VI. *On the General Theory of Integration.*

By W. H. Young, *Sc.D., St. Peter's College, Cambridge.*

Communicated by Dr. E. W. Hobson, *F.R.S.*

Received April 23,—Read May 19, 1904.

*Introductory.**

Riemann was the first to consider the theory of integration of non-continuous functions. As is well known, his definition of the integral of a function between the limits a and b is as follows:—Divide the segment (a, b) into any finite number of intervals, each less, say, than a positive quantity, or norm d; take the product of each such interval by the value of the function at any point of that interval, and form the sum of all these products; if this sum has a limit, when d is indefinitely diminished which is independent of the mode of division into intervals, and of the choice of the points in those intervals at which the values of the function are considered, this limit is called the integral of the function from a to b.

The most convenient mode, however, of defining a Riemann (that is an ordinary) integral of a function, is due to Darboux; it is based on the introduction of *upper and lower integrals* (intégrale par excès, par défaut; oberes, unteres Integral). The definitions of these are as follows:—It may be shown that, if the interval (a, b) be divided as before, and the sum of the products taken as before, but with this difference, that instead of the value of the function at an arbitrary point of the part, the upper (lower) limit of the values of the function in the part be taken and multiplied by the length of the corresponding part, these summations have, whatever be the type of function, each of them a definite limit, independent of the mode of division and the mode in which d approaches the value zero. This limit is called the upper (lower) integral of the function. In the special case in which these two limits agree, the common value is called *the integral of the function.*

The progress of the modern theory of sets of points (Théorie des ensembles; Mengenlehre), due, as is well known, chiefly to G. Cantor, though taking its origin in Riemann's paper 'Ueber die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe,' naturally leads us to put the question how far these definitions

* An abridged statement of the contents of this memoir will be found in the "Abstract" published in the 'Proceedings of the Royal Society,' vol. 73, pp. 445–449.

The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to
Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character.
www.jstor.org

can be generalised. This theory has in fact taught us on the one hand that many of the theorems hitherto stated for finite numbers are true with or without modification for a countably infinite number, and on the other hand that closed sets of points possess many of the properties of intervals. We may, in accordance with these facts, divide the segment (1) into an infinite number of non-overlapping intervals, in which case, however, seeing that such a set of intervals always has points which are external or semi-external to them, we must in general add a set of points to the set of intervals, if the division of the segment is to be properly performed, that is, if all the points of the segment are to be accounted for in our division; or, more generally, (2) into a finite or countably infinite number of sets of points.

What would be the effect on the Riemann and Darboux definitions, if in those definitions the word "finite" were replaced by "countably infinite," and the word "interval" by "set of points"? A further question suggests itself:—Are we at liberty to replace the segment (a, b) itself by a closed set of points, and so define integration with respect to any closed set of points?

Going one step further, recognising that the theory of the content of open sets quite recently developed by M. LEBESGUE* has enabled us to deal with all known open sets in much the same way as with closed sets as regards the very properties which here come into consideration, we may attempt to replace both the segment and the intervals of the segment by any kinds of measurable sets.

In the Riemann and Darboux definitions it is tacitly assumed that the interval (a, b) is finite, and that the function is throughout the interval finite and possesses finite upper and lower limits. The discussion of the integration of a function which is not necessarily finite, over an interval not necessarily finite in length, requires separate consideration, and the definitions of such integrals, called improper integrals, are of the nature of extensions of the definitions of ordinary integrals. Bearing in mind the somewhat unsatisfactory and artificial character of such extensions, we may hope finally that our discussion† may throw light on improper integrals also.

In M. LEBESGUE'S valuable memoir, already referred to, a striking addition has been made to the previously existing knowledge of the subjects dealt with. He has shown that a more general definition than that of RIEMANN, available for all known functions, one moreover coinciding with that of RIEMANN in the case of all functions integrable in the Riemann sense, may be given; a definition possessing, among others, the remarkable property of permitting passage back from the derivatives of a continuous function to the continuous function itself.

In the present paper I attempt to discuss the whole matter, and take occasion in the proper place to bring LEBESGUE'S work into connection with my own. Some special cases of the results I obtain have been given by me in the paper presented to

* "Intégrale; Longueur; Aire," 'Ann. di Mat.,' 1902. *Cf.* also a paper by the Author, "Open Sets and the Theory of Content," 'Proc. Lond. Math. Soc.,' Ser. 2, vol. 2, Part I., p. 16.

† In the instalment now presented to the Society I confine my attention however to proper integrals.

the L.M.S. already cited, and I apply one of the results of this paper, viz., that the content of any closed n dimensional set may be expressed both as an ordinary integral in terms of an $\overline{n-1}$ dimensional content, or as the upper integral of the content of sets of lower dimensions, to obtain the corresponding theorem for any measurable set (contained in a closed set of finite content).

PART I.—OF FINITE PROPER INTEGRALS.

§ 1. The necessary and sufficient condition that a function should satisfy the requirements of RIEMANN'S definition is simply that the content of the points of continuity of the function should be equal to the length of the segment (a, b), (or, in the case of a multiple integral, to the content of the region over which the integration is extended), that is to say, when this condition is satisfied, the summations referred to in the Riemann definition have a definite limit, independent of the mode of division, &c. Is this still true when for "interval" the expression "set of points" is substituted,* and for "length of interval" the "content of the set of points"? The following example shows that, when these substitutions have been made, the definition, as it stands, ceases to have any meaning, even in the case of continuous functions.

Example 1.—Take $y = x$ as the function. The Riemann integral $\int_0^1 x\,dx$ has the value $\frac{1}{2}$. If, however, after dividing the segment (0, 1) into n equal intervals, we abstract the n points

$$\tfrac{3}{4}, \tfrac{7}{8}, \ldots \frac{2^n-1}{2^n},$$

and add them singly to what remains of the intervals, we obtain n measurable sets of points, each of the same content as before, viz., $\dfrac{1}{n}$; in each, however, there is a point at which the function has a value greater than $\frac{3}{4}$, except possibly in one of the sets in which there is a point at which the function has the value $\frac{3}{4}$. The summation is therefore always greater than $\frac{3}{4}$, so that it is clear that we do not get the same limit as before when n is indefinitely increased.

The principle of this example shows that in the general case also, except in the single case when the function is a constant, different modes of division of the segment into a finite number of sets of points each of content less than a given norm d, and different modes of proceeding to the limit, will certainly not always give the same limiting value of the summations. Thus, suppose the function to be continuous at its upper limit, then we can arrange that the mode of division is such that to every partial set at every stage a point belongs for which the function differs from its

* I shall always, except when the contrary is stated, suppose that the sets employed are measurable sets, so that the sum of two non-overlapping sets has for content the sum of their separate contents.

maximum value by a quantity as small as we please, and there is nothing to prevent our taking precisely this point as that at which the value of the function is to be taken in forming the summation; similarly for the minimum value, and it is clear that the limits obtained in these two cases could only agree when the function is a constant.

It is plain, moreover, that there is no room for discussion of the range of possible values of limits corresponding to the various conceivable modes of division. Speaking generally, the range will be from SM to Sm, where S is the length of the segment, or more generally n-dimensional volume of the region, and M and m are the upper and lower limits of the values of the function. Thus the Riemann definition completely breaks down when we attempt to generalise it in this direction.

§ 2. There is another direction, however, in which an important generalisation of the Riemann definition is possible. If we change the words "finite number of intervals" into "set of intervals" and add "such that the content of the external points is zero," the definition still holds good; we get a perfectly definite limit, which is, of course, the Riemann integral.

To prove this, we notice first that the content of the set of intervals is in this case, and in this case alone, the same as that of the segment (or region) under discussion, say S.

Let the intervals be arranged in any way in countable order $d_1, d_2 \ldots$ Then, since the d's are all positive, their sum is an absolutely convergent series; therefore the same is true when the content of each d_r is multiplied by a quantity f_r which, for all values of r, lies between finite upper and lower limits, say between $\pm$M, as is the case in forming our summations.*

If we consider n of these intervals in order $d_1, d_2 \ldots d_n$, these leave over a finite number of complementary intervals, say $d'_1, d'_2 \ldots d'_m$, and we can so choose n that the sum of these latter intervals d'_r is less than $\frac{e}{2\mathrm{M}}$, while the contribution to our summation over the remaining intervals $d_{n+1}, d_{n+2} \ldots$ is numerically less than $\frac{1}{2}e$.

If now we form the summation in RIEMANN's way over the finite number of intervals $d_1, d_2 \ldots d_n, d'_1, d'_2 \ldots d'_m$, and compare it with the corresponding summation over the set of intervals $d_1, d_2 \ldots$ *ad inf.*, we see that the difference between the two summations is less than e. Since e is at our disposal, and we can insure that both the intervals d_r and d'_r are less than any assigned norm, this proves the statement embodied again in the following theorem:—

* I take this first opportunity of emphasising the fact that, though it is convenient, indeed necessary, in forming the sum of an infinite number of terms to arrange them in some sort of order, in doing so here we do not introduce the idea of order into the concept of integration. Indeed, from the definitions it is evident that the concept of integration no more of itself involves the idea of order than do the concepts of length, area, and volume. The distinction of the two notions has, perhaps, not always been present to the mind of some writers.

Theorem 1.—*If the interval* (a, b) *be divided into a set of intervals and a set of points of zero content, the length of each interval being less than some assigned norm* d; *and if the product of the length of each interval by the value of a given integrable function at any point of that interval be formed and the summation of all such products calculated, this summation has a definite limit when* d *is indefinitely diminished; this limit is, of course, the integral of the function.*

Corollary.—The value of the integral of an integrable function is unaltered if, at the points of a set of zero content, we arbitrarily change the values of the function.

