$$

Therefore

$$f(t) = Ae^{-\rho t} + t - \frac{1}{\rho} + (-)^{\rho-1}te^{-\rho t},$$

where A is a constant of integration.

Now when $t=0$,

$$f(t) = \sum_{s=-k}^{k}{}'' \frac{(-)^{s-1}}{s\left(\frac{s}{\rho}+1\right)} = \sum_{s=-k}^{k}{}'' (-)^{s-1}\left(\frac{1}{s} - \frac{1}{s+\rho}\right),$$

and, by putting $\theta = 0$ in the FOURIER's series, which we considered in the preceding paragraph, we see that this is equal to $-\frac{1}{\rho} + \frac{(-)^{\rho-1}}{\rho}$.

Therefore

$$f(t) = (-)^{\rho-1} e^{-\rho t} \left[\frac{1}{\rho} + t\right] + t - \frac{1}{\rho},$$

so that

$$\sum_{s=-k}^{k}{}'' \frac{(-)^{s-1}}{sz^s} \frac{m^{s/\rho}}{\frac{s}{\rho} + 1} = (-)^{\rho-1} \frac{z^\rho}{m} \left[\frac{1}{\rho} + \log \frac{m^{1/\rho}}{z}\right] + \log \frac{m^{1/\rho}}{z} - \frac{1}{\rho}.$$

Revert now to the asymptotic expansion for $\log R_\rho(z)$.

We find on substitution that

$$\begin{aligned}\log R_\rho(z) = (m-1)\log z &- \frac{1}{\rho}\{(m - \tfrac{1}{2})\log m - m + \tfrac{1}{2}\log 2\pi\} \\ &+ \sum_{s=-\rho}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} F\left(\frac{s}{\rho}\right) + (-)^{\rho-1} z^\rho \left\{\log \frac{m^{1/\rho}}{z} + \frac{1}{\rho}\right\} \\ &+ m\left(\log \frac{m^{1/\rho}}{z} - \frac{1}{\rho}\right) + \frac{(-z)^\rho}{\rho} \log m - \tfrac{1}{2} \log \frac{m^{1/\rho}}{z}.\end{aligned}$$

And thus, when ρ is an integer, $|z|$ very large, and $-\pi < \arg z < \pi$,

$$\log R_\rho(z) = -\tfrac{1}{2}\log z + \sum_{s=-\rho}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} F\left(\frac{s}{\rho}\right) - \frac{1}{2\rho}\log 2\pi + (-z)^\rho \log z + (-)^{\rho-1}\frac{z}{\rho}.$$

As formerly, this expansion is, in form, independent of the argument of z.

§ 71. We may easily deduce this theorem independently as the limit of our former results.

Take the asymptotic equality

$$\prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^{1/\rho}}\right) e^{-\frac{z}{n^{1/\rho}} + \dots + \frac{(-z)^{p-1}}{\overline{p-1}\, n^{\frac{p-1}{\rho}}}}\right] = (2\pi)^{-\frac{1}{2\rho}} z^{-\frac{1}{2}} e^{\frac{\pi}{\sin \pi\rho} z^\rho + \sum_{s=-p+1}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} F\left(\frac{s}{\rho}\right)},$$

where ρ lies between $p - 1$ and p.

Put now $\rho = p - \epsilon$; then $R_p(z)$ is the limit, when ϵ vanishes, of

$$\prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^{\frac{1}{p-\epsilon}}}\right) e^{-\frac{z}{n^{\frac{1}{p-\epsilon}}} + \dots + \frac{(-z)^{p-1}}{\overline{p-1}\, n^{\frac{p-1}{p-\epsilon}}}}\right] \times \prod_{n=1}^{\infty} e^{\frac{(-z)^p}{p\, n^{\frac{p}{p-\epsilon}}}}.$$

It therefore possesses an asymptotic expansion which is the limit, when ϵ vanishes, of

$$(2\pi)^{-\frac{1}{2p-\epsilon}} z^{-\frac{1}{2}} \exp\left\{\frac{\pi z^{p-\epsilon}}{\sin\pi(p-\epsilon)} + \frac{(-z)^p}{p}\zeta\left(\frac{p}{p-\epsilon}\right) + \sum_{s=-p+1}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} \mathrm{F}\left(\frac{s}{p-\epsilon}\right)\right\}$$

$$= (2\pi)^{-\frac{1}{2p}} z^{-\frac{1}{2}} \exp\left\{(-z)^p \log z + \frac{(-z)^p(\gamma-1)}{p} + \sum_{s=-p+1}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} \mathrm{F}\left(\frac{s}{p}\right)\right\}$$

$$= (2\pi)^{-\frac{1}{2p}} z^{-\frac{1}{2}} \exp\left\{(-z)^p \log z + (-)^{p-1}\frac{z^p}{p} + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} \mathrm{F}\left(\frac{s}{p}\right)\right\},$$

remembering that $\mathrm{F}(-1) = \gamma$, and that $\underset{s=1}{\mathrm{Lt}}\left[\zeta(s) + \frac{1}{1-s}\right] = \gamma +$ terms which vanish when $s = 1$.

We thus obtain the same asymptotic expansion as in the previous paragraph.

Note that we have obtained our expansion by making ρ increase up to the nearest integer. If, on the contrary, we make ρ decrease *down* to the nearest integer, there is no breach of continuity in the introduction of an additional exponential factor. Thus we have

$$\mathrm{R}_p(z) = \underset{\epsilon=0}{\mathrm{Lt}} \prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{n^{\frac{1}{p+\epsilon}}}\right) e^{-\frac{z}{n^{\frac{1}{p+\epsilon}}} + \ldots + \frac{(-z)^p}{p n^{\frac{p}{p+\epsilon}}}}\right],$$

and therefore we have the asymptotic expansion

$$\mathrm{R}_p(z) = \underset{\epsilon=0}{\mathrm{Lt}} (2\pi)^{-\frac{1}{2p+\epsilon}} z^{-\frac{1}{2}} \exp\left\{\frac{\pi z^{p+\epsilon}}{\sin\pi(p+\epsilon)} + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} \mathrm{F}\left(\frac{s}{p+\epsilon}\right)\right\}.$$

Now, unless $s = -1$, $\mathrm{F}(s) = \zeta(-s)$; and therefore we have asymptotically

$$\mathrm{R}_p(z) = \underset{\epsilon=0}{\mathrm{Lt}} (2\pi)^{-\frac{1}{2p+\epsilon}} z^{-\frac{1}{2}} \exp\left\{\frac{(-z)^p(1 + \epsilon\log z + \ldots)}{\epsilon} + \frac{(-z)^p}{p}\zeta\left(\frac{p}{p+\epsilon}\right)\right.$$
$$\left. + \sum_{s=-p+1}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} \mathrm{F}\left(\frac{s}{p+\epsilon}\right)\right\}$$

$$= (2\pi)^{-\frac{1}{2p}} z^{-\frac{1}{2}} \exp\left\{(-z)^p \log z + \frac{(-z)^p(\gamma-1)}{p} + \sum_{s=-p+1}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} \mathrm{F}\left(\frac{s}{p}\right)\right\},$$

the same expansion as before.

This paragraph is instructive in that it shows how the asymptotic expansion calls for another exponential factor in each term of WEIERSTRASS' product as the order passes through an integral value.

§ 72. If now it is desired to construct a function which is the natural extension among simple integral functions of the ordinary gamma function, we take

$$\frac{1}{\Gamma(z|\rho)} = e^{\sum_{s=1}^{\rho} \frac{(-)^{s-1}}{s} z^s \mathrm{F}\left(-\frac{s}{\rho}\right)} . z . \prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{n^{1/\rho}}\right) e^{-\frac{z}{n^{1/\rho}} + \ldots + \frac{(-z)^\rho}{\rho n^{\rho/\rho}}}\right].$$

And now the asymptotic expansion of $\Gamma(z|\rho)$ when $|z|$ is very large and $-\pi < \arg z < \pi$ is given by

$$(2\pi)^{\frac{1}{2\rho}} z^{(-)^{\rho-1} z^{\rho}-\frac{1}{2}} e^{\sum_{s=1}^{\infty} \frac{(-)^s}{sz^s} F\left(\frac{s}{\rho}\right) + \frac{(-z)^{\rho}}{\rho}}.$$

When $\rho = 1$, this formula is exactly the asymptotic expansion of $\Gamma(z)$ for complex values of z, which, as stated in § 3, was first obtained by STIELTJES.

For, when s is an even positive integer, $F(s) = 0$.

When s is an odd positive integer $= 2t + 1$, let us say, $F(s) = \frac{(-)^{t-1} B_{t+1}}{2t+2}$. $t \geqq 0$. And $F(-1) = \gamma$.

Thus, when $\rho = 1$, $\Gamma(z|\rho)$ becomes $\Gamma(z)$, and the series $\sum_{s=1}^{\infty} \frac{(-)^s}{sz^s} F\left(\frac{s}{\rho}\right)$ becomes $\sum_{t=0}^{\infty} \frac{(-)^t B_{t+1}}{2t+1 \,.\, 2t+2} \cdot \frac{1}{z^{2t+1}}$, which accords with the usual result.

