PHILOSOPHICAL TRANSACTIONS.

I. *Further Consideration of the Stability of the Pear-shaped Figure of a Rotating Mass of Liquid.*

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Introduction.

In vol. 17, No. 3 (1905), of the 'Memoirs of the Imperial Academy of St. Petersburg,' M. Liapounoff has published an abstract of his work on figures of equilibrium of rotating liquid under the title "Sur un Problème de Tchebychef." In this paper he explains how he has obtained a rigorous solution for the figure and stability of the pear-shaped figure, and he pronounces it to be unstable. In my paper in the 'Philosophical Transactions'* I had arrived at an opposite conclusion.

The stability or instability depends, in fact, on the sign of a certain function which M. Liapounoff calls A, and which I denote $A_0 + \Sigma (B_i^s)^2/C_i^s$, where A_0 is equal to $\mathfrak{A}_3[\frac{1}{3}(\sigma_2)^2 + 2\zeta_4] - \frac{1}{3}\sigma_4 + \Sigma[i, s]$.

M. Liapounoff tells us that, after having seen my conclusion he repeated all his computations and confirmed his former result. He attributes the disagreement between us to the fact that I have only computed portion of an infinite series, and only used approximate forms for the elliptic integrals in the several terms. He believes that the inclusion of the neglected residue of the infinite series would lead to an opposite conclusion.

In my computation the function $\mathfrak{A}_3[\frac{1}{3}(\sigma_2)^2 + 2\zeta_4] - \frac{1}{3}\sigma_4$ is decisively negative, and being numerically greater than $\Sigma\{(B_i^s)^2/C_i^s + [i, s]\}$, which is positive, the sum of the two is negative. The inclusion of the neglected residue undoubtedly tends to make this whole function positive, but after making the revision, explained in the present paper, it remains incredible, to me at least, that the neglected residue can amount to the total needed to invert the sign.

It may be worth mentioning that in revising my work I notice that $\mathfrak{A}_3[\frac{1}{3}(\sigma_2)^2 + 2\zeta_4] - \frac{1}{3}\sigma_4$ owes its negative sign to the term $-\frac{1}{3}\sigma_4$. This term arises from the energy of the double layer, called $\frac{1}{2}$DD. It comes from the portion of the term $\frac{2}{3}\pi\rho^2\int\epsilon^3(1-\lambda\epsilon)\,d\sigma$, which gives rise to a term in e^4 with a negative sign. This

* Series A, vol. 200, pp. 251–314.

term involves under the integral sign the factor $\frac{5}{2}\left(\frac{1}{\Gamma_1^2}+\frac{1}{\Delta_1^2}\right)-3G$, all the other factors being positive. If we attribute to θ and to ϕ various values between $\frac{1}{2}\pi$ and zero, we see that in part of the range the factor is positive and in other parts negative. A general inspection does not suffice to determine whether the positive portion outweighs the negative, as in fact it does. Therefore, in order to feel abundantly sure that no gross mistake had been made, I computed by quadratures the eight constituent integrals involved in the final result, and confirmed the correctness of the value found by the rigorous evaluation.

The analysis of the investigation has been carefully examined throughout, and I have, besides, applied the same method to the investigation of MACLAURIN'S spheroid, where the solution can be verified by the known exact result.*

As a further check, the formulæ of the paper on the Pear-shaped Figure have been examined on the hypothesis that the ellipsoid of reference reduces to a sphere. The several terms correctly reproduce the analogous terms in the paper on MACLAURIN'S spheroid, but in effecting the comparison it is necessary to note that the variable τ of the Pear-shaped Figure reduces to $\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)$, whereas in the paper on MACLAURIN'S spheroid the corresponding variable τ denotes $\frac{1}{3}\left(1-\frac{r^3}{a^3}\right)$, where r is radius vector and a the radius of the sphere.

Dissent from so distinguished a mathematician as M. LIAPOUNOFF is not to be undertaken lightly, and I have, as explained, taken especial pains to ensure correctness. Having made my revision, and completed the computations as set forth hereafter, I feel a conviction that the source of our disagreement will be found in some matter of principle, and not in the neglected residue of this series. I can now only express a hope that some one else will take up the question.

In the revision of the computations, the methods now used are much better than the old ones. In as far as this paper is a mere repetition of the former work with improved methods, the results will only be stated in outline, but I now show how any of the ellipsoidal harmonic functions may be computed without approximation, and how the functions of the second kind may be found rigorously.

The Cambridge University Press is now bringing out a collection of my mathematical papers, and when we come to the paper on the stability of the pear-shaped figure, the new methods of computation will be substituted for the old.

This paper is supplementary to the former one on Stability,† and it will only be intelligible in connection therewith. As before, I refer to the papers in the 'Philosophical Transactions'‡ as "Harmonics" and the "Pear-shaped Figure."

* 'Amer. Math. Soc. Trans.,' 1903, vol. 4, p. 113, on "The Approximate Determination of the Form of MACLAURIN'S Spheroid," and a further note on the same subject, recently sent to the same Society.

† 'Phil. Trans.,' A, vol. 200, p. 251.

‡ Vol. 197, A, p. 461, and vol. 198, A, p. 301.

§ 1. *The Rigorous Expression for the Ellipsoidal Harmonic Functions.*

Rigorous forms have been found in previous papers for all the harmonics of orders up to the third inclusive. For harmonics of the fourth order rigorous algebraic forms may be obtained in all cases except when $s = 0, 2, 4$, but these are exactly the cases to be considered in this investigation. We have, then, to show how these functions of higher orders may be evaluated rigorously for an ellipsoid of known ellipticity.

The only case required is that in which both i and s are even, and although all the forms might be evaluated by processes similar to those indicated below, I shall confine myself to this case.

We have seen in "Harmonics" that if β denotes $(1-\kappa^2)/(1+\kappa^2)$ of this paper, and μ denotes $\sin\theta$,

$$\mathfrak{P}_i^s(\mu) = P_i^s(\mu) - \beta q_{s+2} P_i^{s+2}(\mu) + \beta^2 q_{s+4} P_i^{s+4}(\mu) - \ldots + (-)^{\frac{1}{2}(i-s)} \beta^{\frac{1}{2}i} q_i P_i^i(\mu)$$
$$- \beta q_{s-2} P_i^{s-2}(\mu) + \beta^2 q_{s-4} P_i^{s-4}(\mu) - \ldots + (-)^{\frac{1}{2}s} \beta^{\frac{1}{2}s} q_0 P_i(\mu).$$

It is well known that

$$P_i(\mu) = \frac{1}{2^i i!} \frac{d^i}{d\mu} (\mu^2-1)^i,$$

and

$$P_i^s(\mu) = \frac{(i+s)(i+s-1)\ldots(i-s+1)}{2^s . s!} (1-\mu^2)^{\frac{1}{2}s} \left[\mu^{i-s} - \frac{(i-s)(i-s-1)}{1!\, 2^2 (s+1)} \mu^{i-s-2} (1-\mu^2) \right.$$
$$\left. + \frac{(i-s)(i-s-1)(i-s-2)(i-s-3)}{2!\, 2^4 (s+1)(s+2)} \mu^{i-s-4} (1-\mu^2)^2 - \ldots \right].$$

Hence we may clearly write $\mathfrak{P}_i^s$ in the form

$$\mathfrak{P}_i^s(\mu) = f_0 \sin^i\theta - f_2 \sin^{i-2}\theta \cos^2\theta + f_4 \sin^{i-4}\theta \cos^4\theta - \ldots.$$

Since when s is not zero $P_i^s(1) = 0$, and when s is zero $P_i(1) = 1$, it follows that $f_0 = (-)^{\frac{1}{2}s} \beta^{\frac{1}{2}s} q_0$. For the zonal harmonics ($s = 0$) this gives $f_0 = 1$. The determination of the other f's depends on that of the q's, which we shall consider later.

