V. *The Gravitational Stability of the Earth.*

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PART I.

Introduction.

1. If in a gravitating body there occurs a displacement which involves alteration of density, there must be a tendency for the material to move towards the places where the density is increased, and away from the places where the density is diminished. The effect of this tendency, if it were not held in check, would be to accentuate local

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alterations of density. In any body the tendency is partially held in check by the elasticity of the body, and, in particular, by the elastic resistance which the body offers to compression. If this resistance is sufficiently great, the body is stable, in spite of the tendency to instability which arises from gravitation. It is important to determine the conditions of stability for bodies of various forms and constitutions, with various distributions of density. The problem of the stability of spherically symmetrical configurations of a quantity of gravitating gas has been investigated by J. H. JEANS,* and he has drawn from his investigations some interesting conclusions in regard to the course of evolution of stellar and planetary systems. In a subsequent memoir† he proceeded to investigate a similar problem in regard to gravitating bodies of a more coherent character. A gravitating solid body, such as a planet may be conceived to be, might exist in a spherical shape with a spherically symmetrical distribution of density. In the absence of gravitation there could be no question of instability. The effect of any local condensation would be to set up vibrations, and the frequency of the vibration of any spherical harmonic type would depend upon the elasticity of the material. If the resistance of the material to compression is sufficiently high the stability persists in spite of gravitation. There are thus two competing agencies: gravitation, tending to instability, and the elasticity of the material, tending to stability. In a general way it is clear that, as the elasticity diminishes, the frequency of vibration of any type also diminishes; and, if the frequency can vanish for sufficiently small elasticity, the planetary body possessing such elasticity cannot continue to exist in the spherically symmetrical configuration. The problem is to determine the conditions as regards elasticity in which the instability occurs.

A grave difficulty presents itself at the outset. In the equilibrium configuration the gravitating planet is in a state of stress; and, in a body of such dimensions as the Earth, this stress is so great that the total stress existing in the body when it vibrates cannot be calculated by the ordinary methods of the theory of elasticity. In that theory it is ordinarily assumed that the body under investigation is in a state so little removed from one of zero stress that the strain, measured from this state as a zero of reckoning, is proportional to the stress existing at any instant. In order that this assumption may be valid, it is necessary that the strain which is calculated by means of it should be so small that its square may be neglected. Now if we apply the equations of the ordinary theory to the problem of a solid sphere strained by its own gravitation, and if we take the sphere to be of the same size and mass as the Earth, and the material of which it is composed to possess moduluses of elasticity as great as those of ordinary steel, we find that the strains may be as great as $\frac{1}{3}$, and thus the strains are much too great for the assumption to be valid. The *initial stress* existing

* "The Stability of a Spherical Nebula," London, 'Phil. Trans. Roy. Soc.,' A, vol. 199 (1902), p. 1.

† J. H. JEANS, "On the Vibrations and Stability of a Gravitating Planet," London, 'Phil. Trans. Roy. Soc.,' A, vol. 201 (1903), p. 157. Quoted below as "JEANS (1903)."

in the gravitating planet, the stress by which the self-attraction of the body is equilibrated, is much too great to permit of the application of the ordinary theory. The same difficulty presents itself in every problem concerning the elasticity of a gravitating planet, for example, in the problem of tidal deformation or of the stress produced in the interior by the weight of continents. In these problems the difficulty was turned by Lord KELVIN* and Sir G. H. DARWIN† by taking the modulus of compression to be much greater than that of any known material, in other words, by taking the material to be incompressible. Their object was to determine the degree of rigidity which must be assigned to the Earth, and for that object it is permissible to turn the difficulty in this way. When the problem is that of gravitational instability this artifice cannot be adopted, because the whole question is that of the degree of compressibility which is admissible if the gravitating planet is to be stable in a spherically symmetrical configuration. The artifice adopted by JEANS (1903) consisted in annulling the initial stress by introducing an imagined external field of force to equilibrate the self-attraction of the planet.

The problem thus posed is an artificial one, which may, nevertheless, throw light on the actual problem. When the initial configuration is taken to be one of uniform density, the analysis of the problem is of the same kind as that which presents itself in the problem of the vibrations of an elastic sphere, a problem which has been worked out very completely by H. LAMB.‡ The determination of the effect produced by gravitation in lowering the frequencies of the various modes of vibration is reduced to a question of troublesome analytical computation. JEANS worked out the problem on the basis of the ordinary theory of elasticity, using the elastic constants λ and μ of LAMÉ. The constant μ is the modulus of rigidity, and the constant λ is such that $\lambda+\frac{2}{3}\mu$ is the modulus of compression. In the case of the Earth the values of these constants can be inferred from the observed rates of propagation of the various types of disturbance which are perceived as earthquake shocks. He concluded that, when the proper values are attributed to these constants, the Earth must be held to be in a state far removed from one of gravitational instability; but he suggested that, if the resistance to compression was at one time considerably smaller than it is now, the spherically symmetrical configuration would then have been unstable; and he held that there are traces of the instability in the distribution of land and water on the surface of the globe.