§ 73. It is obvious that we can now at once write down the asymptotic expansion for $G(z) = \prod_1^{\infty} \left[\left(1 + \frac{z}{a_n}\right) e^{-\frac{z}{a_n} + \ldots + \frac{(-)^p z^p}{a_n^p}}\right]$, where $a_n = n^{\frac{1}{p}} \left[1 + \frac{b_1}{n^{\epsilon_1}} + \frac{b_2}{n^{\epsilon_2}} + \ldots\right]$ and p is an integer, from the corresponding expansion for the function in which $a_n = n^{\frac{1}{\rho}} \left[1 + \frac{b_1}{n^{\epsilon_1}} + \ldots\right]$ and ρ is not integral. The ϵ's, of course, are assumed to be positive and in ascending order of magnitude.

The result is

$$\begin{aligned}
&\log \prod_1^{\infty} \left[\left(1 + \frac{z}{a_n}\right) e^{-\frac{z}{a_n} + \ldots + \frac{(-)^p z^p}{a_n^p}}\right] \\
&= (-)^p z^p \log z + (-)^{p-1} \frac{z^p}{p} - \tfrac{1}{2} \log z - \frac{1}{2p} \log 2\pi + \sum_{s=-p}^{\infty}{}' \frac{(-)^s}{sz^s} F\left(\frac{s}{p}\right) \\
&+ (-)^p \frac{\pi p b_1}{\sin \pi p \epsilon_1} z^{p-\epsilon_1 p} + (-)^{p-1} \frac{p(p+1-2p\epsilon_1)}{2} b_1^2 \frac{\pi}{\sin 2\pi p \epsilon_1} z^{p-2\epsilon_1 p} \\
&+ (-)^p \frac{\pi p b_2}{\sin \pi p \epsilon_2} z^{p-\epsilon_2 p} + \ldots + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} Z\left(p, s\,;\, \begin{matrix}\epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots\end{matrix}\right) - Z\left(p, 0\,;\, \begin{matrix}\epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots\end{matrix}\right),
\end{aligned}$$

provided $\frac{p\epsilon_1}{n}, \frac{p\epsilon_2}{n} \ldots$ be not integral $(n = 1, 2, \ldots \infty)$.

Thus, ϵ_1 not being integral, the first term of the asymptotic expansion of the quotient

$$\log \frac{\prod_1^{\infty} \left\{\left(1 + \frac{z}{a_n}\right) e^{-\frac{z}{a_n} + \ldots + \frac{(-)^p z^p}{a_n^p}}\right\}}{\prod_1^{\infty} \left\{\left(1 + \frac{z}{n^{\frac{1}{p}}}\right) e^{-\frac{z}{n^{1/p}} + \ldots + \frac{(-)^p z^p}{n}}\right\}} \quad \text{is} \quad \frac{(-)^p}{p} z^p Z\left(p, p\,;\, \begin{matrix}\epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots\end{matrix}\right).$$

We note that $Z\left(p, p\,;\, \begin{matrix}\epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots\end{matrix}\right) = \sum_{n=1}^{\infty} \left[\frac{1}{a_n^p} - \frac{1}{n}\right]$.

When $p\epsilon_1$ is an integer, we see on evaluating the limit which arises, that the dominant term of the asymptotic expansion is still the one just written down. For, in this case, the only other term which might be considered first in the asymptotic expansion of the quotient is $(-)^{p+1} pb_1 z^{p-\epsilon_1 p} \log z$, which, since ϵ_1 is positive, is of lower order than

$$\frac{(-)^p}{p} z^p \mathrm{Z}\left(p, p\,;\, \begin{matrix}\epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots\end{matrix}\right).$$

§ 74. It is now evident that, if we are given any simple function of finite integral order, we can find its asymptotic expansion. The analysis just given solves completely the case of algebraical zeros. When the zeros are not algebraic we may, and, in fact, we shall have to introduce new analytical functions defined as indefinite integrals; but there will be no essential difference in the theory.

It should be noticed that just as we have to take the principal values of the algebraically many-valued expressions which occur in the asymptotic approximation for functions of non-integral order, so we must assign principal values to the logarithms which occur when the functions are of integral order.

PART IV.

The Asymptotic Expansion of Repeated Integral Functions.

§ 75. As has been stated in the general classification of Part I., an integral function, which is such that its n^{th} zero is repeated a number of times dependent upon n, is called a *repeated* function.

If the number of sequences of zeros be not infinite, the function is called a *simple* repeated function; and it is obvious that such a function may be built up of functions, each of which possesses a single sequence of zeros. We shall limit ourselves to the consideration of such functions. The order of simple repeated functions with a single sequence of zeros has been previously defined. Taking this definition, we consider, in turn, in the ensuing paragraphs, functions

(1) of finite (non-zero or zero) order less than unity,
(2) of finite non-integral order greater than unity,
(3) of finite integral order greater than or equal to unity.

And, finally, an example is given of the asymptotic expansion of a repeated function with a transcendental index.

Inasmuch as the principles which underlie the analysis are exactly the same as those which have been previously discussed, we shall give but a bare outline of the methods by which the results are obtained.

Simple Repeated Functions of Finite Order less than Unity.

§ 76. The most general function of this type may be written

$$F(z) = \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{a_n}\right)^{\mu_n} \right],$$

where the principal value of each term is taken when μ_n is any function of n; and where a_n is a function of n which increases without limit as n increases, and which is such that $\sum_{n=1}^{\infty} \mu_n/a_n$ is absolutely convergent. The function is of finite (or zero) order less than unity; and, when μ_n is an integer, its n^{th} zero is repeated μ_n times.

We take

$$a_n = \phi(n).$$

When $r = \phi(n)$ we suppose that inversely $n = \psi(r)$. Suppose that $z = Re^{i\theta}$, then if we take m to be a large integer such that $m - 1 < \psi(R) \leqq m$, we have

$$\log F(z) = \sum_{n=1}^{m-1} \mu_n \log z - \sum_{n=1}^{m-1} \mu_n \log \phi(n) + \sum_{n=1}^{m-1} \mu_n \log\left(1 + \frac{a_n}{z}\right) + \sum_{n=m}^{\infty} \mu_n \log\left(1 + \frac{z}{a_n}\right).$$

We carry out our analysis in a manner which depends exactly upon the argument previously employed in the corresponding case for non-repeated functions.

We have at once, in the limit when $k = \infty$,

$$\log F(z) = \log z \sum_{n=1}^{m-1} \mu_n - \sum_{n=1}^{m-1} \mu_n \log \phi(n) + \sum_{n=1}^{m-1} \sum_{s=1}^{k} \frac{(-)^{s-1} \mu_n \phi^s(n)}{s z^s} + \sum_{n=m}^{\infty} \sum_{s=1}^{k} \frac{(-)^{s-1} \mu_n z^s}{s \phi^s(n)}.$$

Now, if s be positive,

$$\sum_{n=1}^{m-1} \mu_n \phi^s(n) = \int_{\gamma_s}^{m} \mu_n \phi^s(n)\, dn - \tfrac{1}{2}\mu_m \phi^s(m) \ldots + (-)^t \frac{B_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \mu_m \phi^s(m) + \ldots$$

where γ_s is a constant depending on s and on the forms of μ_n and $\phi(n)$.

We call γ_s the s^{th} Maclaurin integral limit for μ_n and $\phi(n)$. If s be negative, the previous expansion will hold, but in this case $\gamma_{-s} = \infty$, and the constant term vanishes. Again we have

$$\sum_{n=1}^{m-1} \mu_n \log \phi(n) = \int_{\gamma_0}^{m} \mu_n \log \phi(n)\, dn - \tfrac{1}{2} \mu_m \log \phi(m) + \ldots$$
$$+ (-)^t \frac{B_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \mu_m \log \phi(m) + \ldots$$

and

$$\sum_{n=1}^{m-1} \mu_n = \int_{g_0}^{m} \mu_n\, dn - \tfrac{1}{2}\mu_m + \ldots + (-)^t \frac{B_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \mu_m + \ldots$$

We shall find it convenient to put

$$\int^{m} \mu_n \, dn = \chi(m), \quad \text{so that} \quad \int^{g_0} \mu_n \, dn = \chi(g_0) = - \mathrm{M} \text{ (say)}.$$

And now, if the limiting values when $k = \infty$ of the summable divergent series be taken,

$$\log \mathrm{F}(z) = \log z \left[\int^{m} \mu_n \, dn + \mathrm{M} - \tfrac{1}{2}\mu_m\right]$$

$$- \int_{\gamma_0}^{m} \mu_n \log \phi(n) \, dn + \frac{\mu_m}{2} \log \phi(m)$$

$$- \sum_{s=1}^{k} \frac{(-)^s}{sz^s} \left[\int_{\gamma_s}^{m} \mu_n \phi^s(n) \, dn - \tfrac{1}{2} \mu_m \phi^s(m)\right]$$

$$+ \sum_{s=1}^{k} \frac{(-)^s}{s} z^s \left[\int_{\gamma_{-s}}^{m} \frac{\mu_n \, dn}{\phi^s(n)} - \frac{\mu_m}{2\phi^s(m)}\right]$$

$$+ \sum_{t=0}^{\infty} \frac{(-)^{t-1}\mathrm{B}_{t+1}}{2t+2\,!} \frac{d^{2t+1}}{dm^{2t+1}} \left[\mu_m \log \phi(m) - \mu_m \log z + \sum_{s=1}^{k} \frac{(-)^s}{s} \left\{\frac{\mu_m \phi^s(m)}{z^s} - \frac{\mu_m z^s}{\phi^s(m)}\right\}\right].$$

