# V. *On the Vibrations and Stability of a Gravitating Planet.*

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*Communicated by Professor* G. H. Darwin, *F.R.S.*

Received November 8,—Read December 4, 1902.

## *Introduction.*

§ 1. In a former paper* I have considered the effect of gravitation as a factor tending towards instability, in the case of a spherical nebula of gas. The object of the present paper is to investigate the analogous problem in the case of a spherical planet, the planet being supposed composed of solid or fluid matter. The main question at issue is the following.

§ 2. So long as gravitation is neglected there can be no doubt as to the stability of an elastic solid; any displacement increases the potential energy, and an unstressed configuration of equilibrium is therefore necessarily stable. But when gravitation is taken into account, the gravitational energy may be either increased or decreased by a displacement from equilibrium, and if a displacement can be found which effects a decrease in the gravitational potential energy of amount sufficient to outweigh the increase in the potential of the elastic forces, then the equilibrium configuration will be unstable.

Now, in § 2 of the previous paper already referred to, it was shown that for any spherical body displacements can be found such that there is a decrease in the gravitational potential. This is sufficient to prove that a spherical configuration of equilibrium may be unstable.

In the terminology of Poincaré† it appears that on any "linear series" of spherical configurations there may be "points of bifurcation."

We must, therefore, attempt to settle the position of these points of bifurcation.

In particular, it will be of interest to examine whether a sphere of the size and material of the earth may be regarded as being anywhere in the neighbourhood of a point of bifurcation.

* "The Stability of a Spherical Nebula," 'Phil. Trans.,' A, vol. 199, p. 1.

† 'Acta. Math.,' vol. 7, p. 259.

*Preliminary Approximation.*

§ 3. A rough and very simple calculation will give an approximate answer to this latter question.

Let $a$ be the radius of a sphere, which will ultimately be taken to be the earth, M its mass, and $\rho_0$ the mean density given by $M = \frac{4}{3}\pi\rho_0 a^3$.

Let us use the elastic constants $\lambda$; $\mu$,* and let $\lambda_0$ be the mean value of $\lambda$. Since the sphere is supposed to be spherically symmetrical, $\lambda$, $\mu$, and $\rho$ will be functions of the single co-ordinate $r$, the distance from the centre. Imagine $\lambda/\lambda_0$, $\mu/\lambda_0$, and $\rho/\rho_0$ each expressed as functions of $r/a$, and let $c_1, c_2, \ldots$ be the coefficients which occur in these functions, these coefficients being mere numbers and independent of the system of units in which $\lambda$, $\rho$, and $a$ are measured.

Imagine a linear series of equilibrium configurations obtained by varying any one of the quantities $\lambda_0$, $\rho_0$, or $a$, while keeping the remaining two quantities and the coefficients $c_1, c_2, \ldots$ constants. The points of bifurcation on this series will occur when the varying parameter becomes equal to some definite function of the remaining quantities and of $\gamma$, the gravitational constant.

Hence, however the linear series are arrived at, the points of bifurcation will be given by an equation of the form

$$f(\gamma, \lambda_0, \rho_0, a, c_1, c_2, \ldots) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now the coefficients $c_1, c_2, \ldots$ are mere numbers, and the only way in which $\gamma$, $\lambda_0$, $\rho_0$, and $a$ can be combined so as to give a mere number is through the term $\gamma\rho_0^2a^2/\lambda_0$. Hence equation (1) can be expressed in the form

$$f\left(\frac{\gamma\rho_0^2a^2}{\lambda_0}, c_1, c_2, \ldots\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

We have seen that the spherical configuration must be unstable for some values of $\gamma$, $\rho_0$, $a$, and $\lambda$ (*e.g.*, it is always unstable for $\gamma\rho_0^2a^2/\lambda_0 = \infty$), hence equation (2) must have at least one real root between $\gamma\rho_0^2a^2/\lambda_0 = 0$ and $\gamma\rho_0^2a^2/\lambda_0 = \infty$. Let the lowest root be

$$\gamma\rho_0^2a^2/\lambda_0 = \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where $\phi$ is a function of $c_1, c_2, \ldots$; then a spherical configuration is stable so long as $\gamma\rho_0^2a^2/\lambda_0 < \phi$, and becomes unstable as soon as $\gamma\rho_0^2a^2/\lambda_0 > \phi$.

The coefficients $c_1, c_2, \ldots$ will, on the average, be comparable with unity, because $\lambda$, $\rho$ are referred to their *mean* values; they are as likely (speaking somewhat loosely) to be above as to be below unity. Hence $\phi$ itself will be comparable with unity, and

* The notation is that of LOVE'S 'Theory of Elasticity.' The $m$, $n$ of THOMSON and TAIT are given by

$$\lambda + \mu = m, \qquad \mu = n.$$

it is not at present possible to say whether it is more likely to be greater or less than unity.

§ 4. Now, in the case of the earth (THOMSON and TAIT, § 838), we have

$$a = 640 \times 10^6 \text{ centims.}, \qquad \rho_0 = 5{\cdot}5,$$

and the value of $\gamma$ in C.G.S. units is known to be

$$\gamma = 648 \times 10^{-10}.$$

This gives for $\gamma\rho_0^2a^2$ the value

$$\gamma\rho_0^2a^2 = 8 \times 10^{11},$$

whence it appears that for a sphere of the size and mass of the earth the spherical configuration will be unstable unless $\lambda_0$ has a value comparable with $8 \times 10^{11}$.

Now for steel (*cf.* THOMSON and TAIT, p. 435) the values of the elastic constants in absolute units are $n = \mu = 7{\cdot}7 \times 10^{11}$, $m = \lambda + \mu = 16{\cdot}0 \times 10^{11}$, whence $\lambda = 8{\cdot}3 \times 10^{11}$. We therefore see that the critical values of the elastic constants in the case of the earth are comparable with those of steel.

The foregoing calculation is, of course, very rough, but it shows that the critical values for the earth are at least in the neighbourhood of what must be supposed to be the actual values, so that we are driven to attempting a more accurate determination of these values. If the view of the present paper is sound, this approximate equality is more than a mere coincidence; we shall see that it could have been predicted from our hypotheses of planetary evolution.

We now attempt a rigorous mathematical investigation of certain problems which have a bearing upon the astronomical questions in hand. Those readers whose interest lies in the application of the results rather than in the processes by which they are obtained may be recommended to turn at once to § 22.

## THE STABILITY OF A GRAVITATING ELASTIC SOLID.

### *The Equations of Small Vibrations.*

§ 5. We shall begin by discussing the principal vibrations and the frequency equation of a spherically symmetrical solid. The case of a non-gravitating sphere has been fully discussed by Professor LAMB,* but the inclusion of the gravitational terms, as will be seen later, brings about a considerable complication in the analysis. The case of a gravitating but incompressible sphere has been considered by BROMWICH,† but this has no bearing on the present problem, in which the whole

* "On the Vibrations of an Elastic Sphere," 'Proc. Lond. Math. Soc.,' vol. 13, p. 189.

† "On the Influence of Gravity on Elastic Waves, &c.," 'Proc. Lond. Math. Soc.,' vol. 30, p. 98.

interest turns upon the compressibility. The solution which follows is, in its main points, very similar to that of Professor LAMB, so that I have not thought it necessary to give the steps of the argument in great detail.

From the symmetry of the solid it follows that the elastic constants $\lambda$, $\mu$, and the density $\rho$, will be functions of the single co-ordinate $r$, the distance from the centre. Taking the centre as origin, we shall use rectangular co-ordinates, $x$, $y$, $z$, and shall suppose the solid to execute a small vibration, such that the displacement of the element initially at $x$, $y$, $z$ has components, $\xi$, $\eta$, $\zeta$. The component of displacement along the radius will be denoted by $u$ and the cubical dilatation by $\Delta$, so that

$$u = \frac{1}{r}(\xi x + \eta y + \zeta z), \qquad \Delta = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}.$$

§ 6. After displacement the density at $x$, $y$, $z$ is

$$\rho - \frac{d}{dx}(\rho\xi) - \frac{d}{dy}(\rho\eta) - \frac{d}{dz}(\rho\zeta),$$

or, since $\rho$ is a function of $r$ only,

$$\rho - \Delta\rho - u\frac{d\rho}{dr}.$$

Hence the gravitational potential at $x$, $y$, $z$ is changed by displacement from V into V − E, where E is the potential of the following distribution of matter :—

(i.) A volume distribution of density

$$\Delta\rho + u\frac{d\rho}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

(ii.) A surface distribution of which the surface density is

$$u(\rho_0 - \rho_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

this being taken over every surface at which the density changes abruptly, the change being from $\rho_0$ to $\rho_1$ in crossing the surface in the direction of $r$ increasing. In particular this will occur at the outer surface of the solid, the value of $\rho_1$ in this case being zero.*

§ 7. The potential at $x$, $y$, $z$ after displacement being V − E, that at $x + \xi$, $y + \eta$, $z + \zeta$ will be

$$\mathrm{V} + \xi\frac{\partial \mathrm{V}}{\partial x} + \eta\frac{\partial \mathrm{V}}{\partial y} + \zeta\frac{\partial \mathrm{V}}{\partial z} + \tfrac{1}{2}\xi^2\frac{\partial^2 \mathrm{V}}{\partial x^2} + \ldots$$
$$- \mathrm{E} - \xi\frac{\partial \mathrm{E}}{\partial x} - \eta\frac{\partial \mathrm{E}}{\partial y} - \zeta\frac{\partial \mathrm{E}}{\partial \zeta} - \ldots$$

* In the investigations on gravitating spheres given in THOMSON and TAIT'S 'Natural Philosophy,' the course of procedure is tantamount to neglecting the volume distribution (4), and regarding E as the potential of a surface distribution (5) alone. For this reason the result obtained differs from that of the present paper.