The actual problem differs from this artificial problem in the mode of balancing of the internal gravitation. Lord RAYLEIGH§ has proposed a method of meeting the difficulty as to initial stress. He proposed to consider the stress in the vibrating

* See, in particular, KELVIN and TAIT'S 'Natural Philosophy,' Part II., §§ 833–846, Cambridge, 1883.

† "On the Stresses caused in the interior of the Earth by the weight of Continents and Mountains," London, 'Phil. Trans. Roy. Soc.,' 173, 1882, p. 187.

‡ "On the Vibrations of an Elastic Sphere," London, 'Proc. Math. Soc.,' 13, 1882, p. 189.

§ "On the Dilatational Stability of the Earth," London, 'Proc. Roy. Soc.,' A, 77, 1906, p. 486.

gravitating sphere as compounded of two stress-systems: a hydrostatic pressure by which gravitation would be balanced if the sphere were in equilibrium, and an *additional stress.* He proposed to measure the strain, not from the unattainable state of zero stress, but from the equilibrium state; and he proposed to take the additional stress to be determined in terms of the strain by those equations which are commonly used in the theory of elasticity. To simplify the problem he proposed to take the material in the equilibrium state to be homogeneous and the elasticity to be isotropic, so that the equations connecting the additional stress and the strain are of the same form as the ordinary stress-strain relations of isotropic elasticity. In justification of the proposed procedure he brought forward theoretical considerations founded upon the general theory of energy, and other evidence drawn from an interpretation of the experimental results in regard to the behaviour of elastic solid bodies. It is not too much to say that all the evidence there is, is just as strong in favour of Lord RAYLEIGH's proposed method as it is in favour of HOOKE's law, in the sense in which that law is applied in the ordinary theory. The only objection which can be raised against the method, an objection mentioned by Lord RAYLEIGH himself, is that the body to be treated is certainly not homogeneous, and possibly not isotropic. When the proposed method is adopted, the density and the moduluses of elasticity must be taken to have their mean values. The justification for treating the values of these quantities at any point as equal to the mean values, is that it is advisable in the first instance to work out the simplest case.*

In the first part of this paper the mathematical problem proposed by Lord RAYLEIGH is worked out; and the conclusion is drawn that the effective moduluses of elasticity of the Earth, in its present state, are sufficiently great for a homogeneous spherical configuration to be thoroughly stable. The second part of the paper is devoted to developing the consequences of supposing that the elasticity of the material of the Earth was once much less than it is at present.

Statement of the Mathematical Problem.

2. We have before us a perfectly definite mathematical problem, which may be stated as follows:—A sphere of radius a, and of uniform density ρ_0, is in equilibrium under its own gravitation, and the stress within it is hydrostatic pressure of amount p_0 at a distance r from the centre. When any small disturbance takes place, so that

* It may be observed that the method advocated by Lord RAYLEIGH is the same, except for a slight modification, as that which was used in the second edition of my "Treatise on the Mathematical Theory of Elasticity," Cambridge, 1906, in the discussion of the statical problem of a gravitating sphere held strained by external disturbing forces. The modification consists in the assumption, which was there made, that the material might be treated as incompressible. If this assumption is not made, the analysis becomes much more difficult. An earlier indication of the method will be found in a paper by J. LARMOR Cambridge, 'Proc. Phil. Soc.,' 9, 1898, p. 183.

the particle which was initially at (x, y, z) is displaced to $(x+u, y+v, z+w)$, the stress is specified by six stress-components X_x, Y_y, Z_z, Y_z, Z_x, X_y, and these are connected with the initial pressure p_0 and the displacement (u, v, w) by the formulæ

$$\left.\begin{aligned} X_x &= -p_0+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}, \\ Y_y &= -p_0+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial v}{\partial y}, \\ Z_z &= -p_0+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}, \\ Y_z = \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right), &\quad Z_x = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \quad X_y = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \end{aligned}\right\} \quad . \; . \; (1)$$

where λ and μ are constants. It is required to form the equations of vibration, and to solve them, so as to determine the character of the modes of vibration and the equation for the frequencies, and, in particular, to ascertain the relations which must hold among the quantities λ, μ, ρ_0, a in order that any frequency may be reduced to zero. We proceed to express this problem in terms of a system of differential equations which hold at all points of the body, and a system of special conditions which hold at all points of the undisturbed surface.