The last term vanishes as for the corresponding case of non-repeated functions. After reduction, we have

$$\log \mathrm{F}(z) = \log z \left[\int^{m} \mu_n \, dn + \mathrm{M}\right] - \int_{\gamma_0}^{m} \mu_n \log \phi(n) \, dn$$

$$+ \sum_{s=1}^{k} \frac{(-)^{s-1}}{s} \left\{\int_{\gamma_s}^{m} \frac{\mu_n \phi^s(n)}{z^s} \, dn - \int_{\gamma_{-s}}^{m} \frac{\mu_n z^s}{\phi^s(n)} \, dn\right\}$$

$$= \mathrm{M} \log z + \int^{\gamma_0} \mu_n \log \phi(n) \, dn + \sum_{s=1}^{k} \frac{(-)^s}{sz^s} \int^{\gamma_s} \mu_n \phi^s(n) \, dn$$

$$+ \int^{\phi(m)} \frac{\chi[\psi(t)] \, dt}{t} \left[1 + \sum_{s=-k}^{k}{}' \frac{(-)^s t^s}{z^s}\right].$$

The last integral

$$= \underset{k=\infty}{\mathrm{Lt}} \int^{\pi} \chi[\psi(-ze^{i\phi})] \, i \, d\phi \, \frac{\sin(k+\frac{1}{2})\phi}{\sin \frac{1}{2}\phi} = f(z) \text{ say}.$$

If then we put $\log \mathrm{F}_0 = \int^{\gamma_0} \mu_n \log \phi(n) \, dn$, $\quad \mathrm{F}_s = - \int^{\gamma_s} \mu_n \phi^s(n) \, dn$, so that F_0 and F_s may be called the zero and s^{th} Maclaurin constants for μ_n and $\phi(n)$, we shall have the asymptotic approximation

$$\prod_{n=1}^{\infty} \left\{\left(1 + \frac{z}{a_n}\right)^{\mu_n}\right\} \doteq \mathrm{F}_0 z^{\mathrm{M}} e^{f(z) + \sum_{s=1}^{\infty} \frac{(-)^{s-1}\mathrm{F}_s}{sz^s}}.$$

§ 77. Consider now, as an application of the general formula just obtained, the asymptotic expansion of $\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n^{\rho}}\right)^{n^{\sigma}}\right]$, where σ and ρ are real positive quantities such that $\sum_{n=1}^{\infty} n^{\sigma-\rho}$ is convergent, and, therefore, such that $\rho > \sigma + 1$.

With our former notation

$$\mu_n = n^{\sigma},$$

$$\chi(m) = \int^m \mu_n \, dn = \frac{m^{\sigma+1}}{\sigma+1},$$

$$\psi(t) = t^{1/\rho},$$

$$\chi[\psi(t)] = \frac{t^{\frac{\sigma+1}{\rho}}}{\sigma+1}.$$

The constant g_0 arises from the asymptotic equality

$$\sum_{n=1}^{m-1} n^{\sigma} = \int_{g_0}^{m} n^{\sigma}\, dn - \frac{m^{\sigma}}{2} + \sum_{t=0}^{\infty} \frac{(-)^t B_{t+1}}{(2t+2)!}\frac{d^{2t+1}}{dm^{2t+1}} m^{\sigma},$$

and, therefore, $M = -\chi(g_0) = \zeta(-\sigma)$.

Similarly $\int^{\gamma_s} \mu_n \phi^s(n)\, dn = -\zeta(-\overline{\rho s + \sigma})$.

The constant γ_0 arises from the asymptotic equality

$$\sum_{n=1}^{m-1} n^{\sigma}\log n = \int_{\gamma_0}^{m} n^{\sigma}\log n \, dn - \frac{m^{\sigma}}{2}\log m + \sum_{t=0}^{\infty} \frac{(-)^t B_{t+1}}{\overline{2t+2}\,!}\frac{d^{2t+1}}{dm^{2t+1}}(m^{\sigma}\log m).$$

We may readily show that $\int^{\gamma_0} n^{\sigma}\log n\, dn = \zeta'(-\sigma)$.

For, as has been stated, for all values of s,

$$\zeta(s) = \sum_{n=1}^{m-1}\frac{1}{n^s} - \frac{1}{1-s}\frac{1}{m^{s-1}} + \frac{1}{2m^s} + \sum_{t=1}^{\infty}\binom{-s}{2t}\frac{(-)^{t-1}B_t}{(s+2t-1)\,m^{s+2t-1}}.$$

If, then, we put $s = \sigma + t$, and expand each term in powers of t, we may equate coefficients of similar powers in the identity.*

If we equate coefficients of the first power, we find

$$\zeta'(\sigma) = -\sum_{n=1}^{m-1}\frac{\log n}{n^{\sigma}} - \frac{1}{m^{\sigma-1}}\left[\frac{1}{(1-\sigma)^2} - \frac{\log m}{1-\sigma}\right] - \frac{1}{2m^{\sigma}}\log m + \sum_{t=0}^{\infty}\frac{(-)^t B_{t+1}}{\overline{2t+2}\,!}\frac{d^{2t+1}}{dm^{2t+1}}\frac{\log m}{m^{\sigma}},$$

* Compare the process carried out in §§ 27 and 30 of the "Theory of the Gamma Function."

or, changing σ into $-\sigma_1$,

$$\sum_{n=1}^{m-1} n^\sigma \log n = \int^m n^\sigma \log n \, dn - \zeta'(-\sigma) - \frac{m^\sigma}{2} \log m + \sum_{t=0}^{\infty} \frac{(-)^t B_{t+1}}{2t+2!} \frac{d^{2t+1}}{dm^{2t+1}} m^\sigma \log m.$$

Thus

$$\int^{\gamma_0} n^\sigma \log n \, dn = \zeta'(-\sigma).$$

We have, therefore,

$$\log \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^\rho}\right)^{n^\sigma} = \zeta(-\sigma) \log z + \rho\zeta'(-\sigma) + \sum_{s=1}^{k} \frac{(-)^{s-1}}{sz^s} \zeta(-\rho s + \sigma)$$

$$+ \frac{1}{\sigma+1} \int^{\frac{\phi(m)}{z}} z^{\frac{\sigma+1}{\rho}} t^{\frac{\sigma+1}{\rho}-1} dt \left[1 + \sum_{s=-k}^{k}{}' (-)^s t^s\right].$$

The last integral is equal to

$$\frac{[\phi(m)]^{\frac{\sigma+1}{\rho}}}{\sigma+1} \left[\frac{\rho}{\sigma+1} + \sum_{s=-k}^{k}{}' \left\{\frac{-\phi(m)}{z}\right\}^s \frac{1}{s + \dfrac{\sigma+1}{\rho}}\right]$$

$$= \frac{[\phi(m)]^{\frac{\sigma+1}{\rho}}}{\sigma+1} \frac{\pi}{\sin \dfrac{\pi . \sigma+1}{\rho}} \left[\frac{z}{\phi(m)}\right]^{\frac{\sigma+1}{\rho}} = \frac{\pi}{\sin \pi . \dfrac{\sigma+1}{\rho}} \frac{z^{\frac{\sigma+1}{\rho}}}{\sigma+1}.$$

Thus we have the asymptotic expansion

$$\prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^\rho}\right)^{n^\sigma}\right] = z^{\zeta(-\sigma)} e^{\rho\zeta'(-\sigma) + \sum_{s=1}^{\infty} \frac{(-)^{s-1}}{sz^s} \zeta(-\rho \overline{s+\sigma}) + \frac{\pi}{\sin \frac{\pi . \sigma+1}{\rho}} \cdot \frac{z^{\frac{\sigma+1}{\rho}}}{\sigma+1}}.$$

We note that the first term of this product vanishes when σ is an even integer.

§ 78. It is now possible to write down the expansion of

$$\prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^\rho}\right)^{\mu_n}\right],$$

where μ_n is algebraic and of the form $a_0 n^\sigma + a_1 n^{\sigma_1} + a_2 n^{\sigma_2} + \ldots$, in which $\sigma > \sigma_1 > \sigma_2 > \ldots$

For such a function is merely the product of the

$$a_0^{\text{th}} \text{ power of } \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^\rho}\right)^{n^\sigma}\right],$$

$$\text{the } a_1^{\text{th}} \text{ power of } \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n^\rho}\right)^{n^{\sigma_1}}\right], \text{ and so on.}$$

We note that the constants which enter will be expressible in terms of the coefficients of μ_n and values of the Riemann ζ function.

We might now investigate the asymptotic expansion of a repeated function of finite order less than unity with algebraic sequence of zeros of the type

$$a_n = n^\rho \left[1 + \frac{b_1}{n^{\epsilon_1}} + \frac{b_2}{n^{\epsilon_2}} + \ldots\right]$$

where the quantities $\epsilon_1, \epsilon_2, \ldots$ are real, positive, and in ascending order of magnitude.

The analysis is, however, such an obvious extension of the corresponding result of Part III. that it may be at once supplied by the reader.

Repeated Simple Functions of Finite Non-integral Order greater than Unity.