Hence the force at $x+\xi, y+\eta, z+\zeta$ in the direction of $x$ increasing, found by differentiating the foregoing expression with respect to $\xi$, is, neglecting squares of the displacements,

$$\frac{\partial V}{\partial x}+\xi\frac{\partial^2 V}{\partial x^2}+\eta\frac{\partial^2 V}{\partial x\,\partial y}+\zeta\frac{\partial^2 V}{\partial x\,\partial z}-\frac{\partial E}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6).$$

Let us suppose that, in addition to its own gravitation, the sphere is acted upon by an external field of force of potential $V_0$, and let us, in the usual notation, denote the six stresses by P, Q, R, S, T, U. Then the equations of motion of the element at $x+\xi, y+\eta, z+\zeta$ in the displaced configuration are three of the form

$$\rho\frac{\partial^2\xi}{\partial t^2}=\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}+\rho\left(\frac{\partial W}{\partial x}+\xi\frac{\partial^2 W}{\partial x^2}+\eta\frac{\partial^2 W}{\partial x\,\partial y}+\zeta\frac{\partial^2 W}{\partial x\,\partial z}-\frac{\partial E}{\partial x}\right) \quad . \quad . \qquad (7),$$

in which $W=V+V_0$, and all the terms such as $\frac{\partial W}{\partial x}$, $\frac{\partial^2 W}{\partial x^2}$, . . . are evaluated at $x, y, z$, but $\rho$, P, Q . . . are calculated in the displaced configuration at $x+\xi, y+\eta, z+\zeta$.

§ 8. Now the only case in which we have any accurate knowledge as to the values of P, Q, R, S, T, U is when the whole strain is small, *i.e.*, when W is small. In the case of the earth, V is not, in this sense, small.* The only way in which we can proceed with any certainty is, therefore, by taking $V_0=-V$, or $W=0$. That is to say, we must artificially annul gravitation in the equilibrium configuration, so that this equilibrium configuration may be completely unstressed, and each element of matter be in its normal state. In this case it seems justifiable to suppose both the density and rigidity to be constant throughout the sphere, and, indeed, it is only with the help of this simplification that the equations become at all manageable.

The vibrations of this system will be of two kinds. First there are "spherical" vibrations in which the displacement is purely radial at every point, so that the solid remains spherically symmetrical after displacement, and, secondly, there is the larger class of vibrations in which the displacement is not of this simple type, so that the displaced configuration is not one of spherical symmetry.

We hope, by discussing the vibrations of this system, to obtain some insight into the corresponding vibrations of a natural non-homogeneous solid, say the earth. Now it is extremely doubtful whether the spherical vibrations of our artificial system have much in common with those of the natural system, but it will be seen later that this is of no importance. We shall not be in any way concerned with these vibrations. What we shall require is a knowledge of the unsymmetrical vibrations, and this, it is hoped, can be obtained with fair accuracy from a consideration of the corresponding vibrations in the artificial case. There must be some uncertainty even in the case of unsymmetrical vibrations, and, unfortunately, this seems to be inevitable; our

* LOVE, 'Elasticity,' I., p. 220.

artificial case appears to be the only case in which the equations can be solved by ordinary analysis.

We now replace P, Q, R, S, T, U by their ordinarily assumed values, and equation (7), putting $W = 0$, takes the form

$$\rho \frac{d^2\xi}{dt^2} = (\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 \xi - \rho \frac{\partial E}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and there are two similar equations for $\eta$, $\zeta$.

*The Principal Vibrations and Frequency Equations.*

§ 9. Differentiate these three equations of motion with respect to $x$, $y$, $z$ and add; then

$$\rho \frac{d^2\Delta}{dt^2} = (\lambda + 2\mu)\nabla^2\Delta - \rho \nabla^2 E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Now, from the definition of E, we have, in the case in which $\rho$ is constant,

$$\nabla^2 E = -4\pi\rho\Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and hence equation (9) becomes

$$\rho \frac{d^2\Delta}{dt^2} = (\lambda + 2\mu)\nabla^2\Delta + 4\pi\rho^2\Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

If we suppose $\Delta$ proportional to $\cos pt$, this equation assumes the form $(\nabla^2 + \kappa^2)\Delta = 0$, where

$$\kappa^2 = \frac{\rho(p^2 + 4\pi\rho)}{\lambda + 2\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

There is, therefore, a particular solution of (11) of the form

$$\Delta = r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa r) S_n(\theta, \phi) \cos pt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

where $S_n(\theta, \phi)$ is a surface harmonic of order $n$, and the general solution found by summation of solutions of this type is

$$\Delta = \Sigma\Sigma r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa r) S_n(\theta, \phi)(A \cos pt + B \sin pt) \quad . \quad . \quad . \quad . \quad (14),$$

where the summation extends over all possible harmonics, and over all values of $\kappa$.

It will appear later that each term in this solution can be made to satisfy the boundary conditions, and, therefore, that each term represents a normal vibration.

The vibrations may, therefore, be classified into vibrations of *order* 0, 1, 2, &c., the order being that of the harmonic which occurs in the expression for $\Delta$. The vibrations of order $n = 0$ are the spherical vibrations already referred to.

We shall assume this provisionally, in order to avoid the continual repetition of double summation, and now proceed to evaluate $\xi$, $\eta$, $\zeta$ and to form the boundary equations for the simple vibration given by equation (13).

§ 10. From equation (8) it appears that the displacement $\xi$ is given by

$$p^2\rho\xi + \mu\nabla^2\xi = -(\lambda+\mu)\frac{\partial\Delta}{\partial x} + \rho\frac{\partial \mathrm{E}}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

The solution is

$$\xi = \frac{d\phi}{dx} + \xi_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

where $\phi$ is any solution of

$$p^2\rho\phi + \mu\nabla^2\phi = -(\lambda+\mu)\Delta + \rho\mathrm{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

and $\xi_0$ is the most general solution of

$$p^2\rho\xi_0 + \mu\nabla^2\xi_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

It can easily be verified that a solution of equation (17) is

$$\phi = -\frac{1}{p^2}\left(\frac{\lambda+2\mu}{\rho}\Delta - \mathrm{E}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

There will be solutions for $\eta$, $\zeta$ similar to (16), but the three solutions for $\xi$, $\eta$, $\zeta$ must be such that

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

The left-hand member of (20) is, from (16),

$$\nabla^2\phi + \frac{d\xi_0}{dx} + \frac{d\eta_0}{dy} + \frac{d\zeta_0}{dz},$$

and from (19) and (17), $\nabla^2\phi = \Delta$. Hence (20) is satisfied if

$$\frac{d\xi_0}{dx} + \frac{d\eta_0}{dy} + \frac{d\zeta_0}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

§ 11. Write $u$ for $\frac{x}{r}\xi + \frac{y}{r}\eta + \frac{z}{r}\zeta$ as before, and $u_0$ for $\frac{x}{r}\xi_0 + \frac{y}{r}\eta_0 + \frac{z}{r}\zeta_0$. Then we shall verify that the solutions for $u$ and $u_0$ are

$$u = \alpha \mathrm{S}_n, \qquad u_0 = \alpha_0 \mathrm{S}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

in which $\alpha$, $\alpha_0$ are functions of $r$, as yet unknown.

Assuming these solutions, the value of E, calculated as explained in § 6, is

$$\mathrm{E} = \frac{4\pi\rho \mathrm{S}_n}{2n+1}\left\{\frac{1}{r^{n+1}}\int_0^r r^{n+\frac{3}{2}}\mathrm{J}_{n+\frac{1}{2}}(\kappa r)\,dr + r^n\int_r^a r^{-n+\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(\kappa r)\,dr + \frac{r^n}{a^{n-1}}\alpha_a\right\} \quad . \quad (23),$$

where $\alpha_a$ denotes $(\alpha)_{r=a}$.