Another form of $\mathfrak{P}_i^s(\mu)$ will be useful, viz.:

$$\mathfrak{P}_i^s(\mu) = a - b\cos^2\theta + c\cos^4\theta - d\cos^6\theta + e\cos^8\theta - \ldots.$$

It is obvious that

$$a = f_0,$$
$$b = f_2 + \frac{i}{2\,.\,1!} f_0,$$
$$c = f_4 + \frac{i-2}{2\,.\,1!} f_2 + \frac{i(i-2)}{2^2\,.\,2!} f_0,$$
$$d = f_6 + \frac{i-4}{2\,.\,1!} f_4 + \frac{(i-2)(i-4)}{2^2\,.\,2!} f_2 + \frac{i(i-2)(i-4)}{2^3\,.\,3!} f_0,$$

&c., &c., &c.

Thus, when the f's are computed it is easy to obtain the a, b, c, d, &c.

We know that $\mathfrak{C}_i^s(\phi)$ (the cosine function of ϕ) is the same function of $-\kappa'^2 \sin^2\phi$ that $\mathfrak{P}_i^s(\mu)$ is of $\kappa^2\cos^2\theta$, except as regards a constant factor.

Hence it follows that

$$\mathfrak{C}_i^s(\phi) = \lambda\left[a + b\frac{\kappa'^2}{\kappa^2}\sin^2\phi + c\frac{\kappa'^4}{\kappa^4}\sin^4\phi + d\frac{\kappa'^6}{\kappa^6}\sin^6\phi + \ldots\right],$$

where λ is a constant factor.

Now I desire to define $\mathfrak{P}_i^s(\mu)$ and $\mathfrak{C}_i^s(\phi)$ exactly as in "Harmonics."

This definition has already been adopted as regards $\mathfrak{P}_i^s(\mu)$, but it remains to adjust the constant λ so as to attain the same end as regards $\mathfrak{C}_i^s(\phi)$.

When i and s are even, $\mathfrak{C}_i^s(\phi)$ was defined thus:

$$\mathfrak{C}_i^s(\phi) = \cos s\phi + \beta p_{s+2}\cos(s+2)\phi + \beta^2 p_{s+4}\cos(s+4)\phi + \ldots + \beta^{\frac{1}{2}(i-s)}p_i\cos i\phi$$
$$+\beta p_{s-2}\cos(s-2)\phi + \beta^2 p_{s-4}\cos(s-4)\phi + \ldots + \beta^{\frac{1}{2}s}p_0.$$

Since

$$\sin^{2r}\phi = \frac{2r\,!}{2^{2r}(r\,!)^2} - \frac{2r\,!}{2^{2r-1}(r-1)\,!\,(r+1)\,!}\cos 2\phi + \frac{2r\,!}{2^{2r-1}(r-2)\,!\,(r+2)\,!}\cos 4\phi - \ldots,$$

it follows that the term independent of ϕ in $\mathfrak{C}_i(\phi)$ is

$$\lambda\left[a + \tfrac{1}{2}b\frac{\kappa'^2}{\kappa^2} + \frac{1\,.\,3}{2\,.\,4}c\frac{\kappa'^4}{\kappa^4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}d\frac{\kappa'^6}{\kappa^6} + \ldots\right].$$

The term in $\cos 2\phi$ in $\mathfrak{C}_i^2(\phi)$ is

$$-2\lambda\cos 2\phi\left[\frac{1}{2}\cdot\frac{1}{2}b\frac{\kappa'^2}{\kappa^2} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{2}{3}c\frac{\kappa'^4}{\kappa^4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{3}{4}d\frac{\kappa'^6}{\kappa^6} + \ldots\right].$$

The term in $\cos 4\phi$ in $\mathfrak{C}_i^4(\phi)$ is

$$2\lambda\cos 4\phi\left[\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1\,.\,2}{3\,.\,4}c\frac{\kappa'^4}{\kappa^4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{2\,.\,3}{4\,.\,5}d\frac{\kappa'^6}{\kappa^6} + \frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}\cdot\frac{3\,.\,4}{5\,.\,6}e\frac{\kappa'^8}{\kappa^8} + \ldots\right],$$

and so forth.

In accordance with the definition to be adopted, these terms in the three cases respectively are: 1, $\cos 2\phi$, $\cos 4\phi$. Hence λ must be chosen so as to fulfil that condition.

Pursuing only the case of $\mathfrak{C}_i(\phi)$ in detail, we have

$$\frac{1}{\lambda} = a + \tfrac{1}{2}b\frac{\kappa'^2}{\kappa^2} + \frac{1\,.\,3}{2\,.\,4}c\frac{\kappa'^4}{\kappa^4} + \ldots.$$

If, then,

$$\mathfrak{C}_i(\phi) = a' + b' \sin^2 \phi + c' \sin^4 \phi + \dots,$$

we must have

$$a' = \lambda a, \quad b' = \lambda b \frac{\kappa'^2}{\kappa^2}, \quad c' = \lambda c \frac{\kappa'^4}{\kappa^4}, \quad \text{\&c.}$$

Thus when f_0, f_2, f_4, &c., are found, it is easy to compute a, b, c, &c., and a', b', c', &c.

Our formulæ tend to involve the differences between large numbers, and this defect becomes more pronounced as the order of harmonics increases. The fault is mitigated by using the forms

$$\mathfrak{P}_i(\mu) = f_0 \sin^i \theta - f_2 \sin^{i-2} \theta \cos^2 \theta + \dots,$$

$$\mathfrak{C}_i(\phi) = a' + b' \sin^2 \phi + c' \sin^4 \phi + \dots.$$

In the case of a lower harmonic, however, such as the fourth, we may just as well use the form for $\mathfrak{P}$ involving a, b, c, &c., and powers of $\cos^2 \theta$.

We must now show how to complete the evaluation of the f's for the zonal harmonics.

It appears, from p. 486 of "Harmonics," that, when i is even, we have to solve the equation

$$\beta\sigma = \frac{\frac{1}{2}\beta^2 \{i, 1\}\{i, 2\}}{4 \,.\, 1^2 + \beta\sigma -} \; \frac{\frac{1}{4}\beta^2 \{i, 3\}\{i, 4\}}{4 \,.\, 2^2 + \beta\sigma - \dots},$$

ending with

$$\frac{-\frac{1}{4}\beta^2 \{i, i\}\{i, i-1\}}{i^2 + \beta\sigma}, \quad \text{where} \quad \{i, j\} = (i+j)(i-j+1).$$

We are to take that root which vanishes when β vanishes.

Although the equation for $\beta\sigma$ is of order $i-1$, yet at least for such an ellipsoid as we have to deal with, it is very easy to solve it by successive rapid approximations.