§ 79. We next consider the asymptotic expansion of the function

$$\mathrm{F}(z) = \prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{a_n}\right)^{\mu_n} e^{\mu_n \sum_{m=1}^{p} \frac{1}{m}\left(-\frac{z}{a_n}\right)^m}\right],$$

where $p < \rho < p + 1$, and ρ is such that

$$\Sigma \frac{\mu_n}{a_n^{\rho+\epsilon}} \text{ is convergent, and } \Sigma \frac{\mu_n}{a_n^{\rho-\epsilon}} \text{ divergent,}$$

when ϵ is a small real positive quantity.

The analysis is an obvious modification of that employed in § 66.

We find $\mathrm{F}(z) = \mathrm{F}_0 z^{\mathrm{M}} e^{f(z) + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}\mathrm{F}_s}{s z^s}}$, where $a_n = \phi(n)$, $\chi(m) = \int^m \mu_n \, dn$,

$$\sum_{n=1}^{m-1} \mu_n = \int^m \mu_n \, dn + \mathrm{M} - \tfrac{1}{2}\mu_m + \ldots,$$

$$\sum_{n=1}^{m-1} \mu_n \log \phi(n) = \int^m \mu_n \log \phi(n) \, dn - \log \mathrm{F}_0 - \tfrac{1}{2}\mu_m \log \phi(m) + \ldots,$$

$$\sum_{n=1}^{m-1} \mu_n \phi^s(n) = \int^m \mu_n \phi^s(n) \, dn + \mathrm{F}_s - \tfrac{1}{2}\mu_m \phi^s(m) + \ldots \; (s = -p, -(p-1), \ldots -1, 1, 2, \ldots \infty),$$

and

$$f(z) = \underset{k=\infty}{\mathrm{Lt}}\, i \int^{\pi} \chi[\psi(-z e^{i\phi})] \frac{\sin(k + \frac{1}{2})\phi}{\sin \frac{1}{2}\phi} \, d\phi.$$

§ 80. As an example, we may consider the function

$$\mathrm{F}(z) = \prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{n^\tau}\right)^{n^\sigma} e^{n^\sigma \sum_{m=1}^{p} \frac{1}{m}\left(-\frac{z}{n^\tau}\right)^m}\right], \text{ where } \frac{\sigma+1}{\tau} \text{ is not integral, and } p < \frac{\sigma+1}{\tau} < p+1.$$

The order of the function is $\dfrac{\sigma+1}{\tau}$.

We have

$$-\chi(g_0) = \zeta(-\sigma),$$

$$\log F_0 = -\sum_{n=1}^{m-1} \tau n^\sigma \log n + \int^m \tau n^\sigma \log n \, dn - \frac{\tau}{2} n^\sigma \log n + \ldots = \tau \zeta'(\sigma),$$

$$-F_s = -\zeta(\overline{-\tau s + \sigma}).$$

And, as in § 77, $f(z) = \dfrac{\pi}{\sin \dfrac{\pi(\sigma+1)}{\tau}} \cdot \dfrac{z^{\frac{\sigma+1}{\tau}}}{(\sigma+1)}.$

Thus the asymptotic expansion of F (z) may be written

$$z^{\zeta(-\sigma)}\, e^{\frac{\pi}{\sin \pi \left(\frac{\sigma+1}{\tau}\right)} \frac{z^{\frac{\sigma+1}{\tau}}}{\sigma+1} + \tau \zeta'(-\sigma) + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1} \zeta(-\sigma-\tau s)}{s z^s}}.$$

Note that $\qquad \zeta(0)^* = -\frac{1}{2}, \quad \zeta'(0)^\dagger = -\frac{1}{2} \log 2\pi.$

Hence, when $\sigma = 0$, $\tau = \dfrac{1}{\rho}$, we get the asymptotic expansion

$$\prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{n^{\frac{1}{\rho}}}\right) e^{\sum_{m=1}^{p} \frac{1}{m}\left(-\frac{z}{n^{1/\rho}}\right)^m}\right] = z^{-\frac{1}{2}}\, e^{\frac{\pi}{\sin \pi\rho} z^\rho - \frac{1}{2\rho}\log 2\pi + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}\zeta\left(-\frac{s}{\rho}\right)}{s z^s}},$$

which agrees with the expansion of § 65.

Simple Repeated Functions of Finite Integral Order.

§ 81. It is obvious from the investigations of §§ 70–73 that the asymptotic expansion obtained in § 79 for $\prod_{n=1}^{\infty}\left[\left(1 + \dfrac{z}{a_n}\right)^{\mu_n} e^{\mu_n \sum_{m=1}^{p} \frac{1}{m}\left(-\frac{z}{a_n}\right)^m}\right]$, where $p < \rho < p+1$ and ρ is such that $\Sigma \dfrac{\mu_n}{a_n^{\rho+\epsilon}}$ is convergent, and $\Sigma \dfrac{\mu_n}{a_n^{\rho-\epsilon}}$ divergent, will hold in the limit when $\rho = p$, provided that in any terms which become infinite we reject the infinite part and keep only the corresponding finite expression found by applying the usual methods of the calculus of limits to the subsidiary Fourier and Maclaurin series. Consider, for example, the function

$$\prod_{n=1}^{\infty}\left\{\left(1 + \frac{z}{n^\tau}\right)^{n^\sigma} e^{n^\sigma \sum_{m=1}^{p} \frac{1}{m}\left(-\frac{z}{n^\tau}\right)^m}\right\}, \qquad \text{where} \quad \frac{\sigma+1}{\tau} = p.$$

The asymptotic expansion obtained previously was

* "Theory of the Gamma Function," § 27.

† *Ibid.*, § 30.

$$z^{\zeta(-\sigma)}\, e^{\frac{\pi}{\sin\frac{\pi(\sigma+1)}{\tau}} \frac{z^{\frac{\sigma+1}{\tau}}}{(1+\sigma)} + \tau\zeta'(-\sigma) + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}\zeta(-\sigma-\tau s)}{sz^s}}.$$

Now, when $\dfrac{\sigma+1}{\tau} = p =$ an integer, $\dfrac{\pi}{\sin\dfrac{\pi(\sigma+1)}{\tau}} z^{\frac{\sigma+1}{\tau}}$ becomes infinite.

We have then to substitute a corresponding finite expression derived by proceeding to the limit in the infinite terms of the subsidiary Fourier series. When $\dfrac{1}{\tau}$ is not integral, the series and its equivalent value are given by

$$\sum_{s=1}^{\infty} \frac{(-)^{s-1}e^{-si\theta}}{s\left(1+\dfrac{s\tau}{1+\sigma}\right)} + \sum_{s=1}^{\infty} \frac{(-)^s e^{si\theta}}{s\left(1-\dfrac{s\tau}{1+\sigma}\right)} = \frac{\pi e^{i\theta\frac{\sigma+1}{\tau}}}{\sin\pi\dfrac{\sigma+1}{\tau}} - \frac{\tau}{1+\sigma} - i\theta.$$

When $\dfrac{1}{\tau}$ is integral, this series, omitting the term for which $s = \dfrac{1+\sigma}{\tau}$ in the second summation, is equal to the finite part of

$$\frac{e^{i\theta\frac{\sigma+1}{\tau}}(1+\epsilon\iota\theta+\ldots)}{(-)^{\frac{1+\sigma}{\tau}}\epsilon} - \frac{\tau}{1+\sigma} - i\theta - \frac{\tau(-)^{\frac{1+\sigma}{\tau}}}{1+\sigma} e^{i\theta\frac{\sigma+1}{\tau}},$$

when $\epsilon = 0$; that is to say, it is equal to

$$(-)^{\frac{\sigma+1}{\tau}-1} e^{i\theta\frac{\sigma+1}{\tau}}\left(-i\theta+\frac{\tau}{1+\sigma}\right) - \frac{\tau}{1+\sigma} - i\theta.$$

We thus replace

$$\frac{\pi}{\sin\dfrac{\pi(\sigma+1)}{\tau}} \cdot \frac{z^{\frac{\sigma+1}{\tau}}}{\sigma+1} \quad \text{by} \quad \frac{(-z)^{\frac{\sigma+1}{\tau}}}{1+\sigma}\left\{\log z - \frac{\tau}{1+\sigma}\right\}.$$

Again, since σ is a positive integer, the only term of the form

$$\zeta(-\sigma-\tau s)\ s = -p, \ldots -1, 1, 2, \ldots \infty$$

which becomes infinite is that for which $s = -p$.

This term is $\zeta(+1)$, which arises from the Maclaurin series

$$\sum_{n=1}^{m-1} \frac{1}{n} = \left[\zeta(s) + \frac{1}{1-s} \cdot \frac{1}{m^{s-1}}\right]_{s=1} - \frac{1}{2m} + \ldots = \log m + \gamma - \frac{1}{2m} +$$

We have already taken account of the substitution of $\log m$ for $\dfrac{1}{1-s} \cdot \dfrac{1}{m^{s-1}}$; we need, therefore, only replace $\zeta(+1)$ by γ.

We have then the asymptotic expansion

$$\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n^{\tau}}\right)^{n^{\sigma}} e^{n^{\sigma}\sum_{s=1}^{\frac{\sigma+1}{\tau}}\frac{1}{\sigma}\left(\frac{-z}{n^{\tau}}\right)^{s}}\right\}$$

$$= z^{\zeta(-\sigma)}\exp\left[\tau\zeta'(-\sigma)+\frac{(-z)^{\frac{\sigma+1}{\tau}}}{\sigma+1}\left\{\log z+\gamma\tau-\frac{\tau}{1+\sigma}\right\}\right.$$

$$\left.+\sum_{s=1-\frac{\sigma+1}{\tau}}^{\infty}{}' \frac{(-)^{s-1}\zeta(-\sigma-\tau s)}{s z^{s}}\right],$$

in which $\frac{1}{\tau}$ and σ are both integers, and $\frac{\sigma+1}{\tau}$ is the "genre" of the function.