We can calculate the value of the integrals which occur in this expression, and the sum of the first two terms inside the curled brackets is found to be

$$\frac{2n+1}{\kappa^2} r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa r) - \frac{r^n}{\kappa a^{n-\frac{1}{2}}} J_{n-\frac{1}{2}}(\kappa a).$$

Hence we may write (19) in the form

$$\phi = \mathfrak{G} S_n + \frac{4\pi\rho S_n}{(2n+1)p^2} \frac{r^n}{a^{n-1}} \alpha_a,$$

where

$$\mathfrak{G} = C r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa r) + D r^n J_{n-\frac{1}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24),$$

$$C = -\frac{\lambda + 2\mu}{p^2\rho} + \frac{4\pi\rho}{p^2\kappa^2} = -\frac{1}{\kappa^2},$$

$$D = -\frac{4\pi\rho}{(2n+1)p^2\kappa} a^{-n+\frac{1}{2}}.$$

We now have, from equation (16),

$$\xi = \frac{d}{dx}(\mathfrak{G} S_n) + \frac{4\pi\rho}{(2n+1)p^2} \frac{d}{dx}\left[\frac{r^n \alpha_a S_n}{a^{n-1}}\right] + \xi_0 \quad . \quad . \quad . \quad . \quad . \quad (25),$$

and hence

$$u = \frac{d\mathfrak{G}}{dr} S_n + \frac{4\pi\rho n}{(2n+1)p^2} \frac{r^{n-1}\alpha_a S_n}{a^{n-1}} + u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

§ 12. There are three boundary-conditions to be satisfied, expressing that the normal pressure and the two tangential tractions shall vanish at every point of the free surface. As LAMB* shows, these may be represented by three symmetrical equations, to be satisfied at the surface $r = a$, each of the type

$$\lambda x \Delta + \mu \frac{d}{dx}(ru) + \mu\left(r\frac{d\xi}{dr} - \xi\right) = 0.$$

Substituting for $\xi$ and $u$ from (25) and (26) this becomes

$$\lambda \Delta x + \mu\left[\frac{d}{dx}\left(r\frac{d\mathfrak{G}}{dr} S_n\right) + r\frac{d}{dr}\frac{d}{dx}(\mathfrak{G} S_n) - \frac{d}{dx}(\mathfrak{G} S_n)\right]$$
$$+ \mu \frac{4\pi\rho(2n-2)}{(2n+1)p^2} \frac{d}{dx}\left[\frac{r^n\alpha_a S_n}{a^{n-1}}\right] + \mu\left[\frac{d}{dx}(ru_0) + r\frac{d\xi_0}{dr} - \xi_0\right] = 0 \quad . \quad . \quad (27).$$

§ 13. Write

$$\frac{d}{dx}(r^n S_n) = r^{n-1}\omega, \qquad \frac{d}{dx}(r^{-(n+1)} S_n) = r^{-(n+2)}\chi$$

so that the right-hand members are solid harmonics of degrees $n - 1$ and $-(n + 2)$; then

* LAMB, *loc. cit. ante*, p. 191.

$$xS_n = \frac{r}{2n+1}(\omega - \chi),$$

$$\frac{d}{dx}\{f(r)\,S_n\} = \frac{1}{2n+1}\left\{r^{-(n+1)}\frac{d}{dr}(r^{n+1}f)\,\omega - r^n\frac{d}{dr}(r^{-n}f)\,\chi\right\}.$$

From these identities it is clear that if the terms in (27) which do not depend on $\xi_0$ or $u_0$ are expanded in spherical harmonics, they will contain no harmonics other than $\omega$ and $\chi$. We therefore see that the form of $\xi_0$ may be assumed to be

$$\xi_0 = \mathfrak{P}\omega + \mathfrak{Q}\chi \quad \text{(28)},$$

where $\mathfrak{P}$ and $\mathfrak{Q}$ are functions of $r$. The value of $u_0$ is

$$u_0 = (n\mathfrak{P} - (n+1)\,\mathfrak{Q})\,S_n \quad \text{(29)},$$

whence

$$\alpha_0 = n\mathfrak{P} - (n+1)\,\mathfrak{Q}. \quad \text{(30)}.$$

§ 14. Substituting for $\xi_0$ in (27) and equating the coefficients of $\omega$ and $\chi$, we obtain the two following equations which must be satisfied at $r = a$:—

$$\frac{\lambda}{\mu}\frac{r^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r)}{2n+1} + \frac{1}{2n+1}\left\{r^{-(n+1)}\frac{d}{dr}\left(r^{n+2}\frac{d\mathfrak{G}}{dr}\right) + \left(r\frac{d}{dr} - 1\right)\left(r^{-(n+1)}\frac{d}{dr}(r^{n+1}\mathfrak{G})\right)\right\}$$
$$+ \frac{4\pi\rho(2n-2)}{(2n+1)p^2}\left(\frac{r}{a}\right)^{n-1}\alpha_a + \frac{1}{2n+1}r^{-(n+1)}\frac{d}{dr}(r^{n+2}\alpha_0) + r\frac{d\mathfrak{P}}{dr} - \mathfrak{P} = 0 \quad \text{(31)},$$

and a second equation of a similar kind, of which the first line can be obtained from the first line of the above by writing $-(n+1)$ for $n$, and the second line is

$$-\frac{1}{2n+1}r^n\frac{d}{dr}(r^{-(n-1)}\alpha_0) + r\frac{d\mathfrak{Q}}{dr} - \mathfrak{Q} \quad \text{(32)}.$$

The expression which occurs in curled brackets in (31) can be transformed into

$$2\left\{r^{-(n+1)}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r^{n+1}\mathfrak{G})\right) - nr^{-(n+1)}\frac{d}{dr}(r^{n+1}\mathfrak{G})\right\} \quad \text{(33)},$$

while the corresponding expression in (32) is seen to be

$$2\left\{r^{n+2}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r^{-n}\mathfrak{G})\right) + (n+1)\,r^n\frac{d}{dr}(r^{-n}\mathfrak{G})\right\} \quad \text{(34)}.$$

From the value of $\mathfrak{G}$, given by equation (24),

$$\frac{d}{dr}(r^{n+1}\mathfrak{G}) = C\kappa r^{n+\frac{1}{2}}J_{n-\frac{1}{2}}(\kappa r) + (2n+1)\,Dr^{2n}J_{n-\frac{1}{2}}(\kappa a),$$

$$\frac{d}{dr}(r^{-n}\mathfrak{G}) = -C\kappa r^{-(n+\frac{1}{2})}J_{n+\frac{3}{2}}(\kappa r).$$

Hence expression (33) becomes

$$2\left\{C\kappa^2 r^{\frac{1}{2}} J_{n-\frac{3}{2}}(\kappa r) - nC\kappa r^{-\frac{1}{2}} J_{n-\frac{1}{2}}(\kappa r) + (2n+1)(n-1) D r^{n-1} J_{n-\frac{1}{2}}(\kappa a)\right\},$$

of which the value at $r = a$ is

$$\theta_1 \equiv 2a^{\frac{1}{2}} C\kappa^2 J_{n-\frac{3}{2}}(\kappa a) + 2\left[(2n+1)(n-1) D a^{n-1} - na^{-\frac{1}{2}} C\kappa\right] J_{n-\frac{1}{2}}(\kappa a).$$

This is the value at $r = a$ of the term which occurs in curled brackets in equation (31). The value of the similar term in (32), namely expression (34), is seen to be

$$\theta_2 \equiv 2a^{\frac{1}{2}} C\kappa^2 J_{n+\frac{3}{2}}(\kappa a) - 2(n+1) a^{-\frac{1}{2}} C\kappa J_{n+\frac{3}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad (35).$$

Write

$$\mathfrak{A} = \theta_1 + \frac{\lambda}{\mu} a^{\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

$$\mathfrak{B} = \theta_2 + \frac{\lambda}{\mu} a^{\frac{1}{2}} J_{n+\frac{1}{2}}(\kappa a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

then equations (31) and (32) become, at $r = a$,

$$\mathfrak{A} + \frac{4\pi\rho(2n-2)}{p^2}\alpha_a + a^{-(n+1)}\frac{d}{dr}(r^{n+2}\alpha_0) + (2n+1)\left(r\frac{d\mathfrak{P}}{dr} - \mathfrak{P}\right) = 0 \quad . \quad (38),$$

$$\mathfrak{B} + a^n\frac{d}{dr}(r^{-(n-1)}\alpha_0) - (2n+1)\left(r\frac{d\mathfrak{Q}}{dr} - \mathfrak{Q}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39).$$

Now we have, from equation (26),

$$\alpha_a = \left(\frac{d\mathfrak{G}}{dr}\right)_{r=a} + \frac{4\pi\rho n}{(2n+1)p^2}\alpha_a + (\alpha_0)_{r=a}.$$

Write

$$e = \left[1 - \frac{4\pi\rho n}{(2n+1)p^2}\right]^{-1} \equiv \frac{p^2(2n+1)}{(2n+1)p^2 - 4\pi\rho n},$$

then this last equation becomes

$$\alpha_a = e\left(\alpha_0 + \frac{d\mathfrak{G}}{dr}\right)_{r=a}.$$

Now, at $r = a$,

$$a^{-(n+1)}\frac{d}{dr}(r^{n+2}\alpha_0) = (n+2)\alpha_0 - a\frac{d\alpha_0}{dr},$$

$$a^n\frac{d}{dr}(r^{-(n-1)}\alpha_0) = -(n+1)\alpha_0 + a\frac{d\alpha_0}{dr},$$

and equations (38) and (39) become

$$\mathfrak{A} + \frac{4\pi\rho(2n-2)e}{p^2}\left(\alpha_0 + \frac{d\mathfrak{G}}{dr}\right) + (n+2)\alpha_0 - a\frac{d\alpha_0}{dr} + (2n+1)\left(r\frac{d\mathfrak{P}}{dr} - \mathfrak{P}\right) = 0 \quad . \quad (40),$$

$$\mathfrak{B} - (n+1)\alpha_0 + a\frac{d\alpha_0}{dr} - (2n+1)\left(r\frac{d\mathfrak{Q}}{dr} - \mathfrak{Q}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41),$$

in which $r$ must be put equal to $a$.