In other words, if we add to the function an integrable null function, we leave the integral unaltered.

It will be convenient to prove the following theorem, which is, in practice, indispensable; from it the theorem on which DARBOUX's definition of upper (lower) integration is based, can be at once deduced:—

§ 3. *Theorem* 2.—*Given any small positive quantity* e_1, *we can determine a positive quantity* e, *such that, if the segment* S *be divided up in any manner into a finite number of non-overlapping intervals, then, provided only the length of each interval is less than* e, *the upper* (*lower*) *summation of any function over these intervals differs by less than* e_1 *from a definite limiting value, the upper* (*lower*) *integral.*

The following is the proof for the case of the upper integral; with slight modification it holds for the lower integral.

Let I be the lower limit of all such summations; then we can determine a division of S into a finite number n of intervals, such that the upper summation over these intervals lies between I and $\mathrm{I}+\frac{1}{2}e_1 \dots$.

Let e be chosen to satisfy the following equation

$$e = \frac{e_1}{2n\mathrm{M}},$$

where M is any quantity greater than the greatest value of the function, and let us consider any division whatever, into a finite number of non-overlapping intervals each less than e. The number of such intervals which do not lie entirely in *one* of the n intervals previously determined, is at most n, so that the sum of the terms corresponding to these intervals is less than $n\mathrm{M}e$, that is $\frac{1}{2}e_1$.

In each of the remaining intervals the upper limit of f is not greater than the upper limit of f in that one of the n intervals in which it lies, so that the upper summation over the remaining intervals is not greater than that over the n intervals. Hence the summation over our intervals, each being less than e, is less than $\mathrm{I}+e_1$.

[Q.E.D.]

§ 4. We have now to discuss the Darboux form of the definition of integration, that is, in the first instance, to consider the effect of the modifications proposed on DARBOUX's definitions of upper and lower integration.

The following example* shows that the theorems stated in these definitions no

* Example 1, of course, shows this in the case of an integrable function.

longer hold, that is to say, it is no longer true that definite limits, independent of the mode of division, &c., exist, when sets of points are substituted for intervals :—

Example 2.—Let y be a function of x, which is zero everywhere, in the segment $(0, 1)$, except at every point of a perfect set G, nowhere dense, whose content is I, and at every point of G let $y = 1$.

If we divide the segment $(0, 1)$ into n equal parts, each of length less than d, the upper summation, that is, the summation which corresponds to the upper integral, is always greater than I, but approaches the value I as d is indefinitely decreased. If I is zero (in which case the function is integrable in RIEMANN'S sense), the limit is always zero, and in any case it is *less* than 1. We can, however, arrange so that we get 1 as limit.

Let the semi-external points of the intervals be arranged in countable order $P_1, P_2, \ldots$; take the same division as before, abstracting, however, the points $P_1, P_2, \ldots P_n$, and then adding these points singly to the single parts; thus we have a division into n measurable sets, each of content less than d, in each of which the function has the value unity, so that the summation always has the value unity however small d may be; therefore we get 1 as limit.

§ 5. If, on the other hand, we retain the intervals, but drop the restriction that they should be finite in number, there remains over a complementary set of points G, of content, say, I. The following example shows that the upper summations over a set of intervals have not then in general a definite limit, so that this extension of the definition of upper integral cannot be made without some restriction :—

Example 3.—Take a perfect set, nowhere dense, of content I in the segment $(0, 1)$, and let the function be zero everywhere except at the points of this set where the value of the function is unity.

The value of the upper integral as defined by DARBOUX is I.

Now let d be any assigned norm, and let an odd integer m be determined so that $(m-1)\,d > 1$.

Divide the segment $(0, 1)$ into m equal parts, and blacken the middle part. Then divide each of the $(m-1)$ remaining equal parts of the segment into m^2 equal parts, and blacken each of the middle parts, and so on, in the usual manner. The set of intervals, each less than d, which we thus obtain, have content

$$\frac{1}{m}+\frac{m-1}{m^{2+1}}+\frac{(m-1)(m^2-1)}{m^{3+2+1}}+\ldots=\frac{1}{m}+\frac{1}{m^2}\left(1-\frac{1}{m}\right)+\frac{1}{m^3}\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)+\ldots,$$

which is less than d. The complementary set of points is a perfect set nowhere dense of content greater than $(1-d)$. The upper summation of the given function over these intervals is then at most equal to d, and is, therefore, less than the upper integral as soon as the norm is less than I, and has the limit zero.

The principle of this example shows that, *if we omit the condition that the number*

of intervals should be finite, the upper summation can always be made numerically as small as we please. The same is true of the lower summation.

Theorem 3.—*If, however, we make the restriction that the set of points complementary to the intervals always has zero content, or has a content which decreases without limit, as d does so, the limits approached by the upper and lower summations are perfectly definite, and are, of course, the upper and lower integrals.*

The proof of this statement is identical in character with that given in § 3 for the corresponding theorem regarding the integral of an integrable function, which theorem is, of course, a special case of the above.

We should naturally ask whether we cannot correct this discrepancy by adding to the upper or lower summation over the set of intervals the product of the content of the complementary set of points into the upper or lower limit of the function for points of that set; or, if this does not suffice, by dividing up the complementary set itself into components and adding the sum of the corresponding products. That neither of these corrections suffice is shown by the following example :—

Example 4.—Take the same set as before at which the function has the value 1, and inside the largest of its black intervals place a similar set G' of content I', at every point of which internal to that black interval the function has the value 2. Everywhere else the function is to be zero. The upper integral is $I+2I'$.

If now we merely subdivide the black intervals of G' which lie inside the largest black interval of G, and subdivide the remaining black intervals of G, the product of the content of the complementary set of points into the upper limit of the function for points of that set will be at least $2(I+I')$, while the summation over the intervals will be zero. Thus the addition of the term in question would not correct the result.

If, on the other hand, we subdivide the complementary set into components whose contents are themselves less than the norm d, we could, on the principle which has already been employed, insure that each component contained a point at which the function had the value 2, and the result would be the same as before. In neither case do we obtain the upper integral.

§ 6. As we have seen in § 4 and § 5, the Darboux definitions of upper and lower integration require modifications, if they are to be generalised in the manner proposed.

We are naturally led to define upper and lower integration tentatively as follows :—

Let the division of the segment into (measurable) sets be performed in any conceivable way, and let the upper limit of the values of the function in each partial set be multiplied by the content of that set, and let the sum of these products be formed; then the upper integral is defined to be the lower limit of all such sums.

Similarly the lower integral might be defined, the words "upper" and "lower" being throughout interchanged. There is clearly no logical reason to prevent our considering these limits.

The names upper and lower integrals will, however, not be suitable, unless (1) the

upper integral, so defined, is greater than, or equal to, the lower integral; (2) these definitions agree with DARBOUX's in all cases.

It is easy to prove that (1) holds :—

Theorem 4.—*The lower limit of the upper summations is not less than the upper limit of the lower summations.*

For otherwise we could clearly take a quantity L lying between the upper and lower integrals so defined, and find two divisions of the segment so that the upper summation for the first division is less than L and the lower summation for the second greater than L. If, now, we consider the division of the segment got by combining these two divisions, that is, if we divide each of the former sets up into the components which it has in common with each of the latter sets, the upper summation is not increased nor the lower summation diminished, thus for this division the upper summation is less than L and the lower summation greater than L, which is impossible. This proves (1) to hold.

In regard to (2) the following simple example shows that, in the general case, there is no agreement between our tentative definitions and those of DARBOUX. By dividing up into sets of points, instead of into intervals, we get a lower value for the upper integral than that given by DARBOUX's definition, and a greater value than the lower integral.

Example 5.—Take the function which is 1 at all the rational points of the segment (0, 1), and zero everywhere else. The Darboux upper integral, from 0 to 1, has the value 1; the lower limit of all possible upper summations is however 0, since the rational points can be enclosed in a set of intervals whose content is as small as we please.

§ 7. I now proceed to show that in the case of a function which is integrable, in the ordinary sense of the word, so that the ordinary upper and lower integrals coincide, the division into sets of points, instead of merely into a finite number of intervals, leads to the same limit, viz., the integral of the function. To see this, we have merely to remark that the upper and lower summations are respectively less than the ordinary upper integral and greater than the ordinary lower integral, except in the case of equality, and that, as shown in the preceding article, the lower limit of the upper summations is not less than the upper limit of the lower summations.

Thus we have the theorem :—

Theorem 5.—*If a function be integrable in a given segment (region), the value of the integral is equal to the limit obtained as follows: divide the segment (region) into any finite or countably infinite number of measurable sets of points, multiply the content of each set by the upper (lower) limit of the values of the function for points of that set, and sum all such products; then the lower (upper) limit of such summations for every conceivable mode of division is the integral of the function.*

§ 8. I next proceed to show that in the case of an upper (lower) semi-continuous

function, the upper (lower) integrals may be obtained in a similar way, so that, *in the case of an upper semi-continuous function the tentative definition discussed in § 6 of the upper integral can be retained, and the same is true of the lower integral of a lower semi-continuous function.*

To prove this let us suppose the segment (A, B) = S divided up into a finite or countably infinite number of sets of points E_1, E_2, ..., so that the corresponding upper summation

$$E_1\bar{f}_1+E_2\bar{f}_2+\dots$$

differs from the minimum X by less than some assigned small positive quantity e.