It is clear that we may write the equation in the form

$$(\beta\sigma)^2 + \left[4 - \frac{\frac{1}{4}\beta^2 \{i, 3\}\{i, 4\}}{4 \,.\, 2^2 + \beta\sigma -} \; \frac{\frac{1}{4}\beta^2 \{i, 5\}\{i, 6\}}{4 \,.\, 3^2 + \beta\sigma - \dots}\right] \beta\sigma = \tfrac{1}{2}\beta^2 \{i, 1\}\{i, 2\}.$$

An analytical approximation is found by neglecting the continued fraction in the second term on the left, and we then obtain

$$\beta\sigma = -2 + 2\sqrt{[1 + \tfrac{1}{8}\beta^2 (i-1)\, i\, (i+1)(i+2)]}.$$

If this value of $\beta\sigma$ is used in computing the first term of the continued fraction, and if the quadratic is solved again, we obtain a closer approximation. We then use the second approximation and include one more term in the continued fraction, and proceed until $\beta\sigma$ no longer changes.

It is shown on pp. 486, 487 of "Harmonics" that

$$\frac{q_2}{q_0} = \frac{1}{4\,.\,1^2+\beta\sigma-}\;\frac{\frac{1}{4}\beta^2\{i,3\}\{i,4\}}{4\,.\,2^2+\beta\sigma-\ldots},$$

$$\frac{2q_4}{q_2} = \frac{1}{4\,.\,2^2+\beta\sigma-}\;\frac{\frac{1}{4}\beta^2\{i,5\}\{i,6\}}{4\,.\,3^2+\beta\sigma-\ldots},$$

$$\frac{2q_6}{q_4} = \frac{1}{4\,.\,3^2+\beta\sigma-}\;\frac{\frac{1}{4}\beta^2\{i,7\}\{i,8\}}{4\,.\,4^2+\beta\sigma-\ldots},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

It may be remarked that the factor 2 occurs in each of these equations on the left, excepting in the first one; also we are to take $q_0 = 1$.

In the course of the successive approximations for the determination of $\beta\sigma$, each of these fractions is naturally evaluated. Therefore it is only necessary to extract certain numerical values already found in the course of solving the equation for $\beta\sigma$.

As a verification, which shows whether the equation has been correctly solved, we have

$$\frac{q_0}{q_2} = \frac{\frac{1}{2}\beta^2\{i,1\}\{i,2\}}{\beta\sigma}.$$

It is now obvious that we are able to find all the q's in terms of q_0, which is unity. We then multiply each q by its appropriate power of β or $\frac{1-\kappa^2}{1+\kappa^2}$, that is to say, we form $\beta^r q_{2r}$ for $r = 1, 2, \ldots, \frac{1}{2}i$, and introduce the results into the formula for $\mathfrak{P}_i(\mu)$.

A closely analogous method enables us to find all the other types of function for an ellipsoid of known ellipticities, but, except for certain harmonics of the fourth order, it is not possible to obtain rigorous analytical solutions. Approximate analytical forms are given in "Harmonics," and the approximation may be carried further if desired.

The following tables give the coefficients in the several functions for the critical Jacobian ellipsoid with which we are dealing:—

i.	*s.*	*a.*	*b.*	*c.*	*d.*	*e.*	*f.*
*2	0	0·603374	0·923128				
*2	2	−0·039203	0·923128				
4	0	1·000000	5·442161	4·892138			
4	2	1·769147	−36·154264	−44·93584			
4	4	0·083965	− 7·984389	95·562431			
6	0	1·000000	12·45814	29·55340	18·53561		
†6	2	−8·4	121·8	439·425	320·523		
†6	4	3·78	−338·312	3680·303	4482·844		
8	0	1·00000	23·29297	103·90805	155·9554	74·7977	
10	0	1·00000	38·29978	274·94458	721·88640	789·90216	306·12784

i.	*s.*	*a′.*	*b′.*	*c′.*	*d′.*	*e′.*	*f′.*
*2	0	0·603374	0·076872				
*2	2	−0·039203	0·076872				
4	0	0·806905	0·365661	0·027371			
4	2	1·065020	−1·812415	−0·187586			
4	4	1·013640	−8·026680	8·000000			
6	0	0·62544	0·64882	0·12816	0·00669		
†6	2	−1·1408	1·5349	0·4404	0·0264		
†6	4	1·0305	−7·704	6·944	0·704		
8	0	0·440664	0·854891	0·317488	0·396793	0·001585	
10	0	0·289818	0·924288	0·552512	0·120795	0·011006	0·000355

* The second harmonics are here defined by $\mathfrak{P}_2^s(\mu) = \kappa^2 - q_s^2 - \kappa^2 \cos^2\theta$, $\mathfrak{C}_2^s = \kappa^2 - q_s^2 + \kappa'^2 \sin^2\phi$ $(s = 0, 2)$ with $q_s^2 = \frac{1}{3}[1 + \kappa^2 \mp \sqrt{(1 - \kappa^2\kappa'^2)}]$, with the upper sign for $s = 0$ and the lower for $s = 2$. In the case of $i = 4$, $s = 2$, I had in the original paper inadvertently changed the sign *both* of $\mathfrak{P}_4^2$ and $\mathfrak{C}_4^2$ without, of course, introducing any error, since they occur always as a product.

† These functions are only given in their approximate forms.

As it is desirable to use the other form of $\mathfrak{P}$ in the higher zonal harmonics, I give the coefficients f_0, f_2, f_4, &c., in these cases. It will be noticed how much smaller are the numbers involved.

COEFFICIENTS of Terms in $\mathfrak{P}_i(\mu)$ when expressed in Sines and Cosines.

i.	s.	f_0.	f_2.	f_4.	f_6.	f_8.	f_{10}.
6	0	1·00000	9·45814	7·63713	0·44035		
8	0	1·00000	19·29797	40·01416	14·03323	0·45235	
10	0	1·00000	33·29978	131·74546	116·85134	22·76398	0·46728

§ 2. *The Rigorous Expression for the Harmonics of the Second Kind.*

The integral $\mathfrak{A}_i^s$ denotes $\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0)$, and $\mathfrak{B}_i^s$ denotes $\mathfrak{Q}_i^s(\nu_0)\dfrac{d}{d\nu_0}\mathfrak{P}_i^s(\nu_0)$. Thus $\mathfrak{A}_i^s$ is, in fact, the harmonic function of the second kind. $\mathfrak{B}_i^s$ is clearly determinable from $\mathfrak{A}_i^s$.

In the original paper $\mathfrak{A}_i^s$ was found by quadrature, and this defect in my procedure is referred to by M. LIAPOUNOFF as a cause possibly contributory to the discrepancy between our results. Quadrature was not, perhaps, a very satisfactory method, and the defect will now be made good by finding these integrals in terms of the F and E elliptic integrals. It will appear that my former results were sufficiently near to the truth for practical purposes.

The functions $\mathfrak{P}_i^s(\nu)$ or $\mathsf{P}_i^s(\nu)$ are of eight types, determined by the oddness or evenness of i and s, and the association with a cosine or sine function of ϕ. In "Harmonics" the types are indicated by combinations in groups of three of the four letters E, O, C, S—denoting Even, Odd, Cosine, Sine; for example, OES means i odd, s even, associated with a sine function.

All the roots of the equation $\mathfrak{P}_i^s(\nu)$ or $\mathsf{P}_i^s(\nu) = 0$ are real, and when the form of the function has been determined by the method of § 1 the equation may be solved. Hence these functions are expressible as the products of a number of factors; and it is to be noted that it is not necessary to adopt the same definition as in "Harmonics," because the function may be multiplied by any constant factor, without affecting the result.