§ 15. Now $\xi_0$ is known to be a solution of equation (18), and hence, from equation (28), we may assume

$$\mathfrak{P} = \mathfrak{P}_0 r^{-\frac{1}{2}} \mathrm{J}_{n-\frac{1}{2}}(hr), \qquad \mathfrak{Q} = \mathfrak{Q}_0 r^{-\frac{1}{2}} \mathrm{J}_{n+\frac{3}{2}}(hr),$$

where

$$h^2 = p^2\rho/\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).$$

It follows that at $r = a$ we may write

$$a\frac{d\mathfrak{P}}{dr} = \mathrm{A}\mathfrak{P}, \qquad a\frac{d\mathfrak{Q}}{dr} = \mathrm{B}\mathfrak{Q},$$

where

$$\mathrm{A} = a\frac{d}{da}\log\left(a^{-\frac{1}{2}}\mathrm{J}_{n-\frac{1}{2}}(ha)\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43),$$

$$\mathrm{B} = a\frac{d}{da}\log\left(a^{-\frac{1}{2}}\mathrm{J}_{n+\frac{3}{2}}(ha)\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44).$$

Hence, from (30), we have at $r = a$,

$$a\frac{d\alpha_0}{dr} = n\mathrm{A}\mathfrak{P} - (n+1)\,\mathrm{B}\mathfrak{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Lastly, we have from (28)

$$\xi_0 = \mathfrak{P}r^{-n-1}\frac{d}{dx}(r^n\mathrm{S}_n) + \mathfrak{Q}r^{n+2}\frac{d}{dx}(r^{-(n+1)}\mathrm{S}_n).$$

In order that (21) may be satisfied, we must have

$$\frac{d}{dr}(\mathfrak{P}r^{-(n-1)})\,nr^{n-1} - \frac{d}{dr}(\mathfrak{Q}r^{n+2})\,(n+1)\,r^{-(n+2)} = 0 \quad . \quad . \quad . \quad . \quad (46).$$

Substituting for $\mathfrak{P}$ and $\mathfrak{Q}$, we find that this is satisfied for all values of $r$ if

$$n\mathfrak{P}_0 + (n+1)\,\mathfrak{Q}_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47),$$

*i.e.*, if we have, at $r = a$,

$$\mathfrak{Q} = \theta\mathfrak{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48),$$

where

$$\theta = -\frac{n}{n+1}\frac{\mathrm{J}_{n+\frac{3}{2}}(ha)}{\mathrm{J}_{n-\frac{1}{2}}(ha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

Equations (40) and (41) now assume the forms

$$\mathfrak{A} + \frac{4\pi\rho(2n-2)e}{p^2}\frac{d\mathfrak{G}}{dr} + \mathfrak{P}\left[\frac{4\pi\rho(2n-2)e}{p^2}(n-(n+1)\theta) + (n^2-1) + (n^2+3n+2)\right.$$
$$\left. + (3n+1)\,\mathrm{A} - (n+1)\,\mathrm{B}\theta\right] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$\mathfrak{Y} + \mathfrak{P}\left[n - n^2 + (n^2+2n)\,\theta + n\mathrm{A} - (3n+2)\,\mathrm{B}\theta\right] = 0 \quad . \quad . \quad . \quad (51),$$

in which $r$ must be put equal to $a$.

The general frequency equation may be found at once by the elimination of $\mathfrak{P}$, the values of $\mathfrak{A}$ and $\mathfrak{Y}$ being given by equations (36) and (37).

*Points of Bifurcation.*

§ 16. The interest of the question lies in the position of the points of bifurcation; to find these we must put $p^2 = 0$ in the frequency equation. The reason why it was not possible to put $p^2 = 0$ at an earlier stage will be understood by those who have read the former paper "On the Stability of a Spherical Nebula." In the present instance it is, perhaps, sufficient to say that putting $p^2 = 0$ at an earlier stage would have led to an entirely misleading result. Upon putting $p^2 = 0$ in equations (50) and (51) we find that the two brackets multiplying $\mathfrak{P}$ vanish, and we therefore see that $\mathfrak{P}$ must be treated as an infinite quantity of the order of $1/p^2$.

Expanding these brackets as far as $p^2$, and then putting $p^2 = 0$, we find that the two equations become

$$x_1 - \mathfrak{P}p^2 y_1 = 0 \quad . \quad . \quad . \quad . \quad (52), \qquad x_2 + \mathfrak{P}p^2 y_2 = 0 \quad . \quad . \quad . \quad . \quad (53),$$

where

$$x_1 = \mathfrak{A} + \frac{4\pi\rho(2n-2)e}{p^2}\frac{d\mathfrak{G}}{dr}, \qquad x_2 = \mathfrak{F},$$

$$y_1 = \frac{(n-1)(2n+1)^2}{2n\pi\rho} + \frac{\rho a^2}{\mu}\left\{\frac{2(2n^2-1)}{(2n+1)(2n+3)} - \frac{3n+1}{2(n+1)}\right\}$$

$$y_2 = \frac{\rho a^2}{\mu}\frac{n}{2(2n+1)(2n+3)}.$$

The equation giving points of bifurcation is, of course,

$$x_1 y_2 + x_2 y_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).$$

The values of $x_1$ and $x_2$ are found, after some simplification, to be

$$x_1 = \frac{\lambda}{\mu}a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a) + 2C\left\{a^{\frac{1}{2}}\kappa^2 J_{n-\frac{3}{2}}(\kappa a) - na^{-\frac{1}{2}}\kappa J_{n-\frac{1}{2}}(\kappa a) - \frac{(n-1)(2n+1)}{n}\frac{d}{da}\left(a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a)\right)\right\}$$
$$+ \frac{2(n-1)(2n+1)}{n\kappa}a^{-\frac{1}{2}}J_{n-\frac{1}{2}}(\kappa a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

$$x_2 = \frac{\lambda}{\mu}a^{\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa a) + 2a^{\frac{1}{2}}C\kappa^2 J_{n+\frac{1}{2}}(\kappa a) - 2(n+1)a^{-\frac{1}{2}}C\kappa J_{n+\frac{3}{2}}(\kappa a) \quad . \quad . \quad (56).$$

Now, it has already been seen that $C = -\frac{1}{\kappa^2}$ (p. 164). If we substitute this value for C, write $x$ for $\kappa a$, and simplify equations (55) and (56) as far as possible, we have

$$x_1 a^{-\frac{1}{2}} = \frac{\lambda + 2\mu}{\mu}J_{n+\frac{1}{2}}(x) + \frac{2(2n+1)^2(n-1)}{nx^2}J_{n+\frac{1}{2}}(x) - \frac{2(n-1)(3n+2)}{nx}J_{n+\frac{3}{2}}(x) \quad . \quad (57),$$

$$x_2 a^{-\frac{1}{2}} = \frac{\lambda + 2\mu}{\mu}J_{n+\frac{1}{2}}(x) - \frac{2(n+2)}{x}J_{n+\frac{3}{2}}(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58),$$

while the value of $y_1$ and $y_2$ may be written in the form

$$y_1 = \frac{\rho a^2}{\mu}\left\{-\frac{4n^3+20n^2+21n+7}{(2n+1)(2n+2)(2n+3)} + \frac{2(n-1)(2n+1)^2}{nx^2}\frac{\mu}{\lambda+2\mu}\right\} \quad . \quad . \quad (59),$$

$$y_2 = \frac{\rho a^2}{\mu}\frac{n}{2(2n+1)(2n+3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

The equation giving points of bifurcation can now be found by substituting these values in equation (54).

§ 17. This equation will have roots corresponding to the different integral values of $n$, $n = 0, 1, 2 \ldots$; these determine points of bifurcation such that the critical vibrations are of orders $n = 0, 1, 2 \ldots$ respectively.

Of these the points of bifurcation of zero order are of no importance. The reason is exactly similar to that given in the case of a spherical nebula (§ 28 of the paper already quoted), namely, that a point of bifurcation of order $n = 0$ does not indicate a departure from the spherical shape. We therefore will only discuss values of $n$ different from zero.

*Case of $\mu = 0$.*

§ 18. Before discussing the general form assumed by equation (54), it will be well to consider the simple case of $\mu = 0$. Putting $\mu = 0$, we obtain from equations (57) and (58)

$$x_1 = x_2 = \frac{\lambda+\mu}{\mu} a^{\frac{1}{2}} J_{n+\frac{1}{2}}(x).$$

Referring to equations (52) and (53) we see that the equation giving points of bifurcation is

$$J_{n+\frac{1}{2}}(x) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

The lowest roots of various orders other than zero are

| | | | | |
|---|---|---|---|---|
| $n = 1$, | 2, | 3, | 4, | 5, |
| $x = 4\cdot49$, | $5\cdot76$, | $6\cdot98$, | $8\cdot18$, | $9\cdot37$, &c., |

the roots continually increasing with $n$. Thus the first point of bifurcation is given by $x = 4\cdot49$, and the critical vibration is of order $n = 1$.