It is interesting to notice that the constants which enter into the asymptotic expansion of this very general function are all values of the Riemann ζ function.

§ 82. When $\tau = 1$, the function is to an exponential factor an important function which I have proposed to call the σ-ple G function. These G functions are derived from the multiple Gamma functions by the coalescence of the parameters. The theory of the simple G function has been developed elsewhere* in the second of a series of papers on Gamma functions.

In that development I took

$$G(z+1) = (2\pi)^{\frac{z}{2}} e^{-\frac{z.z+1}{2}-\gamma\frac{z^2}{2}}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^{n} e^{-z+\frac{z^2}{2n}}\right]$$

and obtained† the asymptotic expansion

$$\log G(z+1) = \tfrac{1}{12} - \log A + \frac{z}{2}\log 2\pi + \left(\frac{z^2}{2}-\tfrac{1}{12}\right)\log z - \frac{3z^2}{4} + \sum_{s=1}^{\infty}\frac{(-)^{s}B_{s+1}}{2s\,.\,2s+2\,z^{2s}},$$

where A is the Glaisher-Kinkelin constant.

Putting $\sigma = \tau = 1$, the asymptotic expansion which we have obtained for the same function in the present paragraph is

$$\frac{z}{2}\log 2\pi - \frac{z(z+1)}{2} - \frac{\gamma z^2}{2} + \zeta(-1)\log z + \frac{z^2}{2}\log z + \frac{\gamma z^2}{2} - \frac{z^2}{4}$$

$$+\zeta'(-1) + \sum_{s=-1}^{\infty}{}' \frac{(-)^{s-1}\zeta(-s-1)}{s z^{s}}.$$

Now ‡

$$\zeta(-1) = -\tfrac{1}{12}, \quad \zeta(0) = -\tfrac{1}{2},$$

$$\zeta(-s-1) = 0, \text{ when } s \text{ is odd},$$

and

$$= \frac{(-)^{t-1}B_{t+1}}{(2t+2)}, \text{ when } s = 2t.$$

* 'Quarterly Journal of Mathematics,' vol. 31, pp. 264 *et seq.*

† *Ibid.*, § 15. ‡ *Ibid.*, § 23.

And, since A is given by the identity,

$$\sum_{n=1}^{m-1} n \log n = \log A + \left(\frac{m^2}{2} - \frac{m}{2} + \frac{1}{12}\right) \log m - \frac{m^2}{4} + \text{terms which vanish when } m \text{ is}$$

infinite, we have $\log A = -\zeta'(-1) + \frac{1}{12}$.

Thus the asymptotic expansion of the present paragraph may be written

$$\frac{1}{12} - \log A + \frac{z}{2} \log 2\pi + \left(\frac{z^2}{2} - \frac{1}{12}\right) \log z - \frac{3z^2}{4}$$
$$+ \sum_{s=1}^{\infty} \frac{(-)^s B_{s+1}}{2s \,.\, \overline{2s+2}\, z^{2s}}.$$

We thus obtain a valuable verification of our results.

Repeated Simple Integral Functions of Transcendental Index.

§ 83. It is obvious that such a function as

$$\prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{a_n}\right)^{e^n} e^{e^n \sum_{m=1}^{p} \left(-\frac{z}{a_n}\right)^m}\right]$$

is of infinite order when a_n is algebraic.

If, however, a_n is of the same order as e^n, the order of the function is finite, and can therefore be expanded in the neighbourhood of infinity by our methods.

We shall take, as an example of repeated functions of transcendental index, the product

$$\prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{e^{qn}}\right)^{e^{pqn}} e^{e^{pqn} \sum_{m=1}^{\rho} \left(-\frac{z}{e^{qn}}\right)^m}\right].$$

This function is of order ρ, greater than or equal to unity.

Suppose first that p is not an integer, so that ρ is the integer next greater than p. Then without former notation

$$\sum_{n=1}^{m-1} e^{pqn} = \int^m e^{pqn}\, dn + M - \frac{e^{pqn}}{2} + \ldots$$

$$\sum_{n=1}^{m-1} e^{pqn}\, qn = \int^m e^{pqn}\, qn\, dn - \log F_0 - \tfrac{1}{2} e^{pqm}\, qm + \ldots$$

$$\sum_{n=1}^{m-1} e^{pqn + sqn} = \int^m e^{(p+s)\,qn}\, dn - F_s - \frac{e^{\overline{p+s}\,qm}}{2} + \ldots$$

And

$$f(z) = \frac{\log z}{pq} e^{pqm} - \frac{m}{p} e^{pqm} + \frac{1}{p^2 q} e^{pqm} + \sum_{s=-k}^{k}{}' \frac{1}{q}\left[\frac{(-)^{s-1}}{s z^s} \frac{e^{\overline{p+s}\,qm}}{p+s}\right],$$

where we take the limit when $k = \infty$ of the summable divergent series.

Thus

$$p\,\frac{f(z)}{e^{pqm}} = \frac{\log z}{q} - m + \frac{1}{pq} + \frac{1}{q}\sum_{=-k}^{k}{}' \left[\frac{(-)^{s-1}}{s}\left(\frac{e^{qm}}{z}\right)^{s}\right] - \frac{1}{q}\sum_{s=-k}^{k}{}' \left[\frac{(-)^{s-1}}{p+s}\cdot\left(\frac{e^{qm}}{z}\right)^{s}\right]$$

$$= \frac{\log z}{q} - m + \frac{1}{pq} + \frac{1}{q}\left\{\frac{\pi\left(\dfrac{z}{e^{qm}}\right)^{p}}{\sin \pi p} - \frac{1}{p} - \log\frac{z}{e^{qm}}\right\}.$$

Therefore

$$f(z) = \frac{\pi z^{p}}{pq \sin \pi p}.$$

Again, M is given by

$$\frac{e^{\alpha} - e^{\alpha m}}{1 - e^{\alpha}} = \mu + \underset{k=\infty}{\mathrm{Lt}}\, \frac{e^{\alpha m}}{\alpha}\left[1 - \frac{\alpha}{2} + \sum_{t=0}^{k}\frac{(-)^{t}\,\mathrm{B}_{t+1}\,\alpha^{2t+2}}{\overline{2t+2}\,!}\right] \quad . \quad . \quad . \quad . \quad . \qquad (1),$$

$$= \mathrm{M} + \frac{e^{\alpha m}}{\alpha}\cdot\frac{\alpha}{1 - e^{\alpha}}, \text{ when } \alpha = pq.$$

Thus

$$\mathrm{M} = \frac{e^{pq}}{1 - e^{pq}}.$$

Also $-\mathrm{F}_s$ is given by putting $\alpha = \overline{p + s}\,q$ in this same expansion.

Thus

$$-\mathrm{F}_s = \frac{e^{q\,.\,\overline{p+s}}}{1 - e^{q\,.\,\overline{p+s}}}.$$

Again, by substituting $\alpha + \epsilon$ for α, expanding in power of ϵ, and equating coefficients of the first power of ϵ in the asymptotic identity (1), we readily find

$$-\log \mathrm{F}_0 = q\,\frac{e^{pq}}{(1 - e^{pq})^{2}}.$$

If, then, p is not an integer, we have the asymptotic expansion

$$\prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{e^{qn}}\right)^{e^{pqn}} e^{\,e^{pqn}\sum\limits_{m=1}^{\rho}\left(\frac{-z}{e^{qn}}\right)^{m}}\right] = z^{-\left(1-e^{-pq}\right)^{-1}}\, e^{\frac{\pi z^{p}}{pq\sin \pi p} - \frac{q\,e^{pq}}{\left(1-e^{pq}\right)^{2}} + \sum\limits_{s=-\rho}^{\infty}{}' \frac{(-)^{s-1} e^{q\,\overline{p+s}}}{s\left(1-e^{q\,\overline{p+s}}\right) z^{s}}}.$$

§ 84. Suppose next that p is an integer—so that $\rho = p$. The analysis will, of course, be slightly more complicated.

The constant F_{-p} will be given by

$$\underset{s=p}{\mathrm{Lt}}\left\{\sum_{n=1}^{m-1} e^{qn(p-s)}\right\} = \int^{m} dn - \mathrm{F}_{-p} - \tfrac{1}{2}$$

or

$$m - 1 = m - \tfrac{1}{2} - \mathrm{F}_{-p}, \text{ so that } \mathrm{F}_{-p} = \tfrac{1}{2}.$$

And we shall have

$$\frac{p\,f(z)}{e^{pqm}} = \frac{\log z}{q} - m + \frac{1}{pq} + (-)^p \frac{mz^p}{e^{pqm}} + \frac{1}{q} \sum_{s=-k}^{k}{}'' \left[\frac{(-)^{s-1}}{s}\left(\frac{e^{qm}}{z}\right)^s\right] - \frac{1}{q} \sum_{s=-k}^{k}{}'' \left[\frac{(-)^{s-1}}{p+s} \cdot \left(\frac{e^{qm}}{z}\right)^s\right],$$

the double dash denoting that the terms for which $\left.\begin{matrix} s = 0 \\ s = -p \end{matrix}\right\}$ are to be omitted.