3. In the equilibrium state the potential V_0 at any point is given by the equation

$$V_0 = \tfrac{2}{3}\pi\gamma\rho_0(3a^2-r^2), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2)$$

where γ is the constant of gravitation. The equation of equilibrium is

$$-\frac{1}{\rho_0}\frac{\partial p_0}{\partial r}+\frac{\partial V_0}{\partial r} = 0, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (3)$$

or

$$\frac{\partial p_0}{\partial r} = -\tfrac{4}{3}\pi\gamma\rho_0^2 r. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (4)$$

Since $p_0 = 0$ at the surface $r = a$, the value of p_0 at any point is given by the equation

$$p_0 = \tfrac{2}{3}\pi\gamma\rho_0^2(a^2-r^2). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (5)$$

When the sphere vibrates, the equations of motion are three equations of the type

$$\rho\frac{\partial^2 u}{\partial t^2} = \rho\frac{\partial V}{\partial x}+\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial Z_x}{\partial z}, \quad . \; . \; . \; . \; . \; . \; . \; . \; (6)$$

where ρ is the density, and V the potential, in the disturbed state. In the left-hand members of these equations we may ignore the distinction between ρ and ρ_0. In the right-hand members we may put

$$\rho = \rho_0(1-\Delta), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (7)$$

where Δ is the dilatation expressed by the formula

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \qquad (8)$$

Further, we may put

$$V = V_0 + W, \qquad (9)$$

where W is the additional potential due to concentration of density at internal points, and to displacement of mass across the initial bounding surface. We may neglect terms of the type $\rho_0 \Delta \partial W/\partial x$. When we substitute for $X_x, \ldots$ from equations (1), and make these simplifications, the equation (6) becomes

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \rho_0 (1 - \Delta) \frac{\partial V_0}{\partial x} + \rho_0 \frac{\partial W}{\partial x} - \frac{\partial p_0}{\partial x} + (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u.$$

On omitting the terms which cancel each other in virtue of equation (3), we have the first of the three equations (10) below. The remaining two of these equations are obtained in the same way. Thus we have the equations of vibratory motion in the forms

$$\left.\begin{aligned} \rho_0 \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u + \tfrac{4}{3}\pi\gamma\rho_0{}^2 x\Delta + \rho_0 \frac{\partial W}{\partial x}, \\ \rho_0 \frac{\partial^2 v}{\partial t^2} &= (\lambda + \mu) \frac{\partial \Delta}{\partial y} + \mu \nabla^2 v + \tfrac{4}{3}\pi\gamma\rho_0{}^2 y\Delta + \rho_0 \frac{\partial W}{\partial y}, \\ \rho_0 \frac{\partial^2 w}{\partial t^2} &= (\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 w + \tfrac{4}{3}\pi\gamma\rho_0{}^2 z\Delta + \rho_0 \frac{\partial W}{\partial z}. \end{aligned}\right\} \qquad (10)$$

In addition to these equations we have the equation connecting the potential with the density in the form

$$\nabla^2 W = 4\pi\gamma\rho_0\Delta. \qquad (11)$$

The system of equations (10) and (11) are the differential equations of the problem.*

4. Besides satisfying the differential equations (10) and (11), the additional potential W and the components of displacement u, v, w must also satisfy certain conditions at the surface $r = a$. Let U denote the radial component of displacement, so that

$$Ur = xu + yv + zw, \qquad (12)$$

and let U_a denote the value of U at $r = a$. The potential W is that due to a volume distribution of density $-\rho_0\Delta$, together with that due to a superficial distribution $\rho_0 U_a$

* In the problem as formulated by JEANS, when the self-attraction of the body is balanced by an external field of force, the equations of vibratory motion differ from those which are obtained here by the omission of the terms such as $\frac{4}{3}\pi\gamma\rho_0{}^2 x\Delta$. In Lord RAYLEIGH'S paper already cited, the equations given by JEANS are discussed in accordance with the analysis which was developed by LAMB in the paper on the vibrations of a sphere.

on the surface $r = a$. By the method of spherical harmonics we can, when W is known, write down the expression for the function $W^{(0)}$ which is the potential at external points of the same distribution. The surface characteristic equation gives

$$\left(\frac{\partial W^{(0)}}{\partial r}\right)_{r=a} - \left(\frac{\partial W}{\partial r}\right)_{r=a} = -4\pi\gamma\rho_0 U_a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This is one of the conditions which must be satisfied at the surface $r = a$. To obtain the other conditions which must be satisfied at this surface, we observe that the disturbed surface $r = a + U_a$ is free from traction. If l, m, n denote the direction cosines of the outward drawn normal to this surface we have three equations of the type

$$lX_x + mX_y + nZ_x = 0,$$

which hold at the surface $r = a + U_a$. If in this equation we substitute for $X_x, \ldots$ from equations (1), we see that in the terms containing $u, \ldots$ we may replace $l, \ldots$ by the approximate values x/r, y/r, z/r. The only term which does not contain $u, \ldots$ is the term $-lp_0$ arising from $-lX_x$. Now p_0 vanishes at $r = a$, and therefore at $r = a + U_a$ we have

$$p_0 = U_a \left(\frac{\partial p_0}{\partial r}\right)_{r=a}$$

to the first order in u, v, w. Hence in this term also we may replace l by x/r. On substituting for p_0 from (5) we find that the equation

$$\tfrac{4}{3}\pi\gamma\rho_0^2 x U + \frac{x}{r}\left(\lambda\Delta + 2\mu\frac{\partial u}{\partial x}\right) + \frac{y}{r}\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{z}{r}\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