For brevity, let

$$\prod_1^n(\kappa^2\nu^2-q_x^{\,2}) = (\kappa^2\nu^2-q_1^{\,2})(\kappa^2\nu^2-q_2^{\,2})\ldots(\kappa^2\nu^2-q_n^{\,2}),$$

where κ^2 is $(1-\beta)/(1+\beta)$ of "Harmonics." The parameters, κ and γ, as elsewhere, define the form of the ellipsoid.

An alternative notation will be needed, in which we write

$$\nu^2 = \frac{1}{\kappa^2\sin^2\psi}, \quad \Delta_x^{\,2} = 1-q_x^{\,2}\sin^2\psi, \quad \Delta^2 = 1-\kappa^2\sin^2\psi.$$

At the surface of the ellipsoid $\nu = \nu_0$, $\psi = \gamma$, and we shall, as before, write $\sin\beta = \kappa \sin\gamma$. At the surface of the ellipsoid we have then

$$\nu = \operatorname{cosec}\beta, \quad \Delta_x^2 = 1 - q_x^2 \sin^2\gamma, \quad \Delta^2 = \cos^2\beta.$$

In this notation

$$\prod_1^n \left(\frac{\Delta_x^2}{\sin^2\psi}\right) = \frac{\Delta_1^2 \Delta_2^2 \ldots \Delta_n^2}{\sin^{2n}\psi}.$$

A consideration of the eight types of harmonics shows that they may be written as follows:—

Type.

$$\begin{array}{llcl}
\text{EEC,} & \mathfrak{P}_{2n}{}^{2t}(\nu) = & \prod_1^n (\kappa^2\nu^2 - q_x^2) & = \operatorname{cosec}^{2n}\psi \prod_1^n (\Delta_x^2), \\
\text{EES,} & \mathsf{P}_{2n+2}{}^{2t}(\nu) = & (\kappa^2\nu^2-\kappa^2)^{\frac{1}{2}} (\kappa^2\nu^2-1)^{\frac{1}{2}} \prod_1^n (\kappa^2\nu^2 - q_x^2) & = \operatorname{cosec}^{2n+2}\psi\, \Delta \cos\psi \prod_1^n (\Delta_x^2), \\
\text{OOC,} & \mathsf{P}_{2n+1}{}^{2t+1}(\nu) = & (\kappa^2\nu^2-1)^{\frac{1}{2}} \prod_1^n (\kappa^2\nu^2 - q_x^2) & = \operatorname{cosec}^{2n+1}\psi \cos\psi \prod_1^n (\Delta_x^2), \\
\text{OOS,} & \mathfrak{P}_{2n+1}{}^{2t+1}(\nu) = & (\kappa^2\nu^2-\kappa^2)^{\frac{1}{2}} \prod_1^n (\kappa^2\nu^2 - q_x^2) & = \operatorname{cosec}^{2n+1}\psi\, \Delta \prod_1^n (\Delta_x^2), \\
\text{OEC,} & \mathfrak{P}_{2n+1}{}^{2t}(\nu) = & \kappa\nu \prod_1^n (\kappa^2\nu^2 - q_x^2) & = \operatorname{cosec}^{2n+1}\psi \prod_1^n (\Delta_x^2), \\
\text{OES,} & \mathsf{P}_{2n+3}{}^{2t}(\nu) = & \kappa\nu (\kappa^2\nu^2-\kappa^2)^{\frac{1}{2}} (\kappa^2\nu^2-1)^{\frac{1}{2}} \prod_1^n (\kappa^2\nu^2 - q_x^2) & = \operatorname{cosec}^{2n+3}\psi\, \Delta \cos\psi \prod_1^n (\Delta_x^2), \\
\text{EOC,} & \mathsf{P}_{2n+2}{}^{2t+1}(\nu) = & \kappa\nu (\kappa^2\nu^2-1)^{\frac{1}{2}} \prod_1^n (\kappa^2\nu^2 - q_x^2) & = \operatorname{cosec}^{2n+2}\psi \cos\psi \prod_1^n (\Delta_x^2), \\
\text{EOS,} & \mathfrak{P}_{2n+2}{}^{2t+1}(\nu) = & \kappa\nu (\kappa^2\nu^2-\kappa^2)^{\frac{1}{2}} \prod_1^n (\kappa^2\nu^2 - q_x^2) & = \operatorname{cosec}^{2n+2}\psi\, \Delta \prod_1^n (\Delta_x^2).
\end{array}$$

Using $\mathfrak{P}$ and $\mathfrak{Q}$ generically for any one of these and for the corresponding function of the other kind, we have

$$\mathfrak{P}(\nu_0)\,\mathfrak{Q}(\nu_0) = [\mathfrak{P}(\nu_0)]^2 \int_{\nu_0} \frac{d\nu}{[\mathfrak{P}(\nu)]^2 (\nu^2-1)^{\frac{1}{2}} (\nu^2 - 1/\kappa^2)^{\frac{1}{2}}}.$$

Or, changing the variable of integration to ψ,

$$\mathfrak{A} = \kappa [\mathfrak{P}(\nu_0)]^2 \int_0^\gamma \frac{d\psi}{[\mathfrak{P}(\nu)]^2 \Delta}.$$

To effect the integration the reciprocal of the square of $\mathfrak{P}$ must be expressed in partial fractions. Inspection of the eight forms of functions shows that

$$\frac{1}{[\mathfrak{P}(\nu)]^2} = \frac{1}{f}\cdot\frac{1}{\kappa^2\nu^2-\kappa^2} + \frac{1}{g}\cdot\frac{1}{\kappa^2\nu^2-1} + \frac{1}{h}\cdot\frac{1}{\kappa^2\nu^2} + \sum_1^n \frac{1}{A_x}\left[\frac{1}{(\kappa^2\nu^2-q_x^2)^2} - \frac{B_x}{\kappa^2\nu^2-q_x^2}\right],$$

with appropriate values of f, g, h, A_x, B_x to be given hereafter.

In every case but that of OES some or all of f, g, h are infinite.

In terms of ψ

$$\frac{1}{[\mathfrak{P}(\nu)]^2} = \frac{1}{f}\frac{\sin^2\psi}{\Delta^2} + \frac{1}{g}\tan^2\psi + \frac{1}{h}\sin^2\psi + \sum_1^n \frac{1}{A_x}\left[\frac{\sin^4\psi}{\Delta_x^4} - \frac{B_x\sin^2\psi}{\Delta_x^2}\right].$$

This has to be divided by Δ and integrated, and the result will be expressible in terms of the elliptic integrals

$$F(\gamma) = \int_0^\gamma \frac{d\psi}{\Delta}, \quad E(\gamma) = \int_0^\gamma \Delta\, d\psi, \quad \Pi(\gamma, \Delta_x) = \int_0^\gamma \frac{d\psi}{\Delta_x^2\Delta}.$$

Accordingly we require certain integrals, which are given on p. 313 of the "Pear-shaped Figure," but in somewhat different forms. Here and elsewhere κ'^2 denotes $1-\kappa^2$, and q'^2_x denotes $1-q_x^2$.