Now round every point P of E_1 describe a small interval, in which the maximum of f differs from the value of f at P by a quantity less than some assigned small positive quantity e_1. We thus get a definite set of intervals enclosing all the points of E_1; their content is therefore not less than the content E_1, but we can so construct them that it is less than $E_1+e'_1$, where e'_1 is another small positive quantity as small as we please. In each of these intervals the maximum of the function f is less than $\bar{f}_1+e_1$.

Let us do likewise for each set E_r, choosing the small quantities so that

$$e_1+e_2+\dots = e'_1+e'_2+\dots = e.$$

Applying the Heine-Borel theorem to all these intervals, which enclose every point of E_1, E_2, ..., that is of (A, B), we can determine a finite number of these intervals enclosing every point of (A, B), and we can insure that the content of these intervals differs from their sum by less than e.* To each of these intervals we can attach the index of the first of the sets E_1, E_2, from which it was constructed; the content of those which have the index i will then be less than $E_i+e'_i$ and the maximum in each will be less than $\bar{f}_i+e_i$. Hence the upper summation over the non-overlapping intervals, consisting partly of the simple parts of these intervals and partly of the overlapping parts, is less than

$$(E_1+e'_1)(\bar{f}_1+e_1)+(E_2+e'_2)(\bar{f}_2+e_2)+\dots+Me,$$

where M is a positive quantity greater than the numerically greatest values of f. That is, this upper summation is less than $X+e+2Me+Se+e$, which, since e is at our disposal, proves the theorem.

A similar proof can be given for the case of the lower integral of a lower semi-continuous function, or we may deduce the corresponding theorem in this case from the fact that a lower semi-continuous function becomes an upper semi-continuous function when its sign is changed, and at the same time the lower integral becomes the upper integral.

Bearing in mind now that the upper integral of a function is equal to that of

* *Cp.* "An extension of the Heine-Borel Theorem," 'Messenger of Mathematics,' New Series, No. 393, January, 1904.

its associated upper limiting function, and that its lower integral is equal to that of its associated lower limiting function,* we have the following theorems, corresponding to that given at the end of § 7.

Theorem 6.—*To find the upper integral of any function we may proceed as follows: divide the segment or region into any finite or countably infinite number of measurable sets of points, multiply the content of each set by the upper (lower) limit of the maxima* † *(minima) of the function at points of that set and sum all such products; then the lower (upper) limit of all such summations for every conceivable mode of division is the upper (lower) integral of the function for the segment or region.*

§ 9. We are now able to define *upper and lower integration over a set of points*, instead of merely over an ordinary region; we may, indeed, if we please, suppose the function defined only for the set of points. The set will be assumed to be measurable, so that, like an ordinary region, it can be divided into component sets, each of which is measurable.

In the special case in which the content of the set is zero, we define the upper and lower integrals to be zero also. In the general case the definition will be as follows:—

Divide the fundamental set S *into any finite or countably infinite number of measurable components, multiply the content of each component by the upper (lower) limit of the maxima (minima) with respect to* S *of the function at all points of that component and sum all such products; then the lower (upper) limit of all such summations for every conceivable mode of division is the upper (lower) integral of the function for the fundamental set* S.

Further, when the upper and lower integrals for S *are equal, the function may be said to be integrable over that fundamental set* S.

Thus we have defined upper integration, lower integration, and integration over any measurable set S in such a manner that, in the particular case when the set S is a segment or region, we get the ordinary Riemann and Darboux integrals.

§ 10. Summing up our results so far, we saw that, though DARBOUX's form of the definition was preferable to RIEMANN's, it did not at once lend itself to generalisation. I then showed how to modify it so that the number of intervals should not necessarily be finite, provided that their content was equal to that of the segment. We then saw how the introduction of the maximum (minimum) at a point, instead of the value

* *Cp.* Upper and Lower Integration, 'Proc. Lond. Math. Soc.,' Ser. 2, vol. 2, Part I., p. 55, also § 11, below.

† It should be carefully noted that the maximum at a point of a set is the lower limit of the upper limit of the values of the function in a small interval or region containing the point, when that interval or region is indefinitely decreased. Similarly for the minimum. In the enunciation of the above theorem, therefore, the word "maxima (minima)" has nothing to do with the particular set to which the point belongs, the upper (lower) limit, however, is taken with respect to that set, that is, it is the upper limit of the maxima corresponding to the various points of that set. Similarly in the definition of § 9 the set S takes the place of the segment or region.

at a point, permitted of division of the segment into sets of points, instead of merely into sets of intervals; and, finally, that in this form the definition is applicable to integration over a set of points.

There is, however, another mode by which we can define integration with respect to a set of points in such a manner as to get the Darboux integral when the fundamental set reduces to a segment or ordinary region. This new generalised definition is in some respects more closely analogous to the original Darboux definition and brings out, more clearly than the one just given, the distinction between an interval and a set of points in general, throwing light, as it does, on the question why it is that the limit of the upper (lower) summations over a segment is different when the segment is divided into intervals from what it is when the division is into set of points. Like the Darboux definition, it concerns itself with the actual value of the function at a point instead of the maximum (minimum) there, while it divides the fundamental set up into components, closely analogous to the intervals in the Darboux definition.

It will be convenient, and conduce to clearness, to give first a few preliminary explanations and theorems. We require to define and give one or two properties of semi-continuous functions, when the region of existence is a set of points. It will be found that the introduction of these functions materially simplifies the treatment of the subject.

§ 11. The definition of an upper (lower) semi-continuous function, defined for any fundamental measurable set, does not differ from the usual definition for a segment, or ordinary region, the maximum (minimum) in each case is to be estimated with respect to the fundamental set alone, in the usual case the fundamental set being that segment or region, and in the general case that measurable set.

Theorem 7.—*If a function, defined with respect to a measurable set of points* S, *be an upper* (*lower*) *semi-continuous function, the points at which the value of the function is* $\geq k$ ($\leq k$) *form a measurable set.*

Complete the set of points S, *i.e.*, form the smallest closed set H of which S is a component. Attribute to the function at the points of H which are not points of S the upper (lower) limit of the values of the original function at points of S in a small neighbourhood of the point, when that neighbourhood is indefinitely decreased. We thus get a function which is upper (lower) semi-continuous with respect to a closed set H. By an argument of precisely the same nature as that used for the case when H is a segment, it follows that the set of points at which the new function is $\geq k$ ($\leq k$) is a closed set, say Q.

Since both Q and S are measurable, the same is true of their common component, which is none other than the set of points at which the given function, upper (lower) semi-continuous with respect to S, is $\geq k$ ($\leq k$). [Q.E.D.]

Definition.—If at every point of the fundamental set S we take, as the value of a new function at any point, the maximum (minimum) with respect to S of a given

function, I call the new function *the associated upper (lower) limiting function of the given function.* Using the term oscillation at a point for the excess of the maximum over the minimum, we get a third associated function by taking as value at a point the value of the oscillation there, this I call *the associated oscillation function.*

Theorem 8.—*The associated upper (lower) limiting function of any function with respect to a fundamental set* S *is an upper (lower) semi-continuous function with respect to* S.

For, complete the set S, and let H be the closed set so obtained. Form an extended upper (lower) limiting function as in the proof of the preceding theorem. The original discontinuous function may be supposed to have the same values in the points of (H−S) as this extended upper (lower) limiting function. A proof precisely similar to that for the continuum* proves that the new upper (lower) limiting function is upper (lower) semi-continuous.

Now it is plain that, though the points of (H−G) may have points of G for limiting points, the upper (lower) limits of the values of the function in the neighbourhood of points of G are the same for the old and new upper limiting functions. Therefore the values of the extended upper (lower) limiting function at the points of G are the maxima (minima) with respect to either H or G, so that the old upper (lower) limiting function is upper (lower) semi-discontinuous with respect to S. [Q.E.D.]

Corollary.—The associated oscillation function, being the sum of two upper semi-continuous functions,† *is itself an upper semi-continuous function.*

Theorem 9.—*If* S′ *be a component of the fundamental set* S, *such that all the limiting points of* S′ *which belong to* S *are contained in* S′, *then the upper (lower) limit of a function in* (S−S′) *is the same as that of the associated upper (lower) limiting function in* (S−S′).

This follows from the fact that the maximum (minimum) at any point P of (S−S′) is unaffected by the values of the function at the points of S′, since P is not a limiting point of S′. Whence the result is easily deduced.

The fact that the value at a point of an upper (lower) semi-continuous function is the maximum (minimum) at that point in the case of such a function, enables us to substitute the word *value* instead of maximum (minimum) in the definition of integration.

The definition in this case takes the following simplified form :—

Divide the fundamental set S *into a finite or countably infinite number of measurable components, multiply the content of each component by the upper (lower) limit of the values of an upper (lower) semi-continuous function at points of that component and sum all such products; then the lower (upper) limit of all such*

* BAIRE, 'Ann. di Mat.' (3), vol. III., 1899.

† This property is evidently unaffected by the substitution of a fundamental set instead of a segment or region.

summations for every conceivable mode of division is the upper (lower) integral of the semi-continuous function for the fundamental set S.

Taking this as the definition of the upper (lower) integral in the case of an upper (lower) semi-continuous function, the definition in the general case is equivalent to the following:—

Theorem 10.—*The upper (lower) integral of any function with respect to any measurable set* S *is the upper (lower) integral of its associated upper (lower) semi-continuous function.*

§ 12. The division of the continuum adopted by DARBOUX is, as has been proved, a special case of a more general division of the continuum into intervals, by means of which we obtain an upper summation differing by as little as we please from the upper integral. This division was such that the sum of the intervals was equal to that of the segment, while each interval had to be less than a quantity which depended only on the degree of approximation desired.