*Case of $\mu$ Different from Zero.*

§ 19. The general equation in which $\mu$ is not put equal to zero is much more complicated than equation (61), which has just been considered. If we write $u_n$ for $J_{n+\frac{1}{2}}(x)/J_{n-\frac{1}{2}}(x)$, it will be seen that the equation giving points of bifurcation of order $n$ is of the form

$$u_{n+1} = \text{an algebraic function of } x \text{ and of } (\lambda + 2\mu)/\mu.$$

To obtain approximate numerical solutions, my plan has been to draw graphs of the functions $u_n$, and in this way obtain a graphical solution of the equations for different values of $\mu$. There is no difficulty in drawing graphs of the functions $u_n$; these are trigonometrical functions, and we have

$$u_1 = \frac{1}{x} - \cot x,$$

while the successive $u$'s are connected by the relation

$$u_{n+1} = \frac{2n+1}{x} - \frac{1}{u_n}.$$

To save space I have suppressed all details of this somewhat tedious part of the work. The results for $n = 1, 2, 3$ are given in the following table :—

LOWEST Values of $x$.

| | $\mu = 0.$ | $\mu = \frac{1}{2}\lambda.$ | $\mu = \lambda.$ |
|---|---|---|---|
| $n = 1$ | 4·49 | 4·2 | 4·0 |
| $n = 2$ | 5·76 | 5·6 | 5·4 |
| $n = 3$ | 6·98 | 6·8 | 6·7 |

For large values of $n$ it will be found that equation (54) reduces approximately to $x_2 = 0$, and hence that for any value of $\mu$ the lowest value of $x$ is slightly less than the corresponding value in the case in which $\mu = 0$.

*The First Point of Bifurcation.*

§ 20. It therefore appears that the first point of bifurcation may be safely assumed to be of order $n = 1$. The value of $x$ for which it occurs will have some value between 4·0 and 4·49, according to the value of $\mu/\lambda$. Now $x = \kappa a$, and the value of $\kappa^2$ is $4\pi\rho^2/(\lambda + 2\mu)$. Hence the first point of bifurcation is approximately given by

$$\frac{4\pi\rho^2 a^2}{\lambda + 2\mu} = \begin{cases} 4{\cdot}00^2 = 16{\cdot}00, \text{ when } \mu = \lambda, \\ 4{\cdot}49^2 = 20{\cdot}16, \text{ when } \mu = 0. \end{cases}$$

In equation (3) we supposed this point of bifurcation to be given by

$$\gamma\rho_0^2 a^2/\lambda_0 = \phi.$$

In our present analysis we have already taken $\gamma = 1$; if we take $(\lambda + 2\mu)$ to be identical with our former $\lambda_0$, we see that the actual values of $\phi$ are roughly

$$\phi = 1{\cdot}60, \text{ when } \mu = 0, \qquad \phi = 1{\cdot}27, \text{ when } \mu = \lambda.*$$

* It will be found that the first point of bifurcation is given, with great accuracy, by the single equation $\rho^2 a^2/(\lambda + \frac{7}{5}\mu) = 1{\cdot}6$ for all values of $\mu$ between 0 and $\lambda$. This is of interest, as showing the relative importance of $\lambda$ and $\mu$ in maintaining stability. As might be foreseen, the importance of $\mu$ relatively to $\lambda$ increases as $n$ increases, and for $n = \infty$, the factor $\lambda + \frac{7}{5}\mu$ must be replaced by $\lambda + 2\mu$.

We have now found a closer approximation to the value of $\phi$ than that which was given in § 3, and have obtained the additional information that instability first enters through a vibration of order $n = 1$. It must, however, be borne in mind that these results are only true of the special and somewhat artificial case specified in § 8.

### *Comparison with the Case of a Spherical Nebula.*

§ 21. It will be seen that the general argument of § 3 will apply to the case of a gaseous planet or nebula if $\lambda$ be taken to mean the pressure in the gas. In this case, however, the laws of distribution of density and pressure are not independent. If the gas is in conductive equilibrium throughout, the planet or nebula must be supposed to extend to infinity, and for these conditions the criterion of stability was worked out in the former paper already referred to. Calling the elasticity of the gas $\kappa$, the first point of bifurcation was found to be reached when the function $\underset{r=\infty}{\mathrm{L}^t} \dfrac{2\pi\rho r^2}{\kappa}$ attains a certain finite value. Now $\underset{r=\infty}{\mathrm{L}^t}\, \rho$ vanishes in comparison with $\rho_0$, the mean density, so that writing $a$ for the radius of the nebula, and $\lambda_0$ for the mean pressure ($\lambda_0 = \kappa\rho_0$), we have, at this first point of bifurcation

$$2\pi\rho_0{}^2a^2/\lambda_0 = \infty\,.$$

Comparing this with the general result obtained in § 3, we see that in this extreme case the value of $\phi$ becomes infinite. This result is only of importance to the present investigation as showing the tendency of a concentration of density about the centre. It seems to show that as the density becomes more concentrated about the centre, the value of $\phi$ may be expected to increase. We are therefore led to expect that in general $\phi$ will have a value rather greater than that found for it upon the assumption of homogeneity of density.

## Recapitulation and Discussion of Results.

§ 22. It will be well to recapitulate our results before attempting to draw any deductions from them.

We consider a spherically symmetrical mass of solid, liquid, or gaseous matter. We denote the radius of this by $a$, the mean density by $\rho_0$, and the mean value of $\lambda$ by $\lambda_0$, where $\lambda$ denotes an elastic constant or the pressure of the fluid, according as the matter is solid or fluid. We have seen that the stability of this dynamical system depends upon the value of the function $\gamma\rho_0{}^2a^2/\lambda_0$, a pure number. When $\gamma = 0$ (*i.e.*, when we deal with artificial matter which is totally devoid of gravitation) there can be no doubt that the system is stable. We have seen that a point of bifurcation occurs when the number $\gamma\rho_0{}^2a^2/\lambda_0$ has a certain value $\phi$. It has not been proved in the present paper that an exchange of stabilities accompanies this point of bifurcation,

but it will be seen that, with slight alterations, the proof of the exchange of stabilities for the spherical nebula, which was given in § 28 of the earlier paper, can be made to apply to the present case. Admitting this, it appears that the spherical system which is at present under discussion will be stable so long as $\gamma\rho_0^2a^2/\lambda_0$ is less than $\phi$, and becomes unstable so soon as $\gamma\rho_0^2a^2/\lambda_0$ exceeds $\phi$.

§ 23. The next question is as to the exact value of $\phi$, and as to the vibration through which instability enters at the point of bifurcation. To the first part of the question we have not been able to obtain a very definite answer. This matters the less, since the numerical data which would have to be used in making any applications of our results are not themselves very definite. On the whole, the uncertainty in the value of $\phi$ is not much greater than the uncertainty in the value of the numerical data (or, what comes to the same thing for our present purpose, the uncertainty in our knowledge of the law of compressibility and distribution of density in the planets of our system).

The general argument of § 3 showed that $\phi$ must, except in extreme cases, be comparable with unity. We then examined an artificial case: a planet in which the density and elasticity were constant throughout—this system being made mechanically possible by introducing an external field of force, of amount just sufficient to annul gravitation in the equilibrium configuration. For this system $\rho_0$ was, of course, taken equal to $\rho$, the uniform density, and $\lambda_0$ was taken to be equal to $\lambda + 2\mu$ in the notation of LOVE, or $m + n$ in the notation of THOMSON and TAIT. The value of $\phi$ depends, of course, on the ratio $\mu/\lambda$ or $n/m$. For $\mu/\lambda = 0$ we found $\phi = 1{\cdot}6$; for $\mu/\lambda = 1$ we found $\phi = 1{\cdot}27$; for intermediate value of $\mu/\lambda$ we saw that the value of $\phi$ was intermediate between these two values.

The planets to which we wish to apply our results do not possess uniform density: it is almost certain that in every case the mean density is much greater than the surface density. The general argument of § 3 shows that there is still a point of bifurcation corresponding to a value of $\phi$ which is comparable with unity, but affords no evidence as to the change which a concentration of density will effect in the value of $\phi$. We therefore examined a case in which there is an infinite concentration of density—the case of a spherical nebula extending to infinity—and found that in this extreme case the value of $\phi$ becomes infinite. It therefore seems probable that a concentration of density is attended by an increase in the value of $\phi$. As a working hypothesis we shall assume for the planets of the solar system the uniform value $\phi = 2$. It must be left to the reader to form a judgment as to the amount of error involved in this assumption, but it will, perhaps, be admitted that results depending upon it will at least be right as regards order of magnitude. It will be seen later that considerable variation in the value of $\phi$ is possible before the astronomical evidence which we are going to bring forward is seriously invalidated.

§ 24. As regards the nature of the vibration through which instability of the spherical configuration enters, we are able to come to a more definite conclusion. In

each of the cases referred to in the last section this vibration is found to be one of order $n = 1$, *i.e.*, one in which the displacement at every point is proportional to the first harmonic. This is the result which we should naturally expect—just as we expect a mass of liquid to become unstable through long surface waves sooner than through short ones. We shall, therefore, suppose it to be true of the planets in general. It is conceivable that planets could be artificially constructed for which this assumption would not be true, but, at present, since we have not a complete knowledge of the structure of the planets and are therefore compelled to make some assumptions, it seems as if the assumption just made is far and away the best to take as a working hypothesis.