The integrals needed are as follows :—

$$\left.\begin{aligned}
\int_0^\gamma \frac{\sin^2\psi}{\Delta^3}\,d\psi &= -\frac{1}{\kappa^2}\mathrm{F}(\gamma) + \frac{1}{\kappa^2\kappa'^2}\mathrm{E}(\gamma) - \frac{\sin\gamma\cos\gamma}{\kappa'^2\cos\beta} \\
\int_0^\gamma \frac{\tan^2\psi}{\Delta}\,d\psi &= -\frac{1}{\kappa'^2}\mathrm{E}(\gamma) + \frac{\tan\gamma\cos\beta}{\kappa'^2} \\
\int_0^\gamma \frac{\sin^2\psi}{\Delta}\,d\psi &= \frac{1}{\kappa^2}\mathrm{F}(\gamma) - \frac{1}{\kappa^2}\mathrm{E}(\gamma)
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

$$\int_0^\gamma \frac{\sin^2\psi}{\Delta_x^2\Delta}\,d\psi = \frac{1}{q_x^2}\Pi(\gamma, \Delta_x) - \frac{1}{q_x^2}\mathrm{F}(\gamma)$$

$$\int_0^\gamma \frac{\sin^4\psi}{\Delta_x^4\Delta}\,d\psi = \frac{1}{2q_x^2}\left(\frac{1}{q'^2_x} - \frac{1}{q_x^2} + \frac{1}{\kappa^2-q_x^2}\right)\Pi(\gamma, \Delta_x) + \frac{1}{2q_x^2}\left(\frac{1}{q_x^2} - \frac{1}{q'^2_x}\right)\mathrm{F}(\gamma)$$
$$- \frac{1}{2q_x^2 q'^2_x(\kappa^2-q_x^2)}\mathrm{E}(\gamma) + \frac{\sin\gamma\cos\gamma\cos\beta}{2q'^2_x(\kappa^2-q_x^2)\Delta_x^2}.$$

The last two of these admit of considerable simplification, as will now be shown.

It is proved in the "Pear-shaped Figure" that the $\Pi(\gamma, \Delta_x)$ elliptic integral disappears from the expression for $\mathfrak{PQ}$ for all cases up to the third harmonics inclusive. I have also proved numerically that for the Jacobian ellipsoid the like is true for all the even zonal harmonics up to the tenth inclusive, and for the tesseral harmonics $i = 4$, $s = 2$ and $i = 4$, $s = 4$. It is thus certainly true in all cases used by me, and I do not care to spend perhaps much time in proving algebraically the general truth of the law.

The last two of our integrals will occur in the form

$$\int_0^\gamma \frac{\sin^4\psi}{\Delta_x^4\Delta}\,d\psi - B_x\int_0^\gamma \frac{\sin^2\psi}{\Delta_x^2\Delta}\,d\psi,$$

and I *assume* that the coefficient of $\Pi(\gamma, \Delta_x)$ in this expression always vanishes, as is *proved* to be true in all cases actually computed.

Hence

$$2B_x = \frac{1}{q'^2_x} - \frac{1}{q_x^2} + \frac{1}{\kappa^2 - q_x^2}.$$

Now the coefficient of F (γ) in this same combination of integrals is

$$\frac{1}{2q_x^2}\left[\frac{1}{q_x^2} - \frac{1}{q'^2_x} + 2B_x\right].$$

In this we may substitute for B_x its value, and thus find

$$\int_0^\gamma \frac{\sin^4\psi}{\Delta_x^4\Delta}\,d\psi - B_x\int_0^\gamma \frac{\sin^2\gamma}{\Delta_x^2\Delta}\,d\psi = \frac{1}{2q_x^2(\kappa^2-q_x^2)}\,\mathrm{F}(\gamma) - \frac{1}{2q_x^2q'^2_x(\kappa^2-q_x^2)}\,\mathrm{E}(\gamma) + \frac{\sin\gamma\cos\gamma\cos\beta}{2q'^2_x(\kappa^2-q_x^2)\,\Delta_x^2}.$$

This expression together with (1) give all the required integrals, and it only remains to tabulate f, g, h, A_x for the several types of function.

For the sake of brevity I write

$$C_x = (q_x^2 - q_1^2)^2\,(q_x^2-q_2^2)^2 \ldots (q_x^2-q_n^2)^2 \qquad (x = 1, 2 \ldots n),$$

the factor which would vanish being in each case omitted.

When there is only one q, C_1 is to be interpreted as unity.

TABLE of Values of f, g, h, and A_x.

Type.	i order of harmonic.	s rank of harmonic.	f.	g.	h.	A_x.
EEC	$2n$	$2t$	∞	∞	∞	C_x
EES	$2n+2$	$2t$	$-\kappa'^2\prod_1^n(\kappa^2-q_x^2)^2$	$\kappa'^2\prod_1^n q'^4_x$	∞	$q'^2_x(\kappa^2-q_x^2)\,C_x$
OOC	$2n+1$	$2t+1$	∞	$\prod_1^n q'^4_x$	∞	$-q'^2_x C_x$
OOS	$2n+1$	$2t+1$	$\kappa'^2\prod_1^n(\kappa^2-q_x^2)^2$	∞	∞	$-(\kappa^2-q_x^2)\,C_x$
OEC	$2n+1$	$2t$	∞	∞	$\prod_1^n q_x^4$	$q_x^2 C_x$
OES	$2n+3$	$2t$	$-\kappa^2\kappa'^2\prod_1^n(\kappa^2-q_x^2)^2$	$\kappa'^2\prod_1^n q'^4_x$	$\kappa^2\prod_1^n q_x^4$	$q_x^2q'^2_x(\kappa^2-q_x^2)\,C_x$
EOC	$2n+2$	$2t+1$	∞	$\prod_1^n q'^4_x$	$-\prod_1^n q_x^4$	$-q_x^2q'^2_x C_x$
EOS	$2n+2$	$2t+1$	$\kappa^2\prod_1^n(\kappa^2-q_x^2)^2$	∞	$-\kappa^2\prod_1^n q_x^4$	$-q_x^2(\kappa^2-q_x^2)\,C_x$

In all the types

$$B_x = 2\left[\frac{1}{q_x^2-q_1^2}+\frac{1}{q_x^2-q_2^2}+\ldots+\frac{1}{q_x^2-q_n^2}\right],$$

with the omission of the term which would be infinite.

We have generally $\mathfrak{B} = \frac{\mathfrak{A}}{\mathfrak{P}(\nu_0)}\frac{d}{d\nu_0}\mathfrak{P}(\nu_0)$. Hence by logarithmic differentiation of the expressions for the several types of $\mathfrak{P}$ we find results given in the following table :—

TABLE of the $\mathfrak{B}$ Integrals.

Type.	i order of harmonic.	s rank of harmonic.	$\mathfrak{B}_i^s \div 2 \sin \beta \mathfrak{A}_i^s$.
EEC	$2n$	$2t$	$\sum_1^n 1/\Delta_x^2$
EES	$2n+2$	$2t$	$\frac{1}{2}\sec^2\beta+\frac{1}{2}\sec^2\gamma+\sum_1^n 1/\Delta_x^2$
OOC	$2n+1$	$2t+1$	$\frac{1}{2}\sec^2\gamma+\sum_1^n 1/\Delta_x^2$
OOS	$2n+1$	$2t+1$	$\frac{1}{2}\sec^2\beta+\sum_1^n 1/\Delta_x^2$
OEC	$2n+1$	$2t$	$\frac{1}{2}+\sum_1^n 1/\Delta_x^2$
OES	$2n+3$	$2t$	$\frac{1}{2}+\frac{1}{2}\sec^2\beta+\frac{1}{2}\sec^2\gamma+\sum_1^n 1/\Delta_x^2$
EOC	$2n+2$	$2t+1$	$\frac{1}{2}+\frac{1}{2}\sec^2\gamma+\sum_1^n 1/\Delta_x^2$
EOS	$2n+2$	$2t+1$	$\frac{1}{2}+\frac{1}{2}\sec^2\beta+\sum_1^n 1/\Delta_x^2$

In the case of the zonal harmonics ($s = 0$), q_x/κ is always less than unity; for harmonics of rank 2 one of the q_x/κ is greater than unity and the rest are less; for rank 4 two of them are greater than unity and the rest less.