When the fundamental set S is not the continuum, but merely any measurable set, we can, given any small positive quantity e', find a set of intervals enclosing every point of S, that is, having every point of S as an internal point, the content of the intervals lying between S and $S+e'$. If we assign any small positive quantity e, there will only be a finite number of the intervals which are not less than e, since their content is finite. Each of these we can divide into a finite number of parts less than e, or we can in any other way determine inside the intervals a set of intervals enclosing all points of S, with the possible exception of a component of S of zero content.

In each of these intervals there is a measurable component of S of content less than e, and the sum of all these components is S. This division of the set S will be found to correspond very closely to the division of the continuum contemplated above, which is, of course, a special case of such a division; in particular it will be shown to lend itself conveniently to form approximations to the upper integral.

I shall, for convenience, refer to such a division as a *division of* S *by means of segments* (e, e'); and the upper summation of any function over these components I shall call the *upper summation with respect to* S *over the intervals.*

§ 13. *Theorem* 11.—*Given any small positive quantity* e_1, *we can determine a positive quantity* e_2, *such that, if the fundamental set* S *be divided in any manner by means of segments* (e, e'), *then, provided only* $e \leq e_2$, *and* $e' \leq e_2$, *the upper summation of the function with respect to* S *over these segments differs by less than* e_1 *from a definite quantity* I, *the lower limit of all such upper summations, when* S *is divided by means of segments.*

For, I being a lower limit, we can determine a set of intervals enclosing S, such that the upper summation with respect to S over these intervals lies between I and $I+\alpha$, where

$$6\alpha = e_1 \quad \ldots \ldots \ldots \quad (1).$$

Also, by the definition of content, we can determine a set of intervals containing S, whose content lies between S and $S+b$, where

$$Mb = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

M being any quantity greater than the greatest numerical value of the function in question for points of S.

The set of intervals consisting of the common parts of these two sets will contain S, have content lying between S and $S+b$, and also be such that the upper summation with respect to S over it has a value lying between I and $I+a$. From among the intervals of this set, we can determine a finite number n, such that the sum of the remaining intervals is less than b. The content of the component of S which is not external to these intervals is not less than $S-b$. The upper summation over the remaining intervals is, by 2, numerically less than a; therefore the upper summation over these n intervals is less than $I+2a$.

Let us now consider any division of S by means of segments (e, e'). There cannot be more than $2n$ of these which lie partly inside and partly outside the n intervals. The contribution of these to the upper summation will therefore be numerically less than $2nMe$.

Since, by what has been shown, the segments which are not entirely external to the n intervals, enclose a component of S of content not less than $S-b$, the sum of those segments which are external to the n intervals is not greater than $S+e'-(S-b)$, that is $e'+b$; their contribution therefore to the upper summation is less than $M(e'+b)$ numerically.

Finally the contribution made by those segments which lie each inside one of the n intervals cannot be greater than the upper summation over those intervals, that is cannot be greater than $I+2a$. Hence the upper summation over these segments is less than $I+3a+M(e'+2ne)$. Using (1), it follows that we have only to take

$$e_2 = e_1/2M(1+2n)$$

so that the upper summation over the segments may differ from I by less than e_1; this proves the theorem.

§ 14. It is clear from the above proof that, when we divide S by means of segments (e, e'), it is immaterial whether, in estimating the upper limit with respect to S over any segment, we include the end-points of that segment (supposing them to belong to S), or not; the potency of the points of S not included will in either case be the same, and, as the theorem shows, in either case the upper summation will be within e_1 of the limit I, provided only e and e' are sufficiently small.

Suppose, then, we exclude the end-points of every segment. Then the upper limit of the values of the function for points of S inside any segment being the same as that of the associated upper limiting function, we have the following result:—*The quantity* I *is the same for a function and its upper limiting function.*

§ 15. *Theorem* 12.—*The quantity* I *is the upper integral of the function with respect to* S.

The result of the preceding section, together with the fact that the upper integral of a function is the same as that of its upper limiting function, show that it is only necessary to prove the present theorem for upper semi-continuous functions.

Let f be an upper semi-continuous function, and let us suppose the fundamental set S divided up into a finite or countably infinite number of sets of points, E_1, E_2 ... so that the corresponding upper summation

$$E_1 f_1 + E_2 f_2 + \dots$$

differs from the upper integral X by less than some assigned small positive quantity e. Round every point of E_1 we can describe a small interval, in which the maximum of f with respect to S differs from the value of f at P by a quantity less than, say e_1, where

$$e_1 + e_2 + \dots = e\,;$$

and this interval may be decreased indefinitely. That is, we have a set of tiles,* each of which may be chipped as much as we please, and their points of attachment fill up E_1.

Applying the Tile Theorem, we obtain a finite or countably infinite set of the tiles, each less than e, covering up every point of E_1, and the sum of the tiles is less than $E_1 + e'_1$, where

$$e'_1 + e'_2 + \dots = e'.$$

In like manner we get a set of tiles from each set E_i. Applying the Tile Theorem to the set of all these tiles, since their points of attachment fill up S, we obtain a finite or countably infinite set of them, covering every point of S, and the sum of all is less than $S + e'$.

To each of these tiles d_P we make correspond the lowest integer i such that its point of attachment P belongs to E_i; the sum of the tiles corresponding to any particular integer i will then be less than $E_i + e'_i$, and the maximum of f in each will be less than or equal to $f_i + e_i$.

Now, if we include the boundary points of any interval, the set of intervals consisting of (1) the simple parts and (2) the overlapping parts of the tiles, leaves no points of S over, and gives us therefore a division of S by means of segments (e, e'). Of these, the sum of the overlapping parts is less than e', since the content of the tiles is not less than S, and their sum not greater than $S + e'$; thus the contribution of the overlapping parts is numerically less than Me'. The contribution of the simple parts, on the other hand, is less than the upper summation over the tiles themselves, that is, less than

$$(f_1 + e_1)(E_1 + e'_1) + (f_2 + e_2)(E_2 + e'_2) + \dots$$

Thus the upper summation over these segments (e, e') is less than

$$X + e + Se + Me' + ee' + Me'.$$

* *Cp.* "The Tile Theorem," 'Proc. Lond. Math. Soc.,' Ser. 2, vol. 2, Part I., p. 67.

Choose e and e' so small that 1) this is less than $X+\epsilon$, and 2) it differs from I by less than ϵ (by *Theorem* 11), ϵ being an assigned small positive quantity as small as we please, which proves that in this case X and I must be identical, and so, as already pointed out, proves the theorem.

§ 16. It is now possible to give the promised alternative definition of integration; the preceding theorems prove the equivalence of the two definitions :—

Definition.—Let the fundamental set, excluding at most a set of points of zero content, be enclosed in or on the borders of a set of non-overlapping segments each less than e, and of content less than $S+e$. Then let the content of that component of S in any segment be multiplied by the upper (lower) limit of the values of the function at points of that component, and let the summation be formed of all such products. Then it may be shown that this summation has a definite limit when e is indefinitely decreased, independent of the mode of construction of the segments and the mode in which e approaches the value of zero. *This limit is called the upper (lower) limit of the function with respect to the fundamental set* S.

In the special case when the upper and lower integrals agree, the common value is called the integral of the function.

§ 17. To find the condition of integrability of a function with respect to a fundamental set, we require the following theorem :—

Theorem 13.—*The upper (lower) integral of the sum of any finite number of upper (lower) semi-continuous functions with respect to a fundamental set* S *is the sum of the upper integrals of those functions.*

Let F be the sum of two upper semi-continuous functions f_1 and f_2; then F is itself upper semi-continuous. Then, by Theorem 11, e_1 being assigned, we can determine e_2 so that if S be divided by means of segments (e, e') and the corresponding upper summations formed, then provided only e and e' are each less than e_2, the summations, say,

$$f_{11}s_1+f_{12}s_2+\ldots \quad (1),$$

$$f_{21}s_1+f_{22}s_2+\ldots \quad (2),$$

$$F_1s_1+F_2s_2+\ldots \quad (3),$$

differ from the corresponding upper integrals by less than e_1.

Since f_1, f_2, and F are all upper semi-continuous with respect to S, we can, to each point P of S, determine an interval round P, such that the maxima of f_1, f_2, and F in that interval differ from the corresponding values of f_1, f_2, and F, by less than some assigned small positive quantity e_3, and this interval may be indefinitely decreased.

Applying the Tile Theorem, we see that every point of S is covered by one at least of a countable number of these tiles, each less than e, no point of attachment being covered by any other tile, and the sum of the tiles differing from S by less than e'. Thus we may take as our segments (e, e') those respectively covered by the simple and overlapping portions of these tiles, if we include the boundary points of each such segment in it.

The upper summation over the overlapping parts is in each of the three cases numerically less than Me', where M is a quantity greater than the greatest numerical values of f_1, f_2, and F. The simple part s_i of any tile d_{P_i} contains the point of attachment P_i, therefore the upper limit in that part differs from the value at P by less than e_3. Thus the upper summation over the segments lies between

$$s_1 f_1(P_1)+s_2 f_1(P_2)+\ldots-e'M \quad \text{and} \quad s_1 f_1(P_1)+s_2 f_1(P_2)+\ldots+e_3 S+e'M.$$

Thus, denoting by θ with any distinguishing suffix or accent a quantity numerically less than 1,

$$\theta_1 e_1+\overline{\int} f_1 ds = \theta_1' M e'_2+\theta_1'' S e_3+s_1 f_1(P_1)+s_2 f_1(P_2)+\ldots$$

with similar equations for the other two functions.