## Application to the Nebular Hypothesis.

### *Theoretical Conclusions.*

§ 25. In the former paper, already referred to, the suggestion has been put forward that the instability of a nebula, sun or planet, which, upon the nebular hypothesis, is supposed ultimately to result in the ejection of a satellite, may be largely brought about by a gravitational tendency to instability of the kind we have been investigating. Let us, for the moment, take an extreme hypothesis, and imagine that this agency is the preponderating agency, and that the rotational tendency to instability may be disregarded in comparison.

Upon this hypothesis let us consider the case of an approximately spherical planet or sun which is known to have thrown off a satellite. Before the ejection of this satellite commenced, the primary mass would have an approximately spherical form, for which $\rho_0{}^2a^2/\lambda_0$ would be below the critical value $\phi$. When this critical value is reached, a divergence from the spherical form occurs, and a series of new processes begins. We are not now concerned with the details of these processes, but they must be supposed ultimately to result in the ejection of a satellite. It must be noticed that we are not supposing the primary to be devoid of rotation—for this would be inconsistent with the ejection of a satellite—but are supposing the rotation to be so small that the rotational tendency to instability is small in comparison with the gravitational.

If we suppose one or more satellites to have been ejected, and the primary to have regained an approximately spherical form, the new value of $\rho_0{}^2a^2/\lambda_0$ must be less than $\phi$. Now every satellite of which we have any knowledge, in our own system at any rate, is small in comparison with the primary. A legitimate inference seems to be that the ejection of a satellite is only a small part of the life-history of the primary. We shall not, however, need to make any assumption so definite as this, but shall suppose only that the values of $\rho_0$, $a$, $\lambda_0$ for the primary after ejection are

nearly equal (*i.e.*, equal, except for a multiplying factor of, say, $1\frac{1}{4}$ or $1\frac{1}{2}$) to the values of the same quantities before the process of ejection commenced.

All this may be summed up, with sufficient accuracy for the present investigation, in the statement that when the ejection of a satellite is taking place the value of $\rho_0{}^2a^2/\lambda_0$ must be nearly equal to $\phi$.

*Conclusions Tested by the Solar System.*

§ 26. To a certain extent the solar system supplies material for testing this conclusion. Let $\rho_0$, $a$ denote the present mean density and radius of any member of the system which is known to have thrown off a satellite, and let $\lambda_0$ be the mean value of $\lambda$, whether this denotes a pressure or an elastic constant, then we may write

$$\rho_0{}^2a^2/\lambda_0 = \theta\phi.$$

At the time at which the satellite was ejected the same equation ought to hold with $\theta$ nearly equal to unity. Unfortunately we have no knowledge as to the changes which will have taken place in $\rho_0$, $a$, $\lambda_0$ since the ejection of the last satellite. We shall, for a rough approximation, neglect these changes altogether, take $\theta = 1$, $\phi = 2$, and examine to what extent the equation

$$\rho_0{}^2a^2/\lambda_0 = 2 \qquad (62)$$

holds for the solar system as it now stands.

It will be remembered that there are three sources of error in this equation :—

(i.) We are neglecting the effect of rotation in bringing about the ejection of a satellite.

(ii.) We are neglecting the changes which have taken place since the ejection of the last satellite.

(iii.) We are using an arbitrarily chosen value for $\phi$, and applying this to every planet, while we know, from the difference in the physical constitutions of the planets, that the value of $\phi$ must be different for each.

These three sources of error would each be serious if we were attempting to get accurate results, but as our calculations are necessarily only of the roughest kind, we may be content to neglect them.

§ 27. In the accompanying table the masses and radii of the sun, and of those planets which possess satellites, are given in the first two columns; the value of mass/(radius)$^2$ is given in the third. The units are so chosen that the earth is measured by unity in each of these columns. Venus is included for the sake of comparison, although the existence of a satellite is extremely doubtful.

| | Observed. | | | Calculated upon the hypothesis of the present paper ($\phi = 2$). |
|---|---|---|---|---|
| | (1) Mass. | (2) Radius. | (3) $\frac{\text{Mass}}{(\text{Radius})^2}$ | (4) Coefficient $\lambda_0$. Unit $= 10^{11}$ absolute $= 10^8$ grammes weight per sq. centim. |
| Sun . . . . . | 315,000 | 109 | 26 | 2700 |
| Venus . . . . | 0·8 | 1·0 | 0·9 | 3·2 |
| Earth . . . . | 1·0 | 1·0 | 1·0 | 4·0 |
| Mars . . . . | 0·1 | 0·5 | 0·4 | ·6 |
| Jupiter . . . . | 300·0 | 11·0 | 2·5 | 25·0 |
| Saturn . . . . | 90·0 | 9·0 | 1·1 | 5·0 |
| Uranus . . . . | 14·0 | 4·0 | 0·9 | 3·2 |
| Neptune . . . | 16·0 | 4·4 | 0·8 | 2·6 |

If our hypotheses give a fair account of the facts the numbers in this third column will be proportional to $\sqrt{\lambda_0 \phi}$. Assuming for $\phi$ the uniform value $\phi = 2$, we can calculate the actual values of $\lambda_0$, and these are given in the fourth column.

§ 28. Knowing nothing about the variation in $\lambda_0$, we shall be content as a preliminary hypothesis to suppose it to have the same value for each planet. Combining this with the hypotheses already formulated, we notice that $\sqrt{\lambda_0 \phi}$ ought to have the same value for each planet, as therefore ought also the function mass/(radius)$^2$, which is tabulated in column (3).

It will be seen at once that there is a certain amount of uniformity about the numbers in this column, but it requires some consideration to determine how much significance is to be attached to this uniformity.

Now, apart from any hypothesis as to how the solar system originated or reached its present configuration—*i.e.*, regarding the solar system as a fortuitous collection of bodies of varying sizes—we should expect the mean density to be greatest in the greatest planets. We should, therefore, expect the quantity (mass)/(radius)$^2$ to be more variable than the radius. In other words, we should, *à priori*, expect less uniformity in the third column than in the second. Judged by this criterion, the uniformity of the numbers in the third column would be very significant. Further, the variation in these numbers is of the kind we should expect. For instance, it is known that the density of Jupiter is very much greater near the centre than near the surface; we should accordingly expect a large value of $\phi$, and therefore a large entry in the third column. The same argument would apply to the Sun, but the physical constitution of the Sun is probably so different from that of the planets that there could be no surprise at the Sun figuring as an exceptional case. Another exception is that of Mars. Part of the discrepancy might, perhaps, be attributed to the

smallness of the planet, but the figure in the fourth column would seem to suggest that rotational instability must have played a large part in the creation of the Martian satellites.

If, on the other hand, we begin by regarding the planets not as a fortuitous collection of bodies, but as a series of satellites all ejected from the same primary, the case is different. For here we should expect the smaller planets to have cooled more than the heavier ones, and therefore to be at a lower temperature. Against this must be set the fact that the heavier planets will probably have the greatest concentration of density about the centre, and the greatest mean pressure. The first consideration tends to increase the value which we should expect for the mean density of the smaller planets as compared with that of the greater ones; the second consideration tends in the opposite direction. We can hardly profess to estimate the relative weights of these two considerations with any approach to accuracy; perhaps it is best to revert to the argument given in the last paragraph, while bearing in mind that the approximate equality of our numbers may become considerably less significant as soon as the question of relative temperature is taken into account.

§ 29. We now consider the evidence afforded by the absolute value of our figures. After allowing for the exceptional cases, it appears that the value of $\lambda_0$ for the earth and for most of the planets is about $4 \times 10^{11}$. In other words, if we suppose these planets suddenly to resume the molten state, while retaining their present mass and radius, the spherical form will be stable or unstable according as the mean value of $\lambda + 2\mu$ is greater or less than $4 \times 10^{11}$. In the molten state we may take $\mu = 0$, and the value $\lambda = 4 \times 10^{11}$ corresponds to a value equal to about half of that of steel, for which $\lambda = 8{\cdot}3 \times 10^{11}$. If, however, we attempt to trace the history of a planet backward in time, we cannot suppose the mass and radius kept constant: the mass may be constant, but the radius will increase. Under these conditions we find that the critical value of $\lambda_0$ will be inversely proportional to the fourth power of the radius, and will, therefore, be somewhat less than the value $\lambda = 4 \times 10^{11}$. It would be extremely difficult to form a reliable estimate of what this corrected value for $\lambda$ ought to be, and equally difficult to estimate what would be the actual value of $\lambda$ for molten material similar to that of which our planets must have been composed when in the molten state. Our argument is that the two values of $\lambda$ are at least of the same order of magnitude, and probably equal, except for inaccuracies in our calculations.

*Comparison of the Rotational and Gravitational Hypotheses.*

§ 30. We may conclude this part of our work by comparing two extreme hypotheses: the first referring the phenomena of planetary evolution solely to rotational, and the second solely to gravitational instability.