For the zonal harmonics there is some gain in simplicity by putting $\sin^2\theta_x = \frac{q_x^2}{\kappa^2}$. We then take the equation

$$\mathfrak{P}_{2n}(\mu) = a - b\cos^2\theta + c\cos^4\theta - \ldots = 0,$$

and find all the n roots, say $\theta_1, \theta_2 \ldots \theta_n$.

If we solve the corresponding equation $\mathfrak{P}_{2n}^2(\mu) = 0$ for the tesseral harmonic of rank 2, we find one root for $\cos^2\theta$ to be negative. If this root corresponds to

θ_1, we must put $\cos^2\theta_1 = 1-\frac{q_1^2}{\kappa^2}$, so that $q_1^2 = \kappa^2[1+(-\cos^2\theta_1)]$. Similarly for the harmonics of rank 4, two roots correspond with imaginary angles, and so forth.

Subject to this explanation we may now regard the roots as defined by $\theta_1, \theta_2 \ldots \theta_n$.

Since $\Delta_x^2 = 1 - q_x^2 \sin^2\gamma$, we have $\Delta_x^2 = 1-\kappa^2\sin^2\gamma\sin^2\theta_x = 1-\sin^2\beta\sin^2\theta_x$, and $\mathfrak{P}_{2n}^{2t}(\mu) = \operatorname{cosec}^{2n}\gamma\,\Delta_1^2\Delta_2^2\ldots\Delta_n^2$.

If we write

$$D_x = (\sin^2\theta_x - \sin^2\theta_1)^2(\sin^2\theta_x-\sin^2\theta_2)^2\ldots(\sin^2\theta_x-\sin^2\theta_n)^2 \quad (n-1 \text{ factors}),$$

our former C_x may be written in the form $\kappa^{4n-4}D_x$, and the several coefficients in the expression for $\mathfrak{A}_i^s$ may be expressed as trigonometrical functions—some of which may, however, be hyperbolic.

We thus have

$$\begin{aligned}\mathfrak{A}_{2n}^{2t} = \frac{1}{2\kappa^{4n-1}}\prod_1^n(1-\sin^2\beta\sin^2\theta_x)^2\Bigg[\mathrm{F}(\gamma)\sum_1^n\frac{1}{D_x\sin^2\theta_x\cos^2\theta_x}\\ -\mathrm{E}(\gamma)\sum_1^n\frac{1}{D_x\sin^2\theta_x\cos^2\theta_x(1-\kappa^2\sin^2\theta_x)}\\ +\kappa^2\sin\gamma\cos\gamma\cos\beta\sum_1^n\frac{1}{D_x\cos^2\theta_x(1-\kappa^2\sin^2\theta_x)(1-\sin^2\beta\sin^2\theta_x)}\Bigg].\end{aligned}$$

This formula agrees with the result given for $\mathfrak{A}_2^s$ ($s = 0, 2$) in § 4 of "The Pear-shaped Figure," although the formula is there expressed in terms of q^2, and $1/(\kappa^2-q^2)$ is replaced by its equivalent $(1-2q^2)/q^2q'^2$.

In the case of the even zonal harmonics of order i, all the θ's are real angles, and it facilitates the solution of the equation for θ to note that, with rough approximation (improving as the order of harmonic increases),

$$\theta_1 = \frac{\pi}{2i}, \quad \theta_2 = \frac{3\pi}{2i}, \quad \theta_3 = \frac{5\pi}{2i}, \ldots, \quad \theta_{\frac{1}{2}i} = \frac{(i-1)\pi}{2i}.$$

The following numerical values apply to the critical Jacobian ellipsoid:—

For the fourth harmonic $\theta_1 = 20°\ 15'$, $\theta_2 = 61°\ 11'$; the rough approximation gives $22°\ 30'$ and $67°\ 30'$.

For the sixth $\theta_1 = 14°\ 1'\cdot9$, $\theta_2 = 42°\ 12'\cdot2$, $\theta_3 = 71°\ 8'\cdot6$; the rough approximation being $15°$, $45°$, $75°$.

For the eighth $\theta_1 = 10°\ 43'\cdot1$, $\theta_2 = 32°\ 11'\cdot8$, $\theta_3 = 53°\ 51'\cdot2$, $\theta_4 = 76°\ 21'\cdot8$; the approximation being $11°\ 45'$, $34°\ 15'$, $56°\ 45'$, $79°\ 15'$.

For the tenth $\theta_1 = 8°\ 40'$, $\theta_2 = 26°\ 1'$, $\theta_3 = 43°\ 26'$, $\theta_4 = 61°\ 3'$, $\theta_5 = 79°\ 28'$; the approximation being $9°$, $27°$, $45°$, $63°$, $81°$.

The values of the several $\mathfrak{A}$'s found by quadratures were in every case too small;

the correct values are given in the table below. I find that for $\mathfrak{A}_4$ quadratures gave too small a value by a $\frac{1}{300}$th part; for $\mathfrak{A}_6$ by a $\frac{1}{140}$th part; for $\mathfrak{A}_8$ by an $\frac{1}{87}$th part.

The method which I have given above fails for the tenth zonal harmonic, unless we use logarithms of more than seven places; and it is not worth while to undertake so heavy a piece of computation. I conclude by extrapolation that for $\mathfrak{A}_{10}$ quadrature (carried out on exactly the same plan as in all the other cases) gives too small a result by a $\frac{1}{70}$th part of itself. I therefore augment in this case the result of the quadratures and find $\mathfrak{A}_{10} = 0{\cdot}11640$; this enables us also to compute $\mathfrak{B}_{10}$.

The following table gives the results of the whole computation*:—

TABLE of Logarithms of $\mathfrak{A}_i^s$ and $\mathfrak{B}_i^s$ Integrals.

i.	*s*.	$\log \mathfrak{A}_i^s + 10$.	$\log \mathfrak{B}_i^s$.
2	0	9·6931231	·0929494
2	2	9·3330037	·4066504
3	0	9·54617	·20462
4	0	9·4332383	·2657402
4	2	9·24250	·39502
4	4	9·04753	·43121
6	0	9·2701270	·3263106
6	2	9·14462	·39512
6	4	9·00632	·42458
8	0	9·15835	·35745
10	0	9·06595	·36897

§ 3. *Note on* § 15. "*The Determination of certain Integrals.*"

It has been found best to make some changes in this part of the work.

The integrals to be evaluated are denoted

$$ {}^{2p}\Lambda_{2m}{}^{2n} = \int_0^{\frac{1}{2}\pi} \frac{\sin^{2p}\theta \cos^{2n}\theta}{\Delta_1{}^{2m}\Delta}\, d\theta, $$

$$ \Omega_{2m}{}^{2n} = \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n}\phi}{\Gamma_1{}^{2m}\Gamma}\, d\phi. $$

The integral ${}^{2p}\Lambda_{2m}{}^{2n}$ may be made to depend on ${}^{0}\Lambda_{2m}{}^{2n}$ (which is the same as $\Lambda_{2m}{}^{2n}$ of the original paper), and therefore I only evaluate the latter.