Now the summation

$$s_1 F(P_1)+s_2 F(P_2)+\ldots$$

is the sum of the corresponding summations for f_1 and f_2; whence it follows that the upper integral $\overline{\int} F ds$ differs from the sum of the upper integrals of f_1 and f_2 with respect to S by a quantity which is smaller than any assigned positive quantity, which proves the theorem.

§ 18. Denoting by f_1, $-f_2$, and F the associated upper, lower, and oscillation functions of a given function f, the three functions f_1, f_2, and F are all upper semi-continuous with respect to the fundamental set, and F is the sum of f_1 and f_2.

The upper and lower integrals of f with respect to the fundamental set are, by *Theorem* 10, the upper integral of f_1 and the lower integral of $-f_2$ respectively, and the latter is minus the upper integral of f_2. Thus the excess of the upper over the lower integral of f with respect to the fundamental set is the sum of the upper integrals of f_1 and f_2, that is, by *Theorem* 13, the upper integral of F with respect to the fundamental set.

Thus f will be integrable with respect to the fundamental set if, and only if, the upper integral of its associated oscillation function be zero.

Let G_k denote the component of the fundamental set at which F has a value greater than, or equal to, k; by *Theorem* 7, G_k is a measurable set, let its content be I_k. Then if, omitting at most a component of zero content, we enclose the fundamental set by means of segments (e, e'), since the common part of two measurable sets is measurable, there will be at most a component of G_k of zero content not included in the segments, and the remaining component of G_k will have content I_k. Thus the content of the components of the fundamental set in segments containing points of G_k will not be less than I_k, and the upper limit of F in each will not be less than k. Hence the upper summation will not be less than $I_k k$.

Thus it is clear that the upper integral of F cannot vanish unless, for all positive values of k, I_k is zero.

Theorem 14.—*Thus the necessary and sufficient conditions that a function f should*

be integrable with respect to a measurable set S, *is that the content of that component of* S *at every point of which the oscillation of* f *is greater than, or equal to,* k *should, for every value of* k, *be zero.*

Remembering that the outer limiting set of measurable sets is measurable, and has for content the limit of the contents of the defining components, this gives us the following alternative statement :—

The necessary and sufficient condition that a function should be integrable with respect to a measurable set S *is that the content of that component of* S, *at every point of which the function is continuous with respect to* S, *should be equal to the content of* S.

§ 19. With the ordinary definition of upper (lower) integration, or of integration, it was at once evident that if the segment or region of integration were divided into two parts (segments or regions), the sum of the upper (lower) integrals over the separate parts was the upper (lower) integral over the whole segment or region.

Theorem 15.—*More generally, it is evident from Theorem* 1 *that the sum of the upper (lower) integrals over any set of non-overlapping segments or regions is equal to the upper (lower) integral over the whole segment or region, provided only the content of those segments or regions is the same as that of the fundamental segment or region.*

That this is not so for the general case when the fundamental set of points, whether a segment or not, is divided into component sets is shown by the following simple example :—

Example 6.—Let f be zero everywhere except at the rational points of the segment (0, 1), and let f have the value unity at the rational points, and consider the integrals over the rational and irrational points separately ; both of these are zero. The upper integral over the whole segment is however unity, and the lower integral is zero.

The alternative definition of upper (lower) integration, given in § 11, shows that when the two component sets consist of parts of S obtained by means of segments, the sum of the upper (lower) integrals over the two components is the upper (lower) integral over the whole fundamental set.

Theorem 16.—*More generally, the sum of the upper (lower) integrals over any finite or countably infinite number of non-overlapping components of* S *obtained by means of segments* (e, e'), *is the same as the upper (lower) integral over the whole fundamental set, provided only the content of the components is equal to that of the fundamental set.*

An upper (lower) semi-continuous function stands here again in an exceptional position. We have, in fact, the following theorem :—

Theorem 17.—*The upper (lower) integral of an upper (lower) semi-continuous function over any fundamental set* S *is equal to the sum of its integrals over every finite or countably infinite number of sets into which* S *may be divided.*

This follows from the definition at the end of § 11; for if we divide the fundamental set S into (for instance) two measurable components S_1 and S_2, and then subdivide all three by means of segments, the definition shows that, in the case of an upper semi-continuous function, the sum of the upper summations over the segments with respect to S_1 and S_2 is not less than the upper integral. But in each segment the sum of the contents of the parts of S_1 and S_2 is the content of the corresponding part of S, while the upper limit corresponding to S is the greater of the upper limits corresponding to S_1 and S_2, so that the sum of the upper summations for S_1 and S_2 is not greater than the upper summation for S. Thus the sum of the upper summations with respect to S_1 and S_2 lies between the upper integral with respect to S and the upper summation with respect to S. And since, by properly constructing the segments, we can make the upper summations differ by as little as we please from the corresponding upper integrals, this proves the theorem.

The above method of proof shows at the same time why it is that the theorem does not hold for every function, for, if we form the associated upper limiting functions with respect to S_1 and S, the values at the different points of S_1 are not always the same.

§ 20. If in finding upper (lower) integrals we wish to divide the fundamental set up into convenient components, we must first replace the function by its associated upper (lower) limiting function.

Example 6 is a particularly instructive one; the function is integrable over each of the two component sets into which the segment which is the range of variation is divided, and is not integrable over the segment itself. We easily see, however, that the following theorems hold :—

Theorem 18.—*If a function be integrable over the fundamental set* S, *it is integrable over every component set of* S. From Theorem 17 it now follows that

Theorem 19.—*The integral of an integrable function over its fundamental set* S *is equal to the sum of its integrals over every finite or countably infinite number of components into which* S *may be divided.*

For, as has been shown in the preceding section, the upper integral of a function over the fundamental set S cannot be less than the sum of the upper integrals over the component sets; and the lower integral of the function over S cannot be greater than the sum of the lower integrals over the component sets. Since, however, the function is integrable over S, it is also integrable over each of the components. Therefore the integral over S cannot be less nor greater than the sum of the integrals over the components, so that it must be equal to this sum. [Q.E.D.]

In this connection, it should be noted that the integration of an integrable function (which has finite upper and lower limits) involves nowhere the idea of order, even when, for convenience, we determine it as the sum of a countably infinite number of integrals. The series of such integrals is an absolutely convergent one, and it has the same sum however it be arranged.

Theorem 20.—*The sum of the integrals of any finite number of integrable functions*

taken over the same fundamental set S *is equal to the integral of the sum of the functions taken over the set.* This follows from *Theorem* 13, which asserts the same for upper (lower) integrals of upper (lower) semi-continuous functions. The proof is of precisely the same nature as the preceding.

§ 21. We now proceed to obtain a formula for the upper (lower) integral of an upper (lower) semi-continuous function over any measurable set of points, in terms of an ordinary integral. These formulæ of course at the same time give us the upper (lower) integral of any function, by reason of § 11.

Let K' be any quantity not less than the greatest, and K not greater than the least, value of an upper semi-continuous function f defined for a fundamental set S. Divide (K, K') into n parts, and consider the sets of points $G_1, G_2, \dots G_n$, where G_r denotes all those points of S for which f is $\geq K'-\frac{K'-K}{n}r$, and G_n is S, so that

$$S = G_1+(G_2-G_1)+\dots+(G_n-G_{n-1}).$$

By *Theorem* 7, $G_1, G_2, \dots$ are measurable sets; let $I_1, I_2, \dots$ be their contents. Then by the definition at the end of § 11, the upper integral, being not greater than any upper summation, is not greater than

$$K'I_1+\left(K'-\frac{K'-K}{n}\right)(I_2-I_1)+\dots+\left(K'-\frac{K'-K}{n}r\right)(I_{r+1}-I_r)+\dots$$
$$+\left\{K'-\frac{K'-K}{n}(n-1)\right\}(I_n-I_{n-1})+\left(K'-\frac{K'-K}{n}n\right)(-I_n)+KI_n,$$

i.e.,

$$\frac{K'-K}{n}(I_1+I_2+\dots+I_{n-1}+I_n)+KI_n,$$

however great n may be, and is therefore

$$\leq\int_K^{K'} I dk+KS,$$

since I is a monotone function of k and therefore an integrable one.

Here S is the content of the fundamental set, and I that of the set of points for which the values of the function are greater than or equal to k. But it was shown, in § 6, that for any function whatever the greatest value of the lower summations is certainly not greater than the least value of the upper summations, therefore the upper integral is not less than all the lower summations. Hence the upper integral is certainly not less than

$$\left(K'-\frac{K'-K}{n}\right)I_1+\dots+\left(K'-\frac{K'-K}{n}r\right)(I_r-I_{r-1})+\dots+\left\{K'-\frac{K'-K}{n}(n-1)\right\}(I_{n-1}-I_{n-2})$$
$$+\left(K'-\frac{K'-K}{n}n\right)(I_n-I_{n-1}),$$

that is, not less than

$$\frac{K'-K}{n}(I_1+\ldots+I_{n-1})+KI_n,$$

and therefore not less than $\int_K^{K'} Idk+KS.$*

Hence, finally the upper integral is actually equal to $KS+\int_K^{K'} Idk$. In precisely the same way it may be shown that the lower integral of a lower semi-continuous function can be expressed as an ordinary integral. For variety I give the following proof.