Given the approximate values of $\lambda$ and $\rho$ for a planet, and the fact that this

planet has thrown off a satellite, the former hypothesis leads to a certain inference as to the angular momentum of the system; the latter to an inference as to the radius of the primary. The former hypothesis leads to no inference at all as to the size of planets which are to be expected—they are as likely to be of the size of billiard balls as of the size of the planets of our system—while the latter leads to no inference as to the angular momentum of the system, but presupposes it to be small. The contention of the present paper is that the inferences which are drawn from the former hypothesis are not borne out by observations on the planets of our system, while those which are drawn from the latter are borne out as closely as could be expected. The true hypothesis must of necessity lie somewhere between the two extremes which we are comparing, and our evidence seems to show that it is much nearer to the latter (gravitational) than to the former (rotational).*

## Stresses and Vibrations in the Earth.

§ 31. It has already been seen that in dealing with a gravitating sphere of the size of the earth it is necessary to take into account terms which are omitted by Lord Kelvin and others—the terms which introduce into our equations the gravitational tendency to instability.

It is of some importance to know whether the existing solution for the vibrations and displacements of the earth would be altered to an appreciable extent by the inclusion of these terms. The general frequency-equation which is given on p. 167 is too complicated for manipulation, and is, moreover, open to the objection that it does not represent the facts of the case; for, inside the earth, the strains caused by permanent gravitation cannot legitimately be treated as small.†

§ 32. Considerations of a general nature will, however, give us some insight into the question. In an imaginary earth, in which $\lambda$, $\mu$ are infinitely great, the gravitational terms will be of no importance in comparison with those representing the elastic stresses. The true solution will, therefore, become identical with the classical solution in which the gravitational terms are neglected. For smaller values of $\lambda$, $\mu$ the error will become appreciable, and if $\lambda$, $\mu$ continue to decrease this error will become infinite as soon as the first point of bifurcation is reached; for at a point of bifurcation the application of an infinitesimally small external force will produce a finite displacement in the solid. For intermediate values of $\lambda$, $\mu$ the error will be small if $\lambda$, $\mu$ are great compared with the critical values of $\lambda$, $\mu$ at the point of bifurcation, and great if $\lambda$, $\mu$ are near to these critical values.

* In addition to the inference as to the size of the planets, the hypothesis of gravitational instability leads to a further inference as to the distances of the fixed stars. This has been discussed in my former paper, "On the Stability of a Spherical Nebula" (§ 48), and here also the results seem to agree with observation as closely as could reasonably be expected.

† Chree, 'Camb. Phil. Soc. Proc.,' vol. 14, or Love, 'Elasticity,' vol. 1, p. 220.

§ 33. The most reliable evidence as to the actual values of $\lambda$, $\mu$ is to be obtained from the phenomena of earthquake propagation.* From the "time curves" given in the British Association Report presented at the 1902 meeting, there seems to be little doubt that the so-called "large-waves" are propagated merely through a thin crust on the earth's surface, while the "preliminary tremor" is propagated in a sensibly straight line through the earth itself. The average velocity of propagation is found to be about 9·7 kiloms. per second, and this is independent of the length of the path. The inference is that $(\lambda + 2\mu)/\rho$ is nearly constant throughout the earth's interior, and that its value is about $(9{\cdot}7 \times 10^5)^2$ or $9{\cdot}4 \times 10^{11}$. If we suppose the mean value of $\rho$ to be 5·5, this gives for the mean value of $\lambda + 2\mu$,

$$\lambda + 2\mu = 51{\cdot}7 \times 10^{11}.$$

Now, the critical mean value of $\lambda + 2\mu$ which corresponds to the first point of bifurcation has already been seen to be about $4 \times 10^{11}$. It would, therefore, appear that the error introduced in the classical solution for the displacements and stresses is appreciable, although not great—it is probably comparable with the error to which attention has already been attracted by CHREE.†

## FIGURE OF THE EARTH.

### *Theoretical Conclusions.*

§ 34. From the evidence of the last section it will be seen that there is an overwhelming probability that the values of the elastic constants of the earth are such that a state of spherical symmetry would be one of stable equilibrium.

Whether or not the earth is at present in a state of spherical symmetry is a different question; various indications and, in particular, the inequality in the distribution of land between the two hemispheres of the globe suggest that it is not so.

Now, even if the material of the earth is at the present moment of sufficient strength to maintain a spherical configuration in spite of the gravitational tendency to instability, it does not seem probable that it has always been so. Looking backwards in time we must come to a stage in which the material of the earth was plastic, and, further back still, fluid. At this time the value of $\lambda$ would be much smaller than its present value, and, as already pointed out in § 29, would probably be about equal to the critical value for the planet at that period of its existence. There would, therefore, seem to be a sufficient reason for considering the possibility that the earth, at the moment at which consolidation set in, was not in a state of spherical symmetry. Let us examine some of the consequences of this conjecture.

* Professor MILNE has kindly assisted me in this question.

† *Loc. cit. ante.*

It is easy to see that enormous stresses would be set up in the interior of the earth after consolidation. An equilibrium configuration depends in general upon the compressibility of the material, and a configuration which was one of equilibrium for the compressibility which obtained at the moment of solidification would not remain so after the incompressibility and rigidity of the material had increased by cooling. If we suppose the earth to cool in an unsymmetrical configuration the stresses set up will soon become very great. In fact, Professor DARWIN has shown that the stresses which would be produced by the weights of our continents in an earth initially homogeneous (*i.e.*, by an irregularity of less than a thousandth part of the radius) would be so great that the material would be near the breaking point.*

We must therefore suppose that as the earth cools and the elastic constants change there will be a series of ruptures resulting from the stresses set up in the interior. The configuration will become approximately spherical (spheroidal if rotation is taken into account) as soon as the point of bifurcation is passed.

The fact that the ultimate configuration is reached only as the result of a long succession of ruptures puts the whole question outside the range of exact mathematical treatment. We can, however, see that the final configuration (disregarding rotation) will probably not be quite spherical, but will retain traces of the initial unsymmetrical configuration.

§ 35. Before we can attempt to decide whether or not the earth shows traces of a process such as that just described, it will be necessary to form some idea of the unsymmetrical configuration with which the process must have begun. We cannot accurately calculate the "linear series" of unsymmetrical configurations except in the immediate neighbourhood of the point of bifurcation. Near to this point the configuration is spherical except for terms proportional to the first harmonic. The free surface will, therefore, be strictly spherical, and it will, of course, be an equipotential, but its centre will not coincide with the centres of other surfaces of equal potential. If we suppose a fluid mass of this kind to solidify, and then to shrink by cooling, the shrinking being accompanied by a series of ruptures of the kind already explained, we can easily imagine that the free surface would retain an approximately spherical form, but that when the final state is reached this surface would not be quite an equipotential, and the centre of gravity would not quite coincide with the centre of figure. If water is placed on the surface of a planet of this kind, it will form a circular sea, of which the centre will be on the axis of harmonics, while the dry land will form a spherical cap.

## *Evidence from the Distribution of Seas and Land.*

§ 36. Now this is not observed on the earth, and it could not be expected, since we have ignored all the agencies which have contributed to the figure of the earth,

* 'Phil. Trans.,' vol. 173, 1882, p. 187.

except the one with which this paper is specially concerned. The question is not whether we observe the state just described, but whether we can detect any approach to this state, and this, I believe, can be done. Professor DARWIN writes* :—

"It is well known that the earth may be divided into two hemispheres, one of which consists almost entirely of land and the other of sea. If the south of England be taken as the pole of a hemisphere, it will be found that almost the whole of the land, excepting Australia, lies in that hemisphere, whilst the antipodal hemisphere consists almost entirely of sea. This proves that the centre of gravity of the earth's mass is more remote from England than the centre of figure of the solid globe. A deformation of this kind is expressed by a surface harmonic of the first order."

§ 37. We can carry our calculations a step further. The divergence from the initial configuration is only represented by a first harmonic so long as squares of this divergence may be neglected. If these squares are taken into account, we must

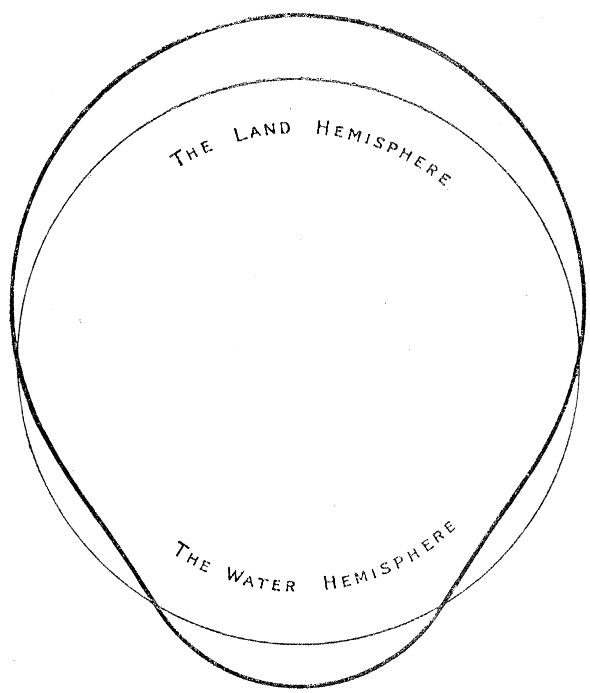

Fig. 1.

include a term proportional to the second harmonic as well as that proportional to the first. This process of successive approximation might be continued to any extent, so that a complete series of unsymmetrical configurations might be calculated in the manner explained in my former paper.† We may, however, be content to stop at the second harmonic. The free surface will now be of the form

* G. H. DARWIN, 'Phil. Trans.,' vol. 173, 1882, p. 230.