We therefore have

$$\frac{p\,f(z)}{e^{pqm}} = \frac{\log z}{q} - m + \frac{1}{pq} + (-)^p \frac{mz^p}{e^{pqm}} - \frac{(-)^p}{pq}\left(\frac{z}{e^{qm}}\right)^p$$
$$+ \frac{1}{q}\left[(-)^p\left(\frac{z}{e^{qm}}\right)^p \log\left(\frac{z}{e^{qm}}\right) - \frac{1}{p} - \log\left(\frac{z}{e^{qm}}\right)\right].$$

Therefore
$$f(z) = \frac{(-)^p}{pq} z^p \log z - \frac{(-)^p}{p^2 q} z^p.$$

We thus have the asymptotic expansion

$$\prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{e^{qn}}\right)^{e^{pqn}} e^{e^{pqn} \sum_{m=1}^{p} \left(\frac{-z}{e^{qn}}\right)^m}\right] = z^{-(1-e^{-pq})^{-1}} e^{\frac{(-z)^p \log z}{pq} - \frac{(-z)^p}{p^2 q} - q\frac{e^{pq}}{(1-e^{pq})^2} + \frac{(-)^{p-1} z^p}{2p}}$$
$$\times\, e^{\sum_{s=-p+1}^{\infty}{}' \frac{(-)^{s-1} e^{q(p+s)}}{s\left(1-e^{q(p+s)}\right) z^s}}.$$

We have now given examples of the asymptotic expansions of repeated simple functions with transcendental index in the cases when the order is or is not integral.

And it is evident that such examples might be multiplied indefinitely. In the more complex cases the difficulties of the analysis will, no doubt, be very great; but such difficulties in no way invalidate the theory which has been developed.

PART V.

Applications of the Previous Asymptotic Expansions.

§ 85. We proceed now to consider some applications of the previous theorems to such questions concerning integral functions as have been raised in the Introduction to the present paper.

In the first place, a knowledge of the asymptotic expansion of a function serves to determine the number of roots which it possesses inside a circle of given large radius.

Let us consider the simple example of the Gamma function, for which we have the asymptotic equality

$$\frac{1}{\Gamma(z)} = (2\pi)^{-\frac{1}{2}} z^{z-\frac{1}{2}} \exp\left\{\gamma z + \sum_{t=0}^{\infty} \frac{(-)^t B_{t+1}}{2t+1.2t+2} \frac{1}{z^{2t+1}}\right\},$$

in which the terms neglected on the right-hand side are of lower exponential order than those retained.

By CAUCHY's theorem the number of roots N within a circle of given large radius r is determined by

$$N = \frac{1}{2\pi\iota}\int \frac{d}{dz} \log \Gamma(z)\, dz,$$

the integral being taken round the circle in question.

Now we may, to terms which vanish exponentially with $\frac{1}{r}$, substitute for $\Gamma(z)$ its value given by the asymptotic expansion. And this expansion is valid for all values of z for which $-\pi < \arg z < \pi$. It is also valid right up to the two limits of $\arg z$, provided the circle on which z lies passes between two consecutive zeros of $\frac{1}{\Gamma(z)}$.

If, now, $z = re^{\iota\theta}$, we have, to terms which vanish exponentially with $\frac{1}{r}$,

$$N = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[-1 + \frac{e^{-\iota\theta}}{2r} - \log r - \iota\theta - \gamma + \sum_{t=0}^{\infty} \frac{(-)^t B_{t+1}}{2t+2}\frac{e^{-(2t+2)\iota\theta}}{r^{2t+2}}\right] re^{\iota\theta}\, d\theta.$$

Now

$$\frac{-\iota r}{2\pi}\int_{-\pi}^{\pi} \theta e^{\iota\theta}\, d\theta = \frac{-\iota r}{2\pi}\left\{\left[\frac{1}{\iota}\theta e^{\iota\theta}\right]_{-\pi}^{\pi} - \frac{1}{\iota}\int_{-\pi}^{\pi} e^{\iota\theta}\, d\theta\right\}$$

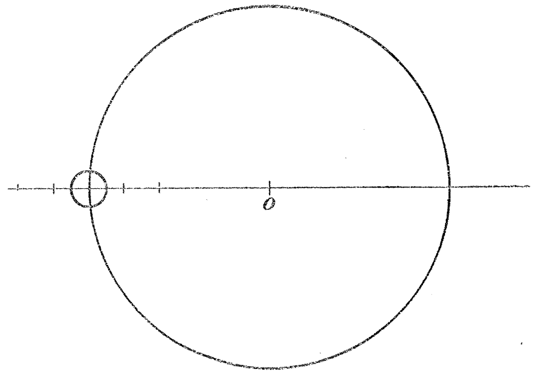
$$= \frac{-r}{2\pi}\{\pi e^{\iota\pi} + \pi e^{-\iota\pi}\}$$

$$= r.$$

Therefore, to terms which are ultimately exponentially small,

$$N = r + \tfrac{1}{2}.$$

Of course we know independently that the number of roots is the greatest integer less than r. And the entrance of the term $\frac{1}{2}$ might have been predicted *à priori*, for when the circle of radius r passes through a zero of $\frac{1}{\Gamma(z)}$ we jump from $-\frac{1}{2}$ to $+\frac{1}{2}$ as we integrate round a small circle enclosing this zero.



§ 86. It is interesting to notice that the analysis verifies itself in the same way for

the function $F(z) = \prod_{n=1}^{\infty}\left[1 + \frac{z}{e^n}\right]$, which, by a change of the independent variable, reduces to LAMBERT's function.

For this function we have obtained the asymptotic expansion

$$\log F(z) = \tfrac{1}{2}(\log z)^2 - \tfrac{1}{2}\log z + \frac{\pi^2}{6} + \sum_{s=1}^{\infty} \frac{(-)^{s-1} C_s}{s z^s}.$$

Therefore $\dfrac{d}{dz}\log F(z) = \dfrac{\log z}{z} - \dfrac{1}{2z} + \sum_{s=1}^{\infty} \dfrac{(-)^s C_s}{z^{s+1}}$.

The number of roots within a circle of radius r is, therefore, to terms which are exponentially small when r is large

$$\frac{1}{2\pi\iota}\int_{-\pi}^{\pi}\left\{\frac{\log r}{r}e^{-\iota\theta} + \frac{e^{-\iota\theta}}{r}(\iota\theta - \tfrac{1}{2})\right\} re^{\iota\theta}\,\iota d\theta = \log r - \tfrac{1}{2}.$$

Since the function has no zero at the origin, we should have predicted the occurrence of the term $-\frac{1}{2}$.

§ 87. We may now prove that, if the dominant term of the zero of an integral function is algebraic and such that the zero is of non-integral order ρ (where ρ is neither zero nor infinite, but greater or less than unity), then the number of roots of the function within a circle of large radius r is to a first approximation

$$\frac{\sin \pi\rho}{\pi}\log \phi(r),$$

where $\phi(r)$ is the maximum value of the modulus of the function on the circle in question.

There are two cases to be considered according as ρ is greater or less than unity. We take the former, the argument will hold in detail for the latter by changing ρ into $\dfrac{1}{\rho}$.

Let $F(z)$ be the function in question; then, under the conditions enunciated, $\log F(z)$ is equal to $\dfrac{\pi}{\sin \pi\rho} z^\rho$ + terms of lower order.

Hence N, the number of roots required is, to a first approximation, given by

$$N = \int_{-\pi}^{\pi} \frac{\pi\rho}{\sin \pi\rho} r^\rho e^{\rho\iota\theta}\,\iota d\theta = r^\rho = \frac{\sin \pi\rho}{\pi}\log \phi(r).$$

We thus complete and prove BOREL's intuition.

§ 88. When ρ is an integer, the preceding theorem ceases to be valid. But we can now prove that the number of roots to a first approximation is $\dfrac{\log \phi(r)}{\log r}$.

For, the same conditions still being supposed to hold good, $\log F(z)$ has to a first approximation been proved to be equal to

$$(-z)^p \log z + (-)^{p-1} \frac{z^p}{p} + \frac{(-)^{p-1} \gamma}{p} z^p + \text{lower terms.}$$

And therefore

$$\frac{d}{dz} \log F(z) = (-)^p \, p z^{p-1} \log z + \text{lower terms.}$$

Hence, to a first approximation,

$$N = \frac{1}{2\pi\iota} \int_{-\pi}^{\pi} p \, (-)^p \, e^{p\iota\theta} \, r^p \left[\log r + \iota\theta\right] \iota \, d\theta$$

$$= (-r)^p \frac{\iota p}{2\pi} \int_{-\pi}^{\pi} e^{p\iota\theta} \, \theta \, d\theta = r^p = \frac{\log \phi(r)}{\log r},$$

which establishes the theorem in question.

§ 89. In the two preceding paragraphs we have assumed that we were dealing with non-repeated functions.

From the analysis of Part IV. it is, however, evident that the theorems hold *in toto* for repeated functions, the order being that which has been assigned to such functions.

We cannot, of course, attempt to prove the theorems for functions of multiple sequence until we have investigated the corresponding asymptotic expansions.

§ 90. We may next write down a number of theorems relating to two or more integral functions.