Let f be a lower semi-continuous function, and $f'=-f$; then f' is an upper semi-continuous function; and if K be not greater than the lower limit of f, and K′ not less than the upper limit of f, then $-K'$ is not greater than the lower limit of f', and $-K$ not less than the upper limit of f'. Hence, by what has just been proved,

$$\underline{\int} fdx = -\overline{\int} f'dx = -(-K')\,S-\int_{-K'}^{-K} Jdk',$$

where J is the content of the set of points of S at which f' has a value $\geq k'$. Writing $k=-k'$, we have for the lower integral of f the expression $K'S-\int_K^{K'} Jdk$, where J is the content of the set of points of S at which f has a value $\leq k$. Summing up, we have the following theorems:—

Theorem.—The upper integral of an upper semi-continuous function with respect to a measurable set S *is* $KS+\int_K^{K'} Idk$, *where* I *is the content of that component of* S *at every point of which the function has a value greater than or equal to k.*

Theorem 21.—*The lower integral of a lower semi-continuous function with respect to a measurable set* S *is* $K'S-\int_K^{K'} Jdk$, *where* J *is the content of that component of* S *at every point of which the function has a value less than or equal to k.*

In both these theorems K and K′ are quantities which are respectively less than or equal to the lower limit, and greater than or equal to the upper limit of the function for points of S. Hence also

Theorem 22.—*The upper integral of any function with respect to a measurable set* S *is* $KS+\int_K^{K'} Idk$, *where* I *is the content of that component of* S *at every point of which the maximum of the function is greater than or equal to k.*

The lower integral is $K'S-\int_K^{K'} Jdk$, *where* J *is the content of that component of* S *at every point of which the minimum of the function is less than or equal to k.*

* This argument shows that in the case of an upper semi-continuous function the upper integral is equal not only to the lower limit of the upper summations, but also to the upper limit of the lower summations the latter is not to be confounded with the lower integral. Similarly for a lower semi-continuous function both limits give the lower integral.

§ 22. By § 18 the condition that a function should be integrable with respect to a measurable set S is that the upper integral of the associated oscillation function should be zero. The latter function is upper semi-continuous, by the corollary to *Theorem* 8, so that we can apply the preceding theorem, putting $K = 0$, since this function is always positive or zero.

Thus the condition of integrability is that $\int_0^{K'} 0\,(k)dk$ should vanish, where $0\,(k)$ is the content of that component of S at every point of which the oscillation of the given function is k. This is only possible if $0\,(k)$ is zero, except possibly at a set of values of k of zero content; as however, if $0\,(k)$ were positive for any value of k, it would be positive for every lesser value of k, and therefore for a set of values of content greater than zero, it is clear that there can be no exception. Thus we get again the condition of integrability obtained in § 18 (*Theorem* 14).

§ 23. When the function is integrable, the upper and lower integrals are equal, otherwise the upper is the greater, thus

$$\mathrm{KS}+\int_{\mathrm{K}}^{\mathrm{K}'}\mathrm{I}\,dk \geq \mathrm{K}'\mathrm{S}-\int_{\mathrm{K}}^{\mathrm{K}'}\mathrm{J}\,dk,$$

therefore

$$\int_{\mathrm{K}}^{\mathrm{K}'}(\mathrm{I}+\mathrm{J})\,dk \geq (\mathrm{K}'-\mathrm{K})\,\mathrm{S}.$$

Now no point can be such that the maximum there is less than k while the minimum is greater than k; thus every point of S belongs to at least one of the two sets I and J. Let L denote the content of the set of points common to both I and J, then*

$$\mathrm{I}+\mathrm{J} = \mathrm{S}+\mathrm{L}.$$

Whence

$$\int_{\mathrm{K}}^{\mathrm{K}'}(\mathrm{S}+\mathrm{L})\,dk \geq (\mathrm{K}'-\mathrm{K})\,\mathrm{S}, \quad \text{that is } \int_{\mathrm{K}}^{\mathrm{K}'}\mathrm{L}\,dk \geq 0,$$

the sign of equality being allowable if, and only if, the function f is integrable.

Now L is the content of the set of points at each of which the maximum $\geq k$, while the minimum is $\leq k$; these points consist of:—

(1.) Points of continuity at which $f = k$;

(2.) Points of discontinuity at which either the maximum is $>k$ and the minimum $\leq k$, or the maximum $= k$ and the minimum $<k$.

But it is clear, as at the end of § 22, that if for any value of k the content of the set (2) were not zero, there would be a set of values of k of positive content, for each of which the content of the set (2) would not be zero; L would then not be a null function and the given function would therefore not be integrable.

Thus we get a new form of the condition of integrability, and also theorems relating to the distribution of the points of continuity of an integrable function, and the values of a continuous function:—

* 'Theory of Content,' p. 49, (5).

Condition of Integrability.—The necessary and sufficient condition for the integrability of a function f over a measurable set S is that the content L of the set of points for which the maximum of the function is $\geq k$ while the minimum $\leq k$, should be a function of k whose value is zero, except for a set of values of k of content zero; for each of these values of k the points of discontinuity of the function still form a set of content zero.

Theorem 23.—The content C of that component of the fundamental set whose points are points of continuity at which an integrable function $= k$, is a function of k whose value is zero, except for a set of values of k of content zero.

Theorem 24.—The content C of that component of the fundamental set at whose points a continuous function $= k$, is a function of k whose value is zero, except for a set of values of k of content zero.

§ 24. We now return to the tentative definitions of § 6, which we saw did not agree with the usual definitions. On the other hand, the definitions we have since constructed seem more artificial than these. It suggests itself, therefore, that the most logical plan is to throw overboard the Riemann and Darboux definitions altogether, and to define an integral as follows:—

Let the fundamental set be divided into measurable components in any conceivable way, and let the content of each component be multiplied by the upper (lower) limit of the values of the function at points of that component, and the sum of all such products be formed; then the outer (inner) measure of the integral is defined to be the lower (upper) limit of all such summations.

If we assume either that all sets are measurable, or that all functions are such that the points for which the values of the function are $\geq k$ ($\leq k$) are measurable, or that the functions with which we are concerned have this property, the argument of § 21 still applies, and we can assert that the outer measure is equal to the inner measure of the integral, and that each can be expressed in either of the two forms

$$\mathrm{KS}+\int_{\mathrm{K}}^{\mathrm{K}'}\mathrm{I}\,dk, \quad \mathrm{K}'\mathrm{S}-\int_{\mathrm{K}}^{\mathrm{K}'}\mathrm{J}\,dk,$$

where I and J are the contents of those components of S, at every point of which the function has values respectively $\geq k$ and $\leq k$.

It is not known whether any but measurable sets exist, or whether any functions can be constructed not having the above property. M. LEBESGUE has, therefore, used the term *summable* to denote the functions under consideration.

Precisely as in § 22 we can now prove the following theorem:—

Theorem.—The content C of that component of the fundamental set at every point of which a summable function has the value k, is a function of k whose value is zero except for a set of values of k of content zero.

In the case of a summable function, therefore, the outer and inner measures of the integral agree, and we may call either the generalised integral of the function. As

we have seen, functions which are integrable in the old sense are integrable in the new sense, and the integrals agree, but the converse is not true. In the case of an upper (lower) semi-continuous function, the generalised integral is the upper (lower) integral.

§ 25. LEBESGUE gives two definitions of his generalised integral, which I shall, for convenience, allude to as the Lebesgue integral.

The first is a geometrical definition, and has the disadvantage that the positive and negative values of the function have to be considered separately. It is as follows :—

Geometrical Definition of the Lebesgue Integral.—Let f be a function defined for every point of a finite segment (a, b); consider the plane set of points defined by the three inequalities*

$$a \leq x \leq b, \quad yf(x) \geq 0, \quad 0 \leq y^2 \leq \overline{f(x)}^2.$$

Let E_1 and E_2 be the two parts of E respectively above and below the axis of x (the points on the axis of x may be considered as belonging to whichever we prefer E_1 or E_2). If E is a measurable set, then both E_1 and E_2 are measurable.† In this case the function f is said to be summable, and the excess of the content of E_1 over that of E_2 is defined to be the Lebesgue integral of f over the segment (a, b).

LEBESGUE'S second definition is analytical.

Analytical Definition of a Summable Function.—A summable function is such that the set of values of x, for which the values of the function lie between any two quantities a and b, is measurable. Conversely, if this condition is satisfied, and the upper and lower limits of the function are finite, the function is summable.

Analytical Definition of the Lebesgue Integral.—Let the region of variation of $f(x)$ be denoted by (k_0, k_n), and let it be divided into n parts each less than α, say, at the point $k_1, k_2, \ldots k_{n-1}$.

Let e_i denote the content of those points x at which $f = k_i$; and e'_i that of the points x at which $k_{i-1} < f(x) < k_i$.

Then it may be shown that the two summations

$$\sum_0^n k_i e_i + \sum_1^n k_{i-1} e'_i \quad \text{and} \quad \sum_0^n k_i e_i + \sum_1^n k_i e'_i$$

have a common limit, when α is decreased indefinitely; this limit is the Lebesgue integral of $f(x)$ from a to b.

The identity of the two definitions is easily proved; in the case of a function which is always positive, the former of the two given expressions evidently represents the

* Here I have corrected an obvious misprint in LEBESGUE'S paper.