These integrals were originally found as the differences of certain other functions, but it is not hard to give formulæ for finding them directly. I have done this and recomputed the whole series of values.†

* The only error of any moment which I have found in my previous work is that in some of the cases I had forgotten to introduce the factor κ in some of the $\mathfrak{A}$'s after effecting the quadratures. The error in my results from this oversight was fortunately not serious.

† There is a misprint on p. 286. The function $U_6{}^0$ should be

$$ \frac{3\pi}{16 \cos\beta \cos\gamma} [\sec^4\beta + \sec^4\gamma + \tfrac{2}{3}\sec^2\beta\sec^2\gamma]. $$

The series of values ${}^0\Lambda_0^{2n}$ were also computed independently from the series given on p. 537 of "Harmonics."

A little consideration will show that the differences of the series of functions ${}^0\Lambda_{2m}^{2n}$, with the signs of the odd differences changed, gives the series of functions ${}^{2p}\Lambda_{2m}^{2n}$. Stated analytically

$$(-)^r \Delta^r \, {}^0\Lambda_{2m}^{2p-2r} = {}^{2r}\Lambda_{2m}^{2p-2r}.$$

A table of the natural numbers ${}^0\Lambda_{2m}^{2n}$ is given, but the table of differences is not reproduced.

The following is the result of the recomputation :—

TABLE of the Λ Functions (Natural Numbers).

n.	${}^0\Lambda_0^n$.	${}^0\Lambda_2^n$.	${}^0\Lambda_4^n$.	${}^0\Lambda_6^n$.
0	2·7024906	8·034600	30·53878	132·38251
2	·9505345	1·4779482	2·866414	7·149917
4	·6559354	·8294118	1·1590804	1·8826818
6	·5285432	·6160958	·7537022	·9929042
8	·4543269	·5084364	·5844941	·6987687
10	·4044492	·4419004	·4909693	·5582507
12	·3680175	·3958484	·4306317	·4755180
14	·3399181	·3616261	·3878603	·4203234
16	·3173978	·3349328	·3556021	
18	·2988272	·3133708	·3295135	
20	·2831734			
22	·2697454			

TABLE of Logarithms of the Ω Functions.

n.	$\log \Omega_0^n$.	$\log \Omega_2^n$.	$\log \Omega^n_4$.	$\log \Omega_6^n$.
0	·2047610	1·0302912	1·8667641	2·7138144
2	9·8993673	·7715375	1·6492558	2·5319084
4	9·7729862	·6613000	1·5528790	2·4473549
6	9·6930884	·5897701	1·4886134	2·3893880
8	9·6346685	·5365117	1·4398970	2·3446714
10	9·5886267	·4939849	1·4005014	2·3080689
12	9·5506357	·4585472	1·3673618	2·2769972
14	9·5182995	·4281522	1·3387296	2·2499642
16	9·4901531	·4015325	1·3135070	
18	9·4652355	·3778480	1·2909612	
20	9·4428427	·3565096	1·2705764	
22	9·4226147			

The results in the original paper were not so accurate as I had thought they were. There was a mistake in the differences which give the Ω_6 series, affecting the values from $n = 10$ onwards, but as no use was made of the Ω_6 series as published, the mistake did not affect the final result.

§ 4. *Note on* §§ 16, 17. *The Integrals* σ_2, σ_4, ζ_4 *and* ω_i^s, ρ_i^s, ϕ_i^s.

An improvement has been made in the method of computing all these. The functions to be integrated were written in every case with a common factor $\operatorname{cosec}^2\gamma\,(\Delta_1^2-\Gamma_1^2)$; now this is equal to $\kappa^2\cos^2\theta+\kappa'^2\sin^2\phi$. In consequence of the substitution of this value for the common factor we are able to obtain the result as the sum of, instead of the difference between, two numbers. Another consequence is that we can dispense with the series of functions denoted Λ_{-2} and Ω_{-2}.

A single example of the way in which this change is applied will suffice. If we write

$$\left.\begin{aligned} f(\Lambda_{2n}^0) &= \alpha\,{}^0\Lambda_{2n}^0-\beta\,{}^0\Lambda_{2n}^2+\gamma\,{}^0\Lambda_{2n}^4-\delta\,{}^0\Lambda_{2n}^6 \\ f(\Lambda_{2n}^2) &= \alpha\,{}^0\Lambda_{2n}^2-\beta\,{}^0\Lambda_{2n}^4+\gamma\,{}^0\Lambda_{2n}^6-\delta\,{}^0\Lambda_{2n}^8 \end{aligned}\right\} n = 1,\ 2,$$

and denote by $f(\Omega_{2n}^0)$, $f(\Omega_{2n}^2)$, corresponding functions with α', β', γ', δ' for α, β, γ, δ and Ω in place of ${}^0\Lambda$; it is easy to show that

$$\sigma_2 = \frac{6}{\pi}\cos^2\beta\cos^2\gamma\,\{\kappa^2[f(\Lambda_2^2)f(\Omega_4^0)+f(\Lambda_4^2)f(\Omega_2^0)-Gf(\Lambda_2^2)f(\Omega_2^0)] \\ +\kappa'^2[f(\Lambda_2^0)f(\Omega_4^2)+f(\Lambda_4^0)f(\Omega_2^2)-Gf(\Lambda_2^0)f(\Omega_2^2)]\}.$$

The computations as revised gave

$$\sigma_2 = \cdot 0136760, \qquad \zeta_4 = \cdot 000092343, \qquad \sigma_4 = \cdot 000176218.$$

Taking $\log \mathfrak{A}_3 = 9\cdot 5461687$, I found

$$\mathfrak{A}_3[\tfrac{1}{3}(\sigma_2)^2+2\zeta_4]-\tfrac{1}{3}\sigma_4 = -\cdot 00050051.$$

This differs by 4 in the seventh place of decimals from the old value.

In evaluating ω_i^s, ρ_i^s, ϕ_i^s when we use the form of $\mathfrak{P}_i^s$ involving f_0, f_2, f_4, &c., we have to put

$$[\mathfrak{P}_3(\mu)]^2 = F_0\sin^6\theta-F_2\sin^4\theta\cos^2\theta+\mathrm{F}_4\sin^2\theta\cos^2\theta,$$

and F_0, F_2, F_4 are easily found from α, β, γ, δ.

I then write

$$L_0 = f_0F_0, \quad L_2 = f_0F_2+f_2F_0, \quad L_4 = f_0F_4+f_2F_2+f_4F_0, \quad \text{\&c.},$$

and, when i denotes the order of the harmonic S_i^s concerned, write

$$f(\Lambda_{2n}^0) = L_0{}^{i+6}\Lambda_{2n}^0-L_2{}^{i+4}\Lambda_{2n}^2+L_4{}^{i+2}\Lambda_{2n}^4-\ldots(n = 1,\ 2),$$

$$f(\Lambda_{2n}^2) = L_0{}^{i+6}\Lambda_{2n}^2-L_2{}^{i+4}\Lambda_{2n}^4+L_4{}^{i+2}\Lambda_{2n}^6-\ldots(n = 1,\ 2).$$

These functions are then combined to give the required integrals.

A similar notation enables us to evaluate ϕ_i^s, which is given by

$$\phi_i^s = \frac{6}{\pi} \iint (\kappa^2 \cos^2 \theta + \kappa'^2 \sin^2 \phi) \frac{(S_i^s)^2}{\Delta\Gamma} d\theta d\phi,$$

so that

$$\phi_i^s = \frac{6}{\pi} \{\kappa^2 f(\Lambda_0^2) f(\Omega_0^0) + \kappa'^2 f(\Lambda_0^0) f(\Omega_0^2)\},$$

appropriate forms being attributed to $f(\Lambda)$, $f(\Omega)$.