must hold at the surface $r = a$. By an easy transformation this equation becomes the first of the three equations written in (14) below. The remaining two of these equations are obtained in the same way. The equations which must hold at the surface $r = a$ are therefore equation (13) and the equations

$$\left.\begin{aligned} \frac{\lambda}{\mu} x\Delta + \frac{\partial}{\partial x}(Ur) + r\frac{\partial u}{\partial r} - u + \tfrac{4}{3}\frac{\pi\gamma\rho_0^2}{\mu} xUr &= 0, \\ \frac{\lambda}{\mu} y\Delta + \frac{\partial}{\partial y}(Ur) + r\frac{\partial v}{\partial r} - v + \tfrac{4}{3}\frac{\pi\gamma\rho_0^2}{\mu} yUr &= 0, \\ \frac{\lambda}{\mu} z\Delta + \frac{\partial}{\partial z}(Ur) + r\frac{\partial w}{\partial r} - w + \tfrac{4}{3}\frac{\pi\gamma\rho_0^2}{\mu} zUr &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

These equations can be interpreted in the statement that the traction on the mean sphere is a pressure equal to the weight per unit of area of the material heaped up to form the inequality U_a.

5. We shall now suppose that the system executes a normal, or principal, vibration of frequency $p/2\pi$, or, in other words, that every component of displacement is proportional to the same simple harmonic function of pt. The equations of vibratory motion become three equations of the type

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2u+\rho_0p^2u+\tfrac{4}{3}\pi\gamma\rho_0{}^2x\Delta+\rho_0\frac{\partial W}{\partial x}=0, \quad . \quad . \quad . \quad . \quad (15)$$

where W satisfies the equation (11). The solutions of this system of equations (11) and (15) must be adjusted to satisfy the conditions (13) and (14) at $r=a$. These conditions can be satisfied only if p has one or other of a certain infinite set of values, which are the roots of the frequency equation. The problem of gravitational instability is solved when we find the conditions that one of the values of p may be zero.

Solution of the Differential Equations by Means of Spherical Harmonics.

6. We introduce the notation

$$\frac{p^2\rho_0}{\lambda+2\mu}=h^2,\qquad \frac{p^2\rho_0}{\mu}=k^2,\qquad \frac{\frac{4}{3}\pi\gamma\rho_0{}^2}{\lambda+2\mu}=s^2,\qquad W=4\pi\gamma\rho_0E\,. \quad . \quad . \quad (16)$$

The equations of motion (15) become three equations of the type

$$(\nabla^2+k^2)u+\left(\frac{k^2}{h^2}-1\right)\frac{\partial\Delta}{\partial x}+\frac{k^2}{h^2}s^2\left(x\Delta+3\frac{\partial E}{\partial x}\right)=0, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and the equation $\nabla^2W=4\pi\gamma\rho_0\Delta$ becomes

$$\nabla^2E=\Delta\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

in these equations Δ stands for

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}.$$

By differentiating the left-hand members of the equations of type (17) with respect to x, y, z, respectively, adding the results, and simplifying by means of (18), we obtain the equation

$$(\nabla^2+h^2)\Delta+6s^2\Delta+s^2r\frac{\partial\Delta}{\partial r}=0.\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The method of solution of the problem is this:—We seek first a solution of the equation (19) in which Δ has the form

$$\Delta=\Sigma f_n(r)\,\omega_n\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where ω_n is a spherical solid harmonic of positive integral degree n, and f_n is a

function of r which is such that $r^n f_n$ is finite at all points within $r = a$, including the origin $r = 0$. We seek next to determine E in the form

$$E = \Sigma [E_n(r) \omega_n + F_n], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where $E_n(r)$ is a certain function of r which is such that $r^n E_n$ is finite at all points within $r = a$, including the origin $r = 0$, and F_n is a spherical solid harmonic of degree n. The equations of motion of type (15) then become three equations of the type

$$(\nabla^2 + k^2) u + \left(\frac{k^2}{h^2} - 1\right) \Sigma \left\{\frac{\partial}{\partial x}(f_n \cdot \omega_n)\right\} + \frac{k^2}{h^2} s^2 x \Sigma (f_n \cdot \omega_n)$$
$$+ \frac{3k^2}{h^2} s^2 \Sigma \left\{\frac{\partial}{\partial x}(E_n \cdot \omega_n)\right\} + \frac{3k^2}{h^2} s^2 \Sigma \left(\frac{\partial F_n}{\partial x}\right) = 0, \quad . \quad . \quad . \quad . \quad (22)$$

in which we must have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \Sigma (f_n \cdot \omega_n). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