† This is stated without proof by LEBESGUE in § 17; it is a special case of the theorem of § 18, p. 251. A more simple proof is afforded by considering E_1 as the common part of E and a sufficiently large rectangle on the given segment as base and lying on the positive side of the x-axis, and E_2 as the difference between E and E_1; since the common part or the difference of two measurable sets is measurable. A similar proof applies to the theorem of § 18, given below here as the analytical definition.

content of a plane set containing the set E, and the latter the content of a plane set contained in E, so that the common limit must be the content of E. In the case when the function is sometimes positive and sometimes negative, the sum of the positive terms of the first summation is the content of a set containing E_1, and the sum of the negative terms, taken with positive sign, is the content of a set contained in E_2; similarly the second summation is the content of a component of E_1 minus the content of a set containing E_2; thus the common limit must be the content of E_1 minus that of E_2. Thus the two definitions are identical.*

The second of LEBESGUE's definitions enables us without difficulty to identify the Lebesgue integral with the generalised integral which I defined in § 24. In fact, comparing LEBESGUE's notation with that used by myself in § 21, it is clear that

$$e_{i-1}+e'_i = \mathrm{I}_{n-i+1}-\mathrm{I}_{n-i}.$$

Whence the former of the two given expressions is equal to

$$k_0(\mathrm{I}_n-\mathrm{I}_{n-1})+k_1(\mathrm{I}_{n-1}-\mathrm{I}_{n-2})+\ldots+k_{n-1}(\mathrm{I}_1-\mathrm{I}_0)+k_n\mathrm{I}_0,$$

that is,

$$k_0\mathrm{S}+(k_1-k_0)\mathrm{I}_{n-1}+(k_2-k_1)\mathrm{I}_{n-2}+\ldots+(k_n-k_{n-1})\mathrm{I}_0,$$

which, as α is decreased indefinitely, approaches the limit

$$\mathrm{KS}+\int_{\mathrm{K}}^{\mathrm{K}'}\mathrm{I}\,dk,$$

that is, the generalised integral of § 24.

Thus *the Lebesgue integral is the same as the generalised integral of* § 24, the fundamental set being a finite segment.

§ 26. Contrasting the definition of § 24 (S being now a finite segment) with the geometrical definition of LEBESGUE, we see that they stand to one another in the same relation as the ordinary definition of integration of, say, a continuous function to its definition as a certain area. Just as, however, the mathematical conception of area is more complex than and indeed depends on that of length, so does the theory of the content of a plane set of points depend naturally on that of the content of a linear set. Just as the determination of area requires the application of the processes explained in the first definition of integration of continuous functions, so with the content of a plane set. Thus the comparative simplicity of the geometrical definition is only apparent. With regard to LEBESGUE's analytical definition, I have pointed out that it is equivalent to what seems to me the much more convenient form in which I have expressed it as an ordinary integral (§ 24).

If we know I as a function of k, which may be the case, all Lebesgue integration and all upper and lower integration reduce to ordinary integration. It may, however, happen that I is not readily found as a function of k. The definition of § 24 seems then the most fundamental, and is, in many respects, very convenient in the theory.

* LEBESGUE, *loc. cit.*, p. 252.

§ 27. It is instructive to show the identity of LEBESGUE's geometrical definition with that of § 24 (S being a finite segment) directly. At the same time it should be remarked that the argument is independent of the fact that S is a finite segment, so that we have some general theorems with respect to the geometrical representation of the processes of integration with respect to a set of points to which I shall return in § 32.

Lemma 1.—*If at every point of a measurable set* G *on the x-axis of content I we place an ordinate of length p, the plane set constituted by this set of ordinates, or, as I shall call them, for definiteness, this set of blocks, is measurable and has content* pI.

For we can enclose G in a set of non-overlapping intervals of content less than $\mathrm{I}+e$. Erecting on these rectangles of height $p+e$, we have enclosed the whole plane set in a set of rectangles of content as near as we please to pI. On the other hand, taking in E a closed set of content greater than $\mathrm{I}-e$, the corresponding set of blocks forms a closed set of content as near as we please to pI. Thus the outer measure of the content of the set of blocks is not greater than pI, and the inner measure of the content is not less than pI, which proves the Lemma.

Suppose now we are given any measurable set S, and on S as base any set of blocks. Let us denote the length of the block at the point x by X, and, for simplicity, let us first consider X as being always positive. Then, if we divide S in any manner into a finite or countably infinite set of measurable components, S_1, S_2..., and at each point of S_i we replace the given block by one of length equal to the upper limit of the lengths of the blocks at points of S_i, we get a new set of blocks which, regarded as a plane set of points, contains the given set of blocks. Since the sum of a finite or countably infinite series of non-overlapping measurable sets is a measurable set whose content is the sum of the contents of its components, it follows that the new set of blocks constitutes a plane set of points which is measurable, and has for content, by the Lemma, the upper summation of X over S corresponding to the mode of division adopted. Hence the outer measure of the content of the plane set of points constituted by the given set of blocks is not greater than this upper summation, and therefore, since this is true for every mode of division, not greater than the outer measure of the integral of the blocks over S. Similarly for the lower measure of the integral.

If, on the other hand, X be sometimes positive and sometimes negative, we must, as LEBESGUE does, consider separately the positive and negative blocks; thus we get the following theorems:—

Theorem 25.—*An upper or lower summation is the difference of the contents of two measurable sets of blocks, one on the positive and one on the negative side of the axis of x.*

Theorem 26.—*Given any set of blocks on a measurable set* S, *the outer measure of the content of the plane set of points constituted by the positive blocks minus the inner*

measure of the content of the negative blocks is not greater than the outer measure of the integral of the blocks over S.

Similarly, the inner measure of the content of the positive blocks minus the outer measure of that of the negative blocks is not less than the inner measure of the integral.

In symbols

$$E_1^o - E_2^i \leq I^o, \quad E_1^i - E_2^o \geq I^i,$$

where E_1 and E_2 are respectively the set of points constituted by the positive and the negative blocks, and I denotes the integral, the indices o and i denote outer and inner measures respectively.

Denoting the sum of E_1 and E_2 by E, and supposing the upper and lower limits of the lengths of the blocks to be finite, E_1 and E_2 are the common parts of E, and the sets of blocks on S as base whose heights are respectively the upper and the lower limits of the lengths of the blocks, both these sets of blocks are measurable. Thus we see that, if E is measurable, E_1 and E_2 are so also, so that I^o and I^i are equal to one another and to the difference of the contents of E_1 and E_2, that is to say, *a summable function is integrable in the generalised sense of* § 24, *and the integral as defined in* LEBESGUE'S *geometrical manner is the integral as defined in* § 24, S being a finite segment.

Conversely, if I^o and I^i are equal,

$$E_1^o - E_2^i \leq I \leq E_1^i - E_2^o,$$

but since the outer content of a set is not less than the inner content,

$$E_1^o - E_2^i \geq E_1^i - E_2^o, \quad \text{whence} \quad E_1^o - E_2^i = E_1^i - E_2^o.$$

Therefore

$$E_1^o + E_2^o = E_1^i + E_2^i,$$

and therefore, since the outer measure of the content of E is not greater than $E_1^o + E_2^o$, and the inner measure is not less than $E_1^i + E_2^i$, the inner and outer measures of the content of E are equal, and E is a measurable set. Thus *a function which is integrable with respect to a finite segment in the generalised sense of* § 24 *is a summable function, and its integral is the Lebesgue integral, by* LEBESGUE'S *geometrical definition.*

Summing up the results of this section, we have the following theorem :—

Theorem.—Lebesgue integration is identical with generalised integration with respect to a finite segment.

§ 28. In general, without confining our attention to finite segments, we have from what has been shown the following geometrical definition of generalised integration with respect to a measurable set S :—

Geometrical Definition.—At each point x of a measurable set S *draw an ordinate (block) equal to the value of a function defined for every point of* S, *the outer measure of the content of the positive blocks minus the inner measure of the negative blocks is called the outer measure of the integral function with respect to the fundamental set.*

Set S.—*If the whole set of blocks is measurable, the function is said to be integrable in the generalised sense* (*or summable*) *with respect to the fundamental set* S. *In this case the positive blocks alone form a measurable set, and so do the negative blocks, and the excess of the content of the positive over that of the negative blocks is called the generalised integral of the function with respect to* S.

LEBESGUE (p. 255) has occasion to use integration with respect to a measurable set, but only in the case when the set is contained in a finite segment. The mode of definition adopted by him is that of completing the function by ascribing to it the value zero at all points of the segment other than those of the set in question, and then defining the integral of the function with respect to the fundamental set as equal to that of the extended function in the whole segment. In this case the generalised integral so defined is evidently the same as that defined above and in § 24; the mode of definition is, however, open to some objection; characteristic properties of the function such as continuity or semi-continuity with respect to the fundamental set, which materially simplify the properties of the integral, are not maintained by the extended function; on the other hand the definition suggests a difficulty, when dealing with fundamental sets of finite content not lying in a finite segment, which is entirely illusory, and gives a pre-eminence to the finite segment as region of operation which it does not in reality in any way possess.

§ 29. LEBESGUE'S theorem that the sum of two summable functions is a summable function, and its integral is the sum of their integrals, is equally true when the fundamental set is any measurable set of finite content. It is an immediate result of any of the definitions that this is the case when one of the functions is a constant, thus, the upper and lower limits being as usual finite, if the theorem is true for positive functions it is true always. In the case of positive functions the theorem is geometrically equivalent to the following:—

Theorem 27.—*Given any set of positive blocks forming a measurable plane set of points, the blocks may be shifted up parallel to themselves without altering the content of the set, provided the amount of shifting at each point x is a summable function of x.*

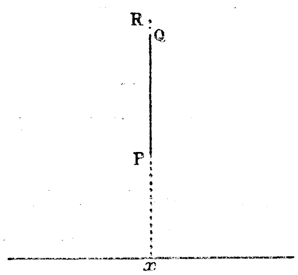

Fig. 1.