The result of the procedure sketched is the following series of values, in which only the cases $i = 6$, $s = 2, 4$, are derived from approximate forms. For the sake of comparison I add the approximate values of ϕ_i^s as computed from the formulæ in "Harmonics." In the cases $i = 2$, $s = 0, 2$, the approximate results, derived from that paper, are multiplied by such factors as to make the approximate formulæ for $\mathfrak{P}_2(\mu)$ and $\mathfrak{C}_2(\phi)$ agree with the exact one when $\mu = 1$, $\phi = 45°$; and for $\mathfrak{P}_2^2(\mu)$ and $\mathfrak{C}_2^2(\phi)$ to make the coefficients of μ^2 and $\cos^2\phi$ agree with the exact formulæ.

TABLE of Logarithms of ω_i^s, ρ_i^s, ϕ_i^s.

i.	s.	$\log \omega_i^s + 10$.	$\log \rho_i^s + 10$.	$\log \phi_i^s$.	Approximate ϕ_i^s from formula in "Harmonics."
0	0	—	7·6310567	—	—
2	0	7·6714241	7·0286816	9·0051748 − 10	9·00518 − 10
2	2	(−) 5·6818162	(−) 5·0264000	7·0397377 − 10	7·03981 − 10
4	0	8·0332932	7·3558076	9·6886735 − 10	9·68861 − 10
4	2	(−) 8·25158	(−) 7·32157	1·72739	1·72729
4	4	8·30779	7·37092	3·81610	3·81612
6	0	7·96786	7·32449	9·69177 − 10	9·69303 − 10
6	2	(−) 8·72778	(−) 7·94094	—	2·20562
6	4	9·10094	8·13161	—	5·29999
8	0	7·78437	6·96857	9·75611 − 10	9·76872 − 10
10	0	(−) 7·9838	6·6024	9·8473 − 10	9·87800 − 10

Note that ω_{10} is negative while ϕ_{10} remains positive.

The calculation of the integrals for $i = 8$ and $i = 10$ was very laborious, and as the results tend to present themselves as the differences between large numbers, it is difficult to obtain accuracy with logarithms of only seven places of decimals. The integrals ϕ are much the most troublesome; indeed I do not claim close accuracy for ϕ_8; and as it appeared to be impossible to compute ϕ_{10} to nearer than 10 per cent. from the formula, I computed the several constituent integrals for the tenth harmonic by quadratures and combined them to find ϕ_{10}. The results derived from the approximate formulæ of "Harmonics" are given for the sake of comparison. They clearly give somewhat too large a value for the higher harmonics. I believe ω_{10} and ρ_{10} to be nearly correct.

If allowance be made for the difference of definition adopted in this paper from that used in "Harmonics" as regards the second zonal harmonic, it will be found that ω_2, ω_4, ω_6, ω_8, ω_{10}, when set out graphically, fall into an evenly flowing curve. The corresponding test for the ρ's is not quite so convincing, but there is nothing which implies a mistake. The values of ρ_0, ρ_2, ρ_4, ρ_6 fall well into line, and so do ρ_4, ρ_6, ρ_8, ρ_{10}, but there is a gentle elevation in the neighbourhood of ρ_6. In consequence of this slight waviness of the curve I recomputed the *whole* again independently, after it had been recomputed and verified once, and special attention was paid to ω_6 and ρ_6.

§ 5. *Final Synthesis of Numerical Results, and Conclusion.*

The several numerical values are combined just as in the original paper, but the numbers, of course, differ a little from those obtained before. The following table gives the final stage, inclusive of the additional terms now computed :—

i.	s.	(1) $[i, s]$.	(2) $(B_i^s)^2/C_i^s$.	(1) + (2).	B_i^s/C_i^s.
2	0	+ ·000138868	− ·000219736	− ·000080868	− ·12382
2	4	·000000717	+ ·000000970	+ ·000001687	− ·08056
4	0	·000092542	·000154732	·000247274	+ ·06273
4	2	·000001908	·000001190	·000003098	− ·000355
4	4	·000000012	·000000000	·000000012	+ ·0000017
6	0	·000031204	·000031146	·000062350	+ ·019564
6	2	·000003422	·000002107	·000005529	− ·000229
6	4	·000000014	·000000000	·000000014	+ ·0000003
8	0	+ ·000012905	·000006671	·000019576	+ ·007505
10	0	− ·000001030	+ ·000007358	+ ·000006328	− ·00667
			Sum =	·000265000	
			$\mathfrak{A}_3\left[\frac{1}{3}(\sigma_2)^2 + 2\zeta_4\right] - \frac{1}{3}\sigma_4$ =	− ·000500513	
			Numerator =	− ·000235513	

I then find $\log D = 9{\cdot}9840165$, $\log L = {\cdot}6454565$, $\log M = {\cdot}9591963$. From these we find $\mathfrak{c} = L\phi_2$, $\mathfrak{d} = M\phi_2^2$; whence

$\dfrac{B_2}{C_2}\mathfrak{c} =$	− ·0553908
$\dfrac{B_2^2}{C_2^2}\mathfrak{d} =$	·0008037
$\mathfrak{h} =$	·0316007
Denominator =	− ·0229864

The Numerator divided by the Denominator is $-\delta\omega^2/4\pi\rho e^2$, whence

$$\log \frac{\delta\omega^2}{4\pi\rho e^2} = (-)\, 8{\cdot}01054.$$

With $\dfrac{\omega^2}{2\pi\rho} = {\cdot}141990$, from § 7 of the "Pear-shaped Figure," we have

$$\omega^2 + \delta\omega^2 = \omega^2[1 - {\cdot}1443066e^2].$$

Thence we find

$$f_2 = {\cdot}195979e^2, \quad f_2^2 = {\cdot}603177e^2.$$

These values differ sensibly from the old ones.

The moment of inertia with $\log \mathfrak{a} = 9{\cdot}8559759$ is given by

$$A_J - A_r = \frac{3M\mathfrak{a}}{2\pi\rho k_0}[1 + {\cdot}157786e^2].$$

The moment of momentum is

$$\frac{3M^2\mathfrak{a}\omega}{2\pi\rho k_0}[1 + {\cdot}085633e^2].$$

As before, we find the pear-shaped figure to be stable, because the moment of momentum is greater than that of the critical Jacobian, provided that the infinite series does not amount to too great a sum.

If ϵ be the uncomputed residue of $\Sigma\left\{[i, s] + \dfrac{(B_i^s)^2}{C_i^s}\right\}$, I then find, as before, that the moment of momentum is

$$\frac{3M^2\mathfrak{a}\omega}{2\pi\rho k_0}[1 + {\cdot}085633e^2 - 499{\cdot}586\epsilon e^2].$$

The coefficient of e^2 will be positive and the pear stable, provided that

$$499{\cdot}586\epsilon < {\cdot}085633,$$

or

$$\epsilon < {\cdot}0001714.$$

The eighth zonal harmonic gave a contribution of ·0000196, and the tenth of ·0000063. These are respectively a ninth and a twenty-seventh of the critical total. The pear is then stable unless the residue of the apparently highly convergent series shall amount to more than 27 times the value of the last term computed. M. Liapounoff claims in effect to prove that this is the case, but to me it seems incredible. I look for the discrepancy between our conclusions in some other direction.