It is obvious that the sum of two or more integral functions of simple sequence is an integral function of order equal to the largest order of the component integral functions. We may replace additive signs by those of subtraction if the two component functions of largest order are not identically equal. The large zeros of the compound expression are to the first order of approximation equal to the large corresponding zeros of the component function of largest order.

The product of two or more integral functions of simple sequence is an integral function of order equal to the largest order of the component integral functions.

The number of zeros of the equation

$$h_1(z) \, F_1(z) + h_2(z) \, F_2(z) + \ldots \qquad + h_n(z) \, F_n(z) = 0,$$

where the F's are integral functions of simple sequence and the h's algebraic polynomials, within a circle of large radius is ultimately to a first approximation equal to the number of zeros within that circle of the function of largest order.

§ 91. The expansions which have been obtained may be utilised to give a proof of BOREL's extension of a theorem due to PICARD.*

* PICARD, 'Annales de l'École Normale Supérieure,' 2 ser., t. 9 (1880). BOREL, 'Acta Mathematica,' t. 20, pp. 382–388.

The identity

$$\sum_{\iota=1}^{n} G_\iota(z)\, e^{H_\iota(z)} = 0,$$

in which the G's are integral functions of simple sequence of any finite or zero order not greater than some number ρ, and the functions $H_\iota - H_k$ are polynomials of order greater than ρ or transcendental integral functions, necessarily involves

$$G_1(z) = G_2(z) = \ldots \qquad = G_n(z) = 0.$$

Since the identity holds for all points in the plane of the complex variable z, we may consider it in the neighbourhood of $z = \infty$.

Suppose first that the G's are functions of simple sequence of non-integral finite order.

If ρ_ι be the order of $G_\iota(z)$, we shall have near $z = \infty$ the identity

$$\sum_{\iota=1}^{n} e^{\frac{\pi}{\sin \pi\rho_\iota} z^{\rho_\iota} + \ldots + H_\iota(z)} = 0,$$

where we have neglected in each term terms of lower exponential order than those retained.

The identity will hold for all values of arg z such that z is not within a finite distance of the zeros of the G's.

The functions H (z) by hypothesis cannot be equal to one another. As z tends to infinity, one of them must become infinite to an order which exceeds the order to which all the others become infinite by a quantity of order greater than z^ρ.

The corresponding term (say) $G_1(z)\, e^{H_1(z)}$ is then infinite to an order greater than the order of any other term of the identity $\sum_{\iota=1}^{n} G_\iota(z)\, e^{H_\iota(z)} = 0$.

Since $e^{H(z)}$ cannot vanish, we must then have $G_1(z) = 0$.

The same argument may now be applied to the identity $\sum_{\iota=2}^{n} G_\iota(z)\, e^{H_\iota(z)} = 0$, and it may be proved successively that all the functions G vanish.

And thus the theorem will be proved.

When any of the quantities ρ are integral, a suitable modification of the formulæ in accordance with § 73 shows that the theorem is still true. When the G's are repeated functions, a corresponding modification again establishes the theorem. When the functions G's reduce to constants c_ι, so that $\rho = 0$, the theorem is still true, the functions H being unequal.

§ 92. We pass now to the consideration of the resemblance between an integral function of simple sequence and its derivative.

And I would remark that, in the same manner as ROLLE's theorem is proved, it may be established that the real zeros of such a function with real coefficients are

separated by zeros of its derivative. [It cannot, however, be proved that the derived function has not other real or a (necessarily even) number of other imaginary zeros.] This theorem I shall not prove, as it is not connected with the main developments of the present paper. We proceed, however, to show that such developments complete and to some extent verify this extension of ROLLE'S theorem, and that incidentally they furnish many criteria as to the nature of the derivative of a given integral function.

§ 93. Let us consider, as an elementary example, the function of genre zero and order $\frac{1}{\rho}$,

$$P_\rho(z) = \prod_{n=1}^{\infty}\left[1 + \frac{z}{n^\rho}\right], \text{ where } \rho > 1.$$

We have the asymptotic equality

$$P_\rho(z) = (2\pi)^{-\frac{\rho}{2}} z^{-\frac{1}{2}} \exp.\left[\frac{\pi}{\sin\frac{\pi}{\rho}} z^{\frac{1}{\rho}} + \sum_{s=1}^{\infty} \frac{(-)^{s-1} F(\rho s)}{z^s}\right].$$

Remember, now, that it has been proved in Part II. that we may differentiate an asymptotic equality of this type, and we obtain

$$\frac{d}{dz} P_\rho(z) = (2\pi)^{-\frac{\rho}{2}} z^{-\frac{1}{2}} \exp\left\{\frac{\pi}{\sin\frac{\pi}{\rho}} z^{\frac{1}{\rho}} + \ldots\right\}\left\{\frac{\pi}{\sin\frac{\pi}{\rho}} z^{\frac{1-\rho}{\rho}} - \frac{1}{2z} + \ldots\right\}$$

$= (2\pi)^{-\frac{\rho}{2}} \frac{\pi}{\sin\frac{\pi}{\rho}} z^{\frac{1+\rho}{\rho}} \exp\left\{\frac{\pi}{\sin\frac{\pi}{\rho}} z^{\frac{1}{\rho}} + \ldots\right\}$ together with terms whose ratio to the terms retained tends to zero as $|z|$ tends to infinity.

From this expansion we see that

(1) $P_\rho'(z)$ is of the same order as $P_\rho(z)$,

(2) The zeros of $P_\rho'(z)$ are such that, when n is large, we have with the usua notation $a_n = n^\rho + (\rho - 1) n^{\rho-1} +$ lower terms.

Not only so, but theoretically, by finding successive terms in the expansion for $P_\rho'(z)$, we ought to be able to determine the form of its n^{th} zero as nearly as we please. Practical difficulties will, of course, arise when we come, in the asymptotic expansion, to a term which arises from a transcendental term in the n^{th} zero of $P_\rho'(z)$.

Note that the formula for a_n may be readily verified when $\rho = z$.

For
$$\prod_{n=1}^{\infty}\left[1 + \frac{z}{n^2}\right] = \frac{\sinh \pi\sqrt{z}}{\pi\sqrt{z}},$$

and, therefore, $P_2'(z) = \frac{1}{2}\frac{\cosh \pi\sqrt{z}}{z} - \frac{1}{2}\frac{\sinh \pi\sqrt{z}}{z^{3/2}} = \frac{e^{\pi\sqrt{z}}}{4z} - \frac{e^{\pi\sqrt{z}}}{4\pi z^{3/2}}$ asymptotically.

The large zeros of $P_z'(z)$ are approximately those of $\cosh \pi\sqrt{z}$, and the latter are such that, with the usual notation, $a_n = n^2 + n + \frac{1}{4}$.

We notice that the general form of a_n given above shows that the large zeros of $P_\rho(z)$ separate and are separated by those of $P_\rho'(z)$, a fact which agrees with the extension of ROLLE's theorem.

§ 94. From the preceding example it is now evident that we are in a position to prove that, for all the types of integral functions of which asymptotic expansions have been obtained in this memoir, the order of the function is equal to the order of its derivative. And not only so, but we are theoretically in a position to determine as nearly as we please a formula for the (large) n^{th} zero of the derivative. It would be tedious to consider in turn all cases which can arise: we will take one or two as typical of the rest.

As an immediate corollary of the preceding example it may be seen that the derivative of a simple non-repeated function of order $\frac{1}{\rho}$ less than unity with algebraic zeros of the type $a_n = n^\rho + \theta n^{\rho-1} + \ldots$ is a similar function of equal order, whose zeros are typified by $b_n = n^\rho + (\theta + \rho - 1)\, n^{\rho-1} + \ldots$

§ 95. As a suggestion of the possibility of extending the expansions of Parts III. and IV. let us next write down the first few terms of the asymptotic expansion of

$P(a+z) = \log \prod_{1}^{\infty} \left[\left(1 + \frac{z+a}{a_n}\right) e^{-\frac{z+a}{a_n} + \ldots + \frac{(-)^p (z+a)^p}{p a_n^p}}\right]$, where a is any quantity of finite modulus, and $a_n = n^{1/\rho}\left[1 + \frac{b_1}{n^{\epsilon_1}} + \frac{b_2}{n^{\epsilon_2}} + \ldots\right]$.