Divide the base S up into any number of measurable components S_1, S_2, ... and consider the set in the shifted position. At any point x of S_i let the block be PQ, so that Px represents the amount of shifting from the position when the lower extremities were all on the x-axes. Prolong the block to R, so that PR is equal to the upper limit of the length of the blocks corresponding to all points of S_i. In this way we get a plane set containing the given set.

Now, by Lemma 1, § 27, the content of a set of blocks each of length PR, erected at all the points of S_i, is PR . S_i. Hence, since we may evidently consider Rx as having been obtained by shifting up the ordinate Px the constant amount PR, the content of the set of blocks Rx = that of the blocks Px+PR . S_i.

Now the blocks Px form a measurable set, by the usual argument, since they consist of the part of a measurable set of blocks on a measurable component S_i as base; therefore the blocks PR at each point of S_i, shifted up to the position of the figure, being the difference of two measurable sets, form a measurable set, and the content is $PR \cdot S_i$, that is, the same as it was before shifting. Since this is true for each component S_i, it is true for the whole set of elongated blocks, that they form a measurable set whose content is the same as it was before shifting.

Now, by § 27, we know that, before shifting, the content of this set was the upper summation of the lengths of the given blocks, that is PQ, over the fundamental set S, divided into $S_1, S_2, \ldots$ Thus this upper summation, being the content of a plane set containing the given set, is not less than the content of the given set (of course in the shifted position).

Similarly the corresponding lower summation is not greater than the same content. But the lower limit of the upper summations is equal to the upper limit of the lower summations, since either of them represents the content of the given blocks before they were shifted up from the x-axes. Thus the same content is itself neither less nor greater than the content of the set in the shifted position, that is to say, the content has been left unaltered by the shifting. [Q.E.D.]

Corollary 1.—*In the shifted position the content is still the generalised integral of the length of the blocks.*

Corollary 2.—*The sum of any finite number of summable functions is a summable function, and its integral is the sum of their integrals, the fundamental set being any measurable set of finite content.*

Corollary 3.—*Since the limit of a sequence of measurable sets is a measurable set, it also follows that the sum of an absolutely convergent series of summable functions is a summable function, and its generalised integral is the sum of their integrals, provided, as usual, the functions have finite upper and lower limits.*

§ 30. By means of a theorem proved in my paper on "Upper and Lower Integration,"* we can extend the results of § 29 still further. The theorem quoted states that, if X' be the content of the section of a closed plane set by the ordinate through the point x of the x-axis, the content of the plane set is $\bar{\int} X' dx$, and that, further, X is an upper semi-continuous function of X. It follows, then, by § 24, that the content is the generalised integral of X'.

The theorem now to be proved is as follows:—

Theorem 28.—*If X^o and X^i be the outer and inner measures of the content of the ordinate section of a measurable set such that the set got by closing it is of finite content, by the ordinate through the point x, X^o and X^i are both summable functions, and the generalised integral of either is the content of the measurable set.*

Let I be the content of the set, and e any assigned small positive quantity. Let us

* 'Proc. Lond. Math. Soc.,' Ser. 2, vol. 2, Part I., p. 60.

take a closed component of the given set of content greater than $\mathrm{I}-e$. Denoting by X' the content of the ordinate section of this set, we have, by the theorem quoted, $\mathrm{I}-e<\int \mathrm{X}'dx$.

Since X' is an upper semi-continuous function, it follows, by § 24, that

$$\mathrm{I}-e< \text{the upper limit of the lower summations of } \mathrm{X}',$$

or, since X' is not greater than X^i,

$$\mathrm{I}-e< \text{the upper limit of the lower summations of } \mathrm{X}^i.$$

Since e may be as small as we please,

$$\begin{aligned}\mathrm{I} &\leq \text{the upper limit of the lower summations of } \mathrm{X}^i,\\ &\leq \text{the upper limit of the lower summations of } \mathrm{X}^o.\end{aligned}$$

Next let the content of the set got by closing the given set be denoted by S, and that of the complementary set by J, so that

$$\mathrm{I}+\mathrm{J} = \mathrm{S}.$$

Denoting by Y' and Z' the quantities for the complementary set and the whole closed set corresponding to X' for the given set, we have, as before,

$$\mathrm{J} \leq \text{the upper limit of the lower summations of } \mathrm{Y}^i.$$

Now $(\mathrm{Z}'-\mathrm{Y}')$ is the content of the difference of two closed sets, that is of an inner limiting set,* containing the set X, therefore,

$$\mathrm{Z}'-\mathrm{Y}' \geq \mathrm{X}^o \geq \mathrm{X}^i.$$

Hence

$$\begin{aligned}\mathrm{J} &\leq \text{the upper limit of the lower summations of } (\mathrm{Z}'-\mathrm{X}^o),\\ &\leq \text{the upper limit of (a lower summation of } \mathrm{Z}' \text{ minus an upper summation of } \mathrm{X}^o).\end{aligned}$$

But the upper limit of the lower summations of Z' is the generalised integral of Z', that is, S; therefore

$$\mathrm{J} \leq \mathrm{S} - \text{the lower limit of the upper summations of } \mathrm{X}^o,$$

that is,

$$\mathrm{I} \geq \text{the lower limit of the upper summations of } \mathrm{X}^o,$$

a fortiori,

$$\mathrm{I} \geq \text{the lower limit of the upper summations of } \mathrm{X}^i.$$

Now (§ 6) every upper summation of a function is greater than, or at least equal to, any lower summation, so that a quantity cannot be less than the upper limit of the

* 'Theory of Content,' p. 36.

lower summations without being less than the lower limit of the upper summations; neither can it be greater than the lower limit of the upper summations without being greater than the upper limit of the lower summations. Thus I must be actually equal to the upper limit of the lower summations as well as to the lower limit of the upper summations in the case of either X^o or X^i; that is to say, I is the generalised integral of either X^o or X^i, and both these functions are summable. [Q.E.D.]

Corollary 1.—*Each ordinate section of a measurable set being moved on its ordinate in such a manner that the (linear) content of the section is unaltered, and that the whole set remains measurable, the content of the whole set is unaltered.*

Corollary 2.—*At each point of a set of points of content* A *draw an ordinate, and on it take any set of points of (linear) content* B, *the content of the whole set so formed is* AB.

Here, as elsewhere, the fundamental set need not be a linear set, but may have a content of any number of dimensions.

§ 31. The preceding section, coupled with § 24, give us $\int_0^{K'} \mathrm{I}dk$ as the content of any measurable set (provided the set got by closing it has finite content), here I is the content of the set of points of the fundamental set at which the inner (or the outer) measure of the content is $\geqq k$. This, together with the preceding section, give the solution of the problem alluded to in § 26, viz., the reduction of the calculation of n-dimensional content to that of $(n-1)$-dimensional content, and so ultimately to that of linear content.

Bearing in mind the definition of a generalised integral, we have the following rule for finding the content of an n-dimensional set:—Take any hyperplane section of the set, project the set on to this hyperplane, and take any measurable set containing this projection as the fundamental set S. Divide S up in any way into a finite or countably infinite set of measurable components, and multiply the content of each component by the upper (lower) limit of the values of the (linear) inner or outer content of the corresponding ordinate sections of the given set; summing all such products, the lower (upper) limit of all such summations is the content of the given set.

§ 32. I have explained the geometrical representation of generalised integration with respect to a set. It is of interest to note the corresponding representations of what I call ordinary upper and lower integration with respect to a set.

Consider first the case where the fundamental set S is a segment, and form the set of blocks corresponding to the generalised integral of the function. If the function is everywhere positive, the geometrical representation of the ordinary upper integral is obtained by closing the plane set of points constituted by the blocks. If the function is not everywhere positive, we can make it so by adding a constant, which is geometrically equivalent to adding a rectangle to the representative set, part of which, viz., E_2, was to be considered as negative, and must be subtracted from the

rectangle; this addition of the rectangle is therefore equivalent to sinking the axis of x to a convenient position, so that all the blocks become positive. The upper integral of the given function, plus the constant, is now represented by the set got by closing the set of blocks. From this we have now to subtract again the rectangle, or, which is the same thing, return the x-axis to its original position, in order to get the geometrical representation of the upper integral of the given function.

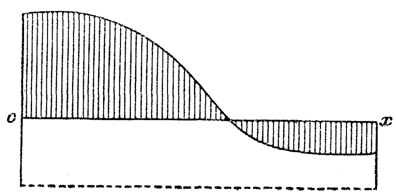

Fig. 2.

Thus, in the general case, the geometrical representation of the upper integral is not precisely a closed set, but a closed set minus a rectangle, as is shown roughly in fig. 2, viz., the shaded region, of which the part below the x-axis, corresponding to E_2, is to be considered as negative, and the other part, corresponding to E_1, is closed.

Similarly, in the case of a function which is always negative, the geometrical representation of the lower integral is a closed plane set constituted by blocks; in the general case it is represented by the excess of a rectangle over such a set.

When the fundamental set S is any measurable set whatever, instead of closing the fundamental set actually, we do so *relative to* S, that is, we take in only those limiting points which lie on ordinates through S. The rectangle to be subtracted in the case of the ordinary upper integral, or from which the relatively closed set is to be subtracted in the case of a lower integral, is then a *relative rectangle*, that is, that part of a rectangle which lies on the ordinates through S.