It appears on trial that we can obtain a solution in which

$$u = u_1 + u_2 + u_3, \quad v = v_1 + v_2 + v_3, \quad w = w_1 + w_2 + w_3, \quad . \quad . \quad . \quad . \quad (24)$$

where u_1, v_1, w_1 satisfy the system of equations

$$\left.\begin{array}{l} (\nabla^2 + k^2) u_1 + \left(\frac{k^2}{h^2} - 1\right) \Sigma \left\{\frac{\partial}{\partial x}(f_n \cdot \omega_n)\right\} + \frac{k^2}{h^2} s^2 x \Sigma (f_n \cdot \omega_n) \\ \qquad + \frac{3k^2}{h^2} s^2 \Sigma \left\{\frac{\partial}{\partial x}(E_n \cdot \omega_n)\right\} = 0, \\ \quad . \quad . \quad . \quad . \quad , \\ \quad . \quad . \quad . \quad . \quad , \\ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = \Sigma (f_n \cdot \omega_n) ; \end{array}\right\} \quad . \quad . \quad . \quad . \quad (25)$$

also u_2, v_2, w_2 satisfy the system of equations

$$\left.\begin{array}{l} (\nabla^2 + k^2) u_2 + \frac{3k^2}{h^2} s^2 \Sigma \left(\frac{\partial F_n}{\partial x}\right) = 0, \quad \ldots, \quad \ldots \\ \qquad \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0 ; \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and u_3, v_3, w_3 are a complementary solution of the system of equations

$$\left.\begin{array}{l} (\nabla^2 + k^2) u_3 = 0, \quad (\nabla^2 + k^2) v_3 = 0, \quad (\nabla^2 + k^2) w_3 = 0, \\ \qquad \frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y} + \frac{\partial w_3}{\partial z} = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (27)$$

7. The sets of functions u_1, v_1, w_1 and u_2, v_2, w_2 can be any particular solutions

of the systems of equations (25) and (26). It appears on trial that u_1, v_1, w_1 can have the forms

$$\left.\begin{aligned} u_1 &= \Sigma\left[\mathrm{P}_n(r)\frac{\partial\omega_n}{\partial x}+\mathrm{Q}_n(r)\,r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right], \\ v_1 &= \Sigma\left[\mathrm{P}_n(r)\frac{\partial\omega_n}{\partial y}+\mathrm{Q}_n(r)\,r^{2n+3}\frac{\partial}{\partial y}\left(\frac{\omega_n}{r^{2n+1}}\right)\right], \\ w_1 &= \Sigma\left[\mathrm{P}_n(r)\frac{\partial\omega_n}{\partial z}+\mathrm{Q}_n(r)\,r^{2n+3}\frac{\partial}{\partial z}\left(\frac{\omega_n}{r^{2n+1}}\right)\right], \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where $\mathrm{P}_n(r)$ and $\mathrm{Q}_n(r)$ are certain functions of r. Also it is clear that u_2, v_2, w_2 can have the forms

$$u_2 = -\frac{3s^2}{h^2}\Sigma\left(\frac{\partial\mathrm{F}_n}{\partial x}\right), \qquad v_2 = -\frac{3s^2}{h^2}\Sigma\left(\frac{\partial\mathrm{F}_n}{\partial y}\right), \qquad w_2 = -\frac{3s^2}{h^2}\Sigma\frac{\partial\mathrm{F}_n}{\partial z}. \quad . \quad . \quad (29)$$

Further, the forms of u_3, v_3, w_3 are known from the analysis of the problem of the vibrating sphere which is free from gravitation. We have

$$u_3 = \Sigma\left[\psi_n(kr)\left(y\frac{\partial\chi_n}{\partial z}-z\frac{\partial\chi_n}{\partial y}\right)+\psi_{n-1}(kr)\frac{\partial\phi_n}{\partial x}-\frac{n}{n+1}\psi_{n+1}(kr)\,k^2r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)\right], \quad . \quad (30)$$

where

$$\psi_n(kr) = \left(\frac{1}{kr}\frac{d}{dkr}\right)^n\left(\frac{\sin kr}{kr}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

χ_n and ϕ_n are spherical solid harmonics of degree n, and the expressions for v_3, w_3 are to be obtained from the expression for u_3 by cyclical interchanges of the letters x, y, z. It appears that to a single term $f_n\omega_n$ in the expression for Δ there corresponds a definite term F_n in the expression for E. Further, when we form the boundary conditions, it appears that the terms of u_3, v_3, w_3 which contain χ_n represent a free vibration, and the frequency of this vibration is determined by the same equation as if the sphere were free from gravitation. Also it appears that to any term $f_n\omega_n$ in the expression for Δ there corresponds a definite function ϕ_n in the expression for u_3, v_3, w_3. The solution expressed by a single term $f_n\omega_n$ of Δ and the corresponding terms of (u_1, v_1, w_1), (u_2, v_2, w_2) and (u_3, v_3, w_3) determines a normal mode of vibration. We shall therefore omit χ_n, and reduce all the summations to typical terms.