† 'Phil. Trans.,' A, vol. 199, p. 41.

$$r = a_0 + a_1 P_1 + a_2 P_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

and we therefore examine whether any traces of the second harmonic term can be found in the earth's surface. Now, if we take $a_2$ positive in equation (63), the equation is that of the pear-shaped curve which was found on p. 46 of this earlier paper. This differs from the spherical shape mainly in possessing a protuberance—the stalk end of the pear—of which the centre is on the axis of harmonics. Traces of this protuberance may, I think, perhaps be found in the Australian continent, the arrangement being that shown in fig. 1. It is true that the centre of Australia does not coincide with the antipodes of England, but the discrepancy becomes less when we take into account the enormous region of ocean shallows which lies to the east of Australia.

[*The discrepancy can be further reduced by taking the rotation of the earth into account. When the rotation of the earth was greater than at present the ellipticity of the earth's surface would be greater, and the transition from this to the present ellipticity would take place through a series of ruptures similar to those already described. The rotation (assumed small) of the pear can be allowed for by adding a term $-\beta P'_2$ to the right-hand side of equation (63), this representing a second harmonic deformation having the axis of rotation for axis of harmonics.

The present rotation of the earth can similarly be represented by a term $-\beta' P'_2$, where $\beta' < \beta$. The equation to the present surface of the sea may accordingly be taken to be

$$r = a'_0 - \beta' P'_2,$$

and hence the height above the present sea-level of the surface of the primæval rotating pear, if restored, would be

$$(a_0 - a'_0) + a_1 P_1 + a_2 P_2 - (\beta - \beta') P'_2.$$

It will be found that the effect of the rotational term $(\beta - \beta') P'_2$ is to move the theoretically predicted Australia nearer to the equator of the earth, and to change its shape from a spherical cap to a sphero-conic.]

Again, we should expect the highest land to be on the axis of harmonics, and, therefore, in or near England. Here, again, the agreement of facts with theory might be closer if we could suppose the continent, which geology shows to have existed at one time in mid-Atlantic, to be restored to its former position. But the agreement of facts with theory can only be expected to be of the roughest kind, and we must always bear in mind that our theory does not lead us to expect that the present figure of the earth will be pear-shaped, but only that it will resemble a pear disfigured by a long series of ruptures.

* Added January 3, 1903. I am indebted to the referee for suggesting this addition.

*Evidence from the Distribution of Earthquake Centres.*

§ 38. It can be seen that the earthquake regions of the world have a reference, as regards their distribution on the earth's surface, to this pear-shaped figure, and this, again, must be considered as evidence.

Let us first examine the facts. MILNE divides the earthquake-areas of the globe into twelve distinct regions, and a map of these is given in the 'British Association Report' for 1902.* These regions are given in the following table. The first figure denotes the number of large earthquakes which have occurred in these regions in the three years 1899–1901. The earthquakes from the three regions printed in italics were small in comparison with the others. In the last column is given the approximate latitude of the centre of each region, referred to Greenwich as pole (the latitude of Greenwich being taken to be $+90°$).

TABLE of Earthquake Regions.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | 25 | Alaskan | +10° | G | 17 | Mauritian | +10° |
| B | 14 | Cordillerean | 0 | *H* | *22* | *N.E. Atlantic* | *+75* |
| C | 16 | Antillean | +25 | *I* | *3* | *N.W.* ,, | *+65* |
| D | 12 | Andean | 0 | *J* | *3* | *N.* ,, | *+70* |
| E | 29 | Japanese | − 5 | K | 14 | Asiatic | +45 |
| F | 41 | Javan | −25 | L | 2 | Antarctic | [small] |

Now, it will be at once noticed that for most of these regions the latitude is small. If we weight the regions according to the corresponding number of earthquakes, giving half-weight to the small earthquakes in regions H, I, J, we find as the mean of the numerical values of the latitude about 20°, whereas if the regions were distributed at random we should expect the mean latitude to be $(\frac{1}{2}\pi - 1)$ radians, or about 33°. We therefore see that the earthquake regions tend to lie near the equator of our pear. The evidence can be put in a more striking way as follows:— Exactly half of the surface of the globe is of a latitude less than 30°. The half for which the latitude is less than 30°, measured from Greenwich as pole, was responsible for 156 earthquakes; the remaining half was responsible only for 42, of which 28 were the small earthquakes from regions H, I, J. There is, therefore, no doubt that the *principal* earthquakes tend to emanate from points near to the equator of the supposed pear.

Now, if we look back to fig. 1, we see that this is equivalent to saying that earthquakes occur where the "slope" in the figure of the earth is steepest. This conclusion is the same as that to which the British Association Committee were led from a

* 'Brit. Assoc., 72nd Report,' Belfast, 1902, "Seismological Investigations," p. 4.

consideration of the actual figure of the earth, and it is that which might naturally be expected. The theory put forward in this paper may, perhaps, suggest a reason why these regions should lie approximately on a great circle of the earth, and why this great circle should approximately divide the earth into two hemispheres of sea and land.

### *Summary and Conclusion.*

§ 39. In conclusion it may be well to summarise those parts of the paper which refer to the figure of the earth.

We saw that at the moment of solidification the earth might be either spherical (except in so far as it was deformed by its rotation) or pear-shaped. Our theoretical calculations and our knowledge of the constants of the earth at the time of solidification were not sufficiently accurate to enable us to decide which of the two alternatives is the more probable. The shape of the earth, whether spherical or pear-shaped, could not be maintained long against the enormous strains which would be set up in the earth as the process of cooling proceeded, and this shape would gradually give place to an approximately spherical shape, the change in shape being produced by a long succession of ruptures. The suggestion of this paper is that the earth, in spite of this series of ruptures, still shows traces of a pear-shaped configuration. Such a configuration would possess a single axis of symmetry, and this, it is suggested, is an axis which meets the earth's surface somewhere in the neighbourhood of England (or, possibly, some hundreds of miles to the S.W. of England). Starting from England we have in the first place a hemisphere which is practically all land; this would be the blunt end of our pear. Bounding this hemisphere we have a great circle of which England is the pole, and it is over this circle that earthquakes and volcanoes are of most frequent occurrence. If we suppose our pear contracting to a spherical shape we notice that it would probably be in the neighbourhood of its equator that the change in curvature and the relative displacements would be greatest, and hence we should expect to find earthquakes and volcanoes in greatest numbers near to this circle. Passing still further from England we come to a great region of deep seas—the Pacific Ocean, the South Atlantic Ocean, and the Indian Ocean: these may mark the place where the "waist" of the pear occurred. Lastly we come, almost at the antipodes of England, to the Australian continent and the shallow seas which lie to the east of it; these may be the remains of the stalk-end of the pear.

§ 40. It may, I am afraid, be thought that the hypotheses upon which the paper is based are too speculative and the results, consequently, too uncertain. In defence it may be said that the object of the paper is not so much to establish new doctrines as to point out possibilities, and that these possibilities seem to be of such a kind that it may be useful to keep them in mind in discussing questions connected with the figure

and structure of the earth, as well as the more general questions of planetary evolution.

In conclusion I have to express my indebtedness to Professor G. H. DARWIN and Professor A. E H. LOVE for advice and assistance which I have received from them.

[NOTE.—*Added February 20th*, 1903. While the above paper was in the press, Professor W. J. SOLLAS read a paper before the Geological Society in which the Figure of the Earth was discussed from a geological standpoint. Professor SOLLAS had arrived, from an examination of the geological features of the earth, at a conclusion very similar to that to which I had been led from theoretical considerations: he had detected an axis of symmetry, other than the axis of rotation, in the earth's figure, and expressed the opinion that "the pear-shaped form, now that it was pointed out, became obvious to mere inspection: it was a geographical fact, and not a speculation."

The axis of Professor SOLLAS' pear does not, however, coincide with that which I tentatively put forward in the above paper, and the object of this note is to accept the alteration suggested by his paper. The conclusion reached in his paper is that the axis of symmetry of the pear-shaped figure passes through a point of latitude and longitude about 6° N. by 30° E. Thus Africa—the continent whose mean height above sea-level is greatest—must be taken to be the centre of the "Land Hemisphere" in fig. 1 of my paper, while the protuberance which formed the stalk of the pear is submerged in the Pacific Ocean, which now forms the "Water Hemisphere." Almost the only remaining evidence of the existence of this protuberance is the fact that the axis of the pear coincides with the earth's greatest diameter. The great circle of earthquake-centres suggested in § 38 of my paper is to be replaced by the line of Pacific folding; this approximately forms a small circle (of radius about 80°) which almost coincides with the proposed great-circle in the northern hemisphere. Further details of Professor SOLLAS' view will be found in his paper ("The Figure of the Earth," 'Quart. Journ. Geol. Soc.,' vol. lix., Part 2).

The fact that Africa is surrounded by a belt of seas, and this again by a belt of land before the Pacific is reached, points perhaps to a bodily subsidence of the blunt end of the pear, the circle of fracture having possibly been the line of Pacific folding. Such a fracture would, of course, displace the centre of gravity of the pear, and probably this would account not only for the feature just mentioned, but also for the non-appearance of the protuberance. It will be noticed that the smallness of the latitude of the extremities of the axis (6°) agrees well with the theory of planetary evolution put forward in §§ 25–30 of the present paper.]