The expansion will be (§ 68)

$$\frac{\pi}{\sin \pi\rho}(z+a)^\rho - \tfrac{1}{2}\log(z+a) - \frac{1}{2\rho}\log 2\pi + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{s(z+a)^s} F\left(\frac{s}{\rho}\right)$$

$$- \frac{\sin \pi\rho b_1}{\sin \pi\rho(1-\epsilon_1)}(z+a)^{\rho(1-\epsilon_1)} + \frac{\rho(\rho+1-2\rho\epsilon_1)}{2} b_1^2 \frac{\pi}{\sin \pi\rho(1-2\epsilon_1)}(z+a)^{\rho(1-2\epsilon_1)}$$

$$- \frac{\sin \pi\rho b_2}{\sin \pi\rho(1-\epsilon_2)}(z+a)^{\rho(1-\epsilon_2)} + \ldots + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{s(z+a)^s} Z\left(\rho, s; \begin{matrix}\epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots\end{matrix}\right)$$

$$- Z\left(0; \begin{matrix}\epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots\end{matrix}\right),$$

and may be transformed into

$$\frac{\pi}{\sin \pi\rho} z^\rho - \tfrac{1}{2}\log z - \frac{1}{2\rho}\log 2\pi + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{s z^s} F\left(\frac{s}{\rho}\right) + \frac{a\rho\pi}{\sin \pi\rho} z^{\rho-1}$$

$$+ \frac{\rho(\rho-1)\pi}{\sin \pi\rho} a^2 z^{\rho-2} + \ldots - \tfrac{1}{2}\frac{a}{z} \ldots + (-)^p z^{p-1} a F\left(-\frac{p}{\rho}\right)$$

$$+ \left[(-)^p \frac{p-1}{2} a^2 F\left(\frac{p}{\rho}\right) + (-)^{p-1} a F\left(\frac{p-1}{\rho}\right)\right] z^{p-2} + \ldots$$

[OVER.

$$- \frac{\pi\rho b_1}{\sin \pi\rho (1 - \epsilon_1)} z^{\rho(1-\epsilon_1)} - \frac{\pi\rho^2 b_1 (1 - \epsilon_1)}{\sin \pi\rho (- \epsilon_1)} a z^{\rho(1-\epsilon_1)-1} + \ldots$$

$$+ \frac{\rho(\rho + 1 - 2\epsilon_1)}{2} b_1^2 \frac{\pi}{\sin \pi\rho (1 - 2\epsilon_1)} z^{\rho(1-2\epsilon_1)} + \frac{\rho^2 (\rho + 1 - 2\epsilon_1)(1 - 2\epsilon_1)}{2} \frac{\pi a z^{\rho(1-2\epsilon_1)-1}}{\sin \pi\rho (1 - 2\epsilon_1)} + \ldots$$

$$- \frac{\pi\rho b_2}{\sin \pi\rho (1 - \epsilon_2)} z^{\rho(1-\epsilon_2)} + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{s z^s} Z\left(\rho, s ; \begin{matrix} \epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots \end{matrix}\right)$$

$$- Z\left(0 ; \begin{matrix} \epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots \end{matrix}\right) + (-)^p z^{p-1} a Z\left(\rho, -p ; \begin{matrix} \epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots \end{matrix}\right) + \ldots.$$

By the employment of extended Riemann ζ functions of parameter a, it is impossible to give a form of this expansion which shall include all powers of a, analogous to the expansion of $\log \Gamma(z + a)$, which involves BERNOULLIAN functions of a as coefficients. For brevity we content ourselves with the preceding first approximation.

§ 96. By differentiating the expansion for $\log P(z)$, given in § 68, we have at once, as is evident by the preceding paragraph,

$$\frac{P'(z)}{P(z)} = \frac{\pi\rho}{\sin \pi\rho} z^{\rho-1} - \frac{1}{2z} \ldots \qquad + (-)^p z^{p-1} F\left(\frac{-p}{\rho}\right) + \ldots$$

$$- \frac{\pi\rho^2 b_1 (1 - \epsilon_1)}{\sin \pi\rho (1 - \epsilon_1)} z^{\rho(1-\epsilon_1)-1} - \frac{\pi\rho^2 b_2 (1 - \epsilon_2)}{\sin \pi\rho (1 - \epsilon_2)} z^{\rho(1-\epsilon_2)-1} - \ldots$$

$$+ (-)^p z^{p-1} Z\left(p, -\rho ; \begin{matrix} \epsilon_1, \epsilon_2, \ldots \\ b_1, b_2, \ldots \end{matrix}\right).$$

Thus the asymptotic expansion for $\log P'(z)$ is given by

$$\frac{\pi}{\sin \pi\rho} z^\rho + \left(\rho - \frac{3}{2}\right) \log z - \frac{1}{2\rho} \log 2\pi + \log \frac{\pi\rho}{\sin \pi\rho} + \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{s z^s} F\left(\frac{s}{\rho}\right)$$

$$- \frac{\pi\rho b_1}{\sin \pi\rho (1 - \epsilon_1)} z^{\rho(1-\epsilon_1)} + \frac{\rho(\rho + 1 - 2\rho\epsilon_1)}{2} b_1^2 \frac{\pi}{\sin \pi\rho (1 - \epsilon_1)} z^{\rho(1-2\epsilon_1)} - \ldots$$

$$+ \sum_{s=-p}^{\infty}{}' \frac{(-)^{s-1}}{s z^s} Z\left(\rho, s ; \begin{matrix} \epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots \end{matrix}\right) - Z\left(0 ; \begin{matrix} \epsilon_1, \epsilon_2 \ldots \\ b_1, b_2 \ldots \end{matrix}\right) + \text{terms involving positive}$$
(fractional) powers of $1/z$.

Thus $\frac{d}{dz} P(z)$ is a function of the type

$$\prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{b_n}\right) e^{-\frac{z}{b_n} + \ldots + \frac{(-)^p z^p}{p b_n^p}}\right],$$

where $b_n = n^{1/\rho}\left[1 + \frac{b_1}{n^{\epsilon_1}} + \frac{b_2}{n^{\epsilon_2}} + \ldots\right] - n^{1/\rho}\left[\frac{\rho - 1}{\rho n} + \text{higher powers of } \frac{1}{n}\right]$.

Thus the differential of an integral function of order ρ (> 1), where ρ is not integral, is itself an integral function of order ρ whose n^{th} zero, when n is large, will

differ from the corresponding zero of the original function by the term $-\frac{\rho-1}{\rho n^{1-1/\rho}}$, together with terms involving lower powers of n.

In an exactly similar manner it may be proved that the function $R'_\rho(z)$ admits an asymptotic expansion of which the dominant term is

$$(-)^\rho \rho z^{(-z)^\rho+\rho-3/2} \exp. \left\{(-)^{\rho-1}\frac{z^\rho}{\rho}+\log\log z-\frac{1}{2\rho}\log 2\pi+\sum_{s=-\rho}^{\infty}{}' \frac{(-)^{s-1}}{sz^s} F\left(\frac{s}{\rho}\right)\right\},$$

so that $R'_\rho(z)$ is of integral order ρ.

The term $\log\log z$ in the exponential just written down shows that we shall come, sooner or later, to a transcendental term in the expansion of the n^{th} zero of $R'_\rho(z)$.

Similarly the theorem may be established for the general simple non-repeated function of finite integral order.

As regards the application of the same methods to simple repeated functions it is only necessary to notice that corresponding to a zero k times repeated of the original function there will be a zero $(k-1)$ times repeated of its derivative.

§ 97. We have now to consider whether the derivative of an integral function, all of whose roots are real, can have zeros other than the real zeros which by the extension of ROLLE's theorem separate the roots of the original function.

For this purpose let us consider the difference between the number of roots of $P(z)$ and of $P'(z)$ within a circle of very large radius r.

This number will be $N = \frac{1}{2\pi\iota}\int\left\{\frac{d}{dz}\log P'(z) - \frac{d}{dz}\log P(z)\right\}dz$

$= \frac{1}{2\pi\iota}\int\frac{d}{dz}\log\frac{P'(z)}{P(z)}dz$, where the integral is taken round the circle in question.

Now by examining the various cases which can arise, it may at once be seen that the asymptotic expansion of $\frac{d}{dz}\log\frac{P'(z)}{P(z)}$ is given by $(\rho-1)\log z +$ terms which vanish when $|z| = \infty$. Therefore to a first approximation we have $N = \rho - 1$.

If then the function $P(z)$ is of genre p, its derivative can at most have only p zeros besides those demanded by the extension of ROLLE's theorem.

And therefore when p is odd, $P'(z)$ can at most have only $(p-1)$ imaginary roots.*
In particular when $P(z)$ is of genre 0 or 1, $P'(z)$ can have no imaginary roots.†

From this theorem coupled with the expansion given in § 5 and the equality

$$\frac{d}{dz}\frac{J_n(z)}{z^n} = -\frac{J_{n+1}(z)}{z^n},$$

* BOREL, 'Fonctions Entières,' p. 44.

† LAGUERRE, 'Œuvres,' t. 1, pp. 167 *et seq.*

we see that all the zeros of $\frac{J_{n+1}(z)}{z^{n+1}}$ are real if the zeros of $\frac{J_n(z)}{z^n}$ are real—a theorem due to MACDONALD.*

If the function $P(z)$ be multiplied by an algebraic polynomial with real coefficients whose degree is q, the derivative of the product can at most have only $p+q$ imaginary roots.

§ 98. So far we have only considered integral functions whose roots are all real and negative. If, however, we have an integral function all of whose roots lie along a line other than the negative half of the real axis, a change of the independent variable will at once reduce it to an integral function all of whose roots are real and negative.

If then an integral function of genre p have all its roots but q lying in a sequence along a straight line through the origin, its derivative will at most have $p+q$ roots which do not lie along this line.

§ 99. We now conclude for the present the applications of the expansions which have been obtained. There are many questions which are still to be discussed—for instance :—

(1) Functions of infinite order ;
(2) Functions of multiple sequence ;
(3) Asymptotic expansions deducible from linear differential equations ;
(4) The rate of increase of the coefficients of the TAYLOR'S series expansion of an integral function ; and so on.

Investigations in connection with each of these questions have been tentatively undertaken—notably by BOREL, HORN, HADAMARD and POINCARÉ. And I find it possible to extend, by the methods of this memoir, many of the results which have hitherto been obtained. But such investigations I leave for future publication.

* MACDONALD, "Zeros of the Bessel Functions," 'Proc. Lond. Math. Soc.,' vol. 29, p. 575.