8. If in the equation

$$(\nabla^2+h^2)\,\Delta+6s^2\Delta+s^2r\frac{\partial\Delta}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19\ bis)$$

we put $f_n(r)\,.\,\omega_n$ for Δ, we find that $f_n(r)$ must satisfy the equation

$$\frac{d^2f_n}{dr^2}+\frac{2}{r}\frac{df_n}{dr}+\frac{2n}{r}\frac{df_n}{dr}+s^2\left(r\frac{df_n}{dr}+nf_n\right)+(h^2+6s^2)f_n = 0,$$

or

$$\frac{d^2f_n}{dr^2}+\left\{\frac{2\,(n+1)}{r}+s^2r\right\}\frac{df_n}{dr}+\{h^2+(n+6)\,s^2\}f_n = 0, \quad . \quad . \quad . \quad . \quad (32)$$

This equation is a linear differential equation of the second order; and the forms of the coefficients show that the point $r = 0$ is a critical point, and that there is no other critical point at any finite value of r. If we seek a solution in series of the form

$$f_n = a_0 r^m + a_1 r^{m+1} + a_2 r^{m+2} + \dots,$$

we find the "indicial equation"

$$m(m-1) + 2(n+1)m = 0,$$

from which either $m = 0$, or $m = -(2n+1)$. We must take the series for which $m = 0$, because $r^n f_n$ must be finite at $r = 0$. Further, the form of the equation shows that this series contains even powers of r only. We assume, therefore, for f_n the form

$$f_n = \mathrm{A}[1 + a_2 r^2 + a_4 r^4 + \dots + a_{2\kappa} r^{2\kappa} + \dots],$$

where A is an arbitrary constant, and then we find the sequence equation

$$a_{2\kappa+2}(2\kappa+2)(2\kappa+1+2n+2) + a_{2\kappa}\{h^2 + (n+6)s^2 + 2\kappa s^2\} = 0,$$

or

$$a_{2\kappa+2} = -a_{2\kappa}\frac{h^2 + (n+6+2\kappa)s^2}{(2\kappa+2)(2\kappa+2n+3)}.$$

Hence we have

$$f_n = \mathrm{A}\Bigg[1 - \frac{h^2 + (n+6)s^2}{2\,.\,(2n+3)}r^2 + \frac{\{h^2 + (n+6)s^2\}\{h^2 + (n+8)s^2\}}{2\,.\,4\,(2n+3)(2n+5)}r^4 - \dots$$
$$(-)^\kappa \frac{\{h^2 + (n+6)s^2\}\{h^2 + (n+8)s^2\}\dots\{h^2 + (n+2\kappa+4)s^2\}}{2\,.\,4\dots 2\kappa\,(2n+3)(2n+5)\dots(2n+2\kappa+1)}r^{2\kappa}\dots\Bigg]. \quad (33)$$

The series is convergent and represents the function f_n for all finite values of r.

9. We must next determine the function $\mathrm{E}_n(r)$ from the equation

$$\nabla^2(\mathrm{E}_n\omega_n) = f_n\omega_n,$$

or

$$\frac{d^2\mathrm{E}_n}{dr^2} + \frac{2(n+1)}{r}\frac{d\mathrm{E}_n}{dr} = f_n, \quad (34)$$

or

$$\frac{d}{dr}\left\{r\frac{d\mathrm{E}_n}{dr} + (2n+1)\mathrm{E}_n\right\} = rf_n.$$

We introduce an intermediate function $\theta_n(r)$ by the equation

$$\theta_n = r\frac{d\mathrm{E}_n}{dr} + (2n+1)\mathrm{E}_n. \quad (35)$$

Then

$$\theta_n = \int rf_n\,dr$$
$$= \mathrm{C} + \mathrm{A}\left[\tfrac{1}{2}r^2 - \frac{h^2 + (n+6)s^2}{2\,.\,4\,.\,(2n+3)}r^4 + \frac{\{h^2 + (n+6)s^2\}\{h^2 + (n+8)s^2\}}{2\,.\,4\,.\,6\,(2n+3)(2n+5)}r^6 - \dots\right],$$

where C is constant of integration. Then we have

$$\frac{d}{dr}(r^{2n+1}\mathrm{E}_n) = r^{2n}\theta_n,$$

and therefore

$$\mathrm{E}_n = \frac{\mathrm{C}'}{r^{2n+1}} + \frac{\mathrm{C}}{2n+1} + \mathrm{A}\left[\frac{r^2}{2\,.\,(2n+3)} - \frac{\{h^2+(n+6)\,s^2\}\,r^4}{2\,.\,4\,(2n+3)\,(2n+5)} + \ldots\right],$$

where C′ is a constant of integration. Since $r^n\mathrm{E}_n$ is finite at $r = 0$, the constant C′ must be zero; but the constant C is in our power, and we may choose it in any way that is convenient. The term contributed to E by C is $(2n+1)^{-1}\,\mathrm{C}\omega_n$, which satisfies LAPLACE's equation, and therefore any change in the chosen value of C is equivalent to borrowing a term of F_n to make up a term of $\mathrm{E}_n\omega_n$.

Now the series

$$1 - \frac{h^2+(n+4)\,s^2}{2\,.\,(2n+1)}r^2 + \frac{\{h^2+(n+4)\,s^2\}\,\{h^2+(n+6)\,s^2\}}{2\,.\,4\,(2n+1)\,(2n+3)}r^4 - \ldots$$

satisfies the equation

$$\left[\frac{d^2}{dr^2} + \left(\frac{2n}{r} + s^2r\right)\frac{d}{dr} + \{h^2+(n+4)\,s^2\}\right]\left\{1 - \frac{h^2+(n+4)\,s^2}{2\,(2n+1)}r^2 + \ldots\right\} = 0;$$

and therefore, if we take for C the value

$$\mathrm{C} = -\frac{2n+1}{h^2+(n+4)\,s^2}\mathrm{A},$$

the function θ_n satisfies the equation

$$\frac{d^2\theta_n}{dr^2} + \left(\frac{2n}{r} + s^2r\right)\frac{d\theta_n}{dr} + \{h^2+(n+4)\,s^2\}\,\theta_n = 0. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

We shall choose this value for C, and thus we shall have

$$\theta_n = \mathrm{A}\left[-\frac{2n+1}{h^2+(n+4)\,s^2} + \tfrac{1}{2}r^2 - \frac{h^2+(n+6)\,s^2}{2\,.\,4\,(2n+3)}r^4 + \ldots \right.$$
$$\left.(-)^{\kappa+1}\frac{\{h^2+(n+6)\,s^2\}\,\{h^2+(n+8)\,s^2\}\ldots\{h^2+(n+2\kappa+2)\,s^2\}}{2\,.\,4\ldots2\kappa\,(2n+3)\,(2n+5)\ldots(2n+2\kappa-1)}r^{2\kappa}\ldots\right], \quad (37)$$

and

$$\mathrm{E}_n = \mathrm{A}\left[-\frac{1}{h^2+(n+4)\,s^2} + \frac{r^2}{2\,(2n+3)} - \frac{\{h^2+(n+6)\,s^2\}\,r^4}{2\,.\,4\,.\,(2n+3)\,(2n+5)} + \ldots \right.$$
$$\left.(-)^{\kappa+1}\frac{\{h^2+(n+6)\,s^2\}\,\{h^2+(n+8)\,s^2\}\ldots\{h^2+(n+2\kappa+2)\,s^2\}}{2\,.\,4\ldots2\kappa\,(2n+3)\,(2n+5)\ldots(2n+2\kappa+1)}r^{2\kappa}\ldots\right]. \quad (38)$$

The function E_n satisfies the equation

$$\frac{d^2\mathrm{E}_n}{dr^2} + \left\{\frac{2\,(n+1)}{r} + s^2r\right\}\frac{d\mathrm{E}_n}{dr} + \{h^2+(n+4)\,s^2\}\,\mathrm{E}_n = 0. \quad . \quad . \quad . \quad . \quad (39)$$

It will be convenient presently to have observed that the equation derived from this one by differentiating the left-hand member with respect to r can be written

$$\left[\frac{d^2}{dr^2}+\left\{\frac{2(n+2)}{r}+s^2r\right\}\frac{d}{dr}+\{h^2+(n+6)s^2\}\right]\left(\frac{1}{r}\frac{d\mathrm{E}_n}{dr}\right)=0. \quad . \quad . \quad (40)$$

10. The forms of u_2, v_2, w_2 and u_3, v_3, w_3 have been put down and it remains to determine u_1, v_1, w_1. We have a system of three equations of the type

$$(\nabla^2+k^2)\,u_1+\left(\frac{k^2}{h^2}-1\right)\left(f_n\frac{\partial\omega_n}{\partial x}+\frac{1}{r}\frac{df_n}{dr}x\omega_n\right)+\frac{k^2}{h^2}s^2f_nx\omega_n$$
$$+\frac{3k^2}{h^2}s^2\left(\mathrm{E}_n\frac{\partial\omega_n}{\partial x}+\frac{1}{r}\frac{d\mathrm{E}_n}{dr}x\omega_n\right)=0. \quad . \quad . \quad . \quad (41)$$

We express $x\omega_n$ in the form

$$x\omega_n=\frac{r^2}{2n+1}\left\{\frac{\partial\omega_n}{\partial x}-r^{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and then the above equation becomes

$$(\nabla^2+k^2)\,u_1=-\frac{\partial\omega_n}{\partial x}\left[\left(\frac{k^2}{h^2}-1\right)\left(f_n+\frac{r}{2n+1}\frac{df_n}{dr}\right)+\frac{k^2}{h^2}s^2\left(\frac{r^2}{2n+1}f_n+3\mathrm{E}_n+\frac{3r}{2n+1}\frac{d\mathrm{E}_n}{dr}\right)\right]$$
$$+\frac{1}{2n+1}r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\left[\left(\frac{k^2}{h^2}-1\right)\frac{1}{r}\frac{df_n}{dr}+\frac{k^2}{h^2}s^2\left(f_n+\frac{3}{r}\frac{d\mathrm{E}_n}{dr}\right)\right]. \quad (43)$$

We seek solutions of the system of three equations of this type in the forms of the type

$$u_1=\mathrm{P}_n\frac{\partial\omega_n}{\partial x}+\mathrm{Q}_nr^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

in which P_n and Q_n are functions of r. We find

$$(\nabla^2+k^2)\,u_1=\left[\frac{d^2\mathrm{P}_n}{dr^2}+\frac{2n}{r}\frac{d\mathrm{P}_n}{dr}+k^2\mathrm{P}_n\right]\frac{\partial\omega_n}{\partial x}$$
$$+\left[\frac{d^2\mathrm{Q}_n}{dr^2}+\frac{2(n+2)}{r}\frac{d\mathrm{Q}_n}{dr}+k^2\mathrm{Q}_n\right]r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right).$$

Hence the assumed forms can be adjusted to satisfy the equations if P_n and Q_n satisfy the linear equations

$$\frac{d^2\mathrm{P}_n}{dr^2}+\frac{2n}{r}\frac{d\mathrm{P}_n}{dr}+k^2\mathrm{P}_n=-\left(\frac{k^2}{h^2}-1\right)\left(f_n+\frac{r}{2n+1}\frac{df_n}{dr}\right)$$
$$-\frac{k^2}{h^2}s^2\left(\frac{r^2}{2n+1}f_n+3\mathrm{E}_n+\frac{3r}{2n+1}\frac{d\mathrm{E}_n}{dr}\right), \quad . \quad . \quad . \quad (45)$$

$$\frac{d^2\mathrm{Q}_n}{dr^2}+\frac{2(n+2)}{r}\frac{d\mathrm{Q}_n}{dr}+k^2\mathrm{Q}_n=\frac{1}{2n+1}\left\{\left(\frac{k^2}{h^2}-1\right)\frac{1}{r}\frac{df_n}{dr}+\frac{k^2}{h^2}s^2\left(f_n+\frac{3}{r}\frac{d\mathrm{E}_n}{dr}\right)\right\}. \quad (46)$$

The right-hand members of these equations can be simplified by introducing the functions θ_n and $r^{-1}\,d\mathrm{E}_n/dr$. We have

$$f_n + \frac{r}{2n+1}\frac{df_n}{dr} = \frac{1}{r}\frac{d\theta_n}{dr} + \frac{r}{2n+1}\frac{d}{dr}\left(\frac{1}{r}\frac{d\theta_n}{dr}\right)$$

$$= \frac{1}{2n+1}\left(\frac{d^2\theta_n}{dr^2} + \frac{2n}{r}\frac{d\theta_n}{dr}\right),$$

and

$$\frac{r^2}{2n+1}f_n + 3\mathrm{E}_n + \frac{3r}{2n+1}\frac{d\mathrm{E}_n}{dr} = \frac{1}{2n+1}\left(r\frac{d\theta_n}{dr} + 3\theta_n\right);$$

and thus the equation for P_n becomes

$$\frac{d^2\mathrm{P}_n}{dr^2} + \frac{2n}{r}\frac{d\mathrm{P}_n}{dr} + k^2\mathrm{P}_n$$

$$= \frac{1}{2n+1}\left(\frac{d^2\theta_n}{dr^2} + \frac{2n}{r}\frac{d\theta_n}{dr}\right) - \frac{1}{2n+1}\frac{k^2}{h^2}\left(\frac{d^2\theta_n}{dr^2} + \frac{2n}{r}\frac{d\theta_n}{dr} + s^2 r\frac{d\theta_n}{dr} + 3s^2\theta_n\right)$$

$$= \frac{1}{2n+1}\left(\frac{d^2\theta_n}{dr^2} + \frac{2n}{r}\frac{d\theta_n}{dr} + k^2\theta_n\right) - \frac{1}{2n+1}\left[k^2\theta_n - \frac{k^2}{h^2}\left\{h^2 + (n+4)s^2\right\}\theta_n + 3\frac{k^2}{h^2}s^2\theta_n\right]$$

by the equation (36). Hence we have

$$\mathrm{P}_n = \frac{\theta_n}{2n+1} + \mathrm{P}'_n, \qquad (47)$$

where P'_n satisfies the equation

$$\frac{d^2\mathrm{P}'_n}{dr^2} + \frac{2n}{r}\frac{d\mathrm{P}'_n}{dr} + k^2\mathrm{P}'_n = \frac{n+1}{2n+1}\frac{k^2}{h^2}s^2\theta_n. \qquad (48)$$

Again, we have

$$\frac{1}{r}\frac{df_n}{dr} = \frac{1}{r}\frac{d}{dr}\left\{\frac{d^2\mathrm{E}_n}{dr^2} + \frac{2(n+1)}{r}\frac{d\mathrm{E}_n}{dr}\right\}$$

$$= \frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}\left\{r\left(\frac{1}{r}\frac{d\mathrm{E}_n}{dr}\right)\right\} + \frac{2(n+1)}{r}\frac{d\mathrm{E}_n}{dr}\right]$$

$$= \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d\mathrm{E}_n}{dr}\right) + \frac{2(n+2)}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\mathrm{E}_n}{dr}\right),$$

and

$$f_n + \frac{3}{r}\frac{d\mathrm{E}_n}{dr} = \frac{d^2\mathrm{E}_n}{dr^2} + \frac{2(n+1)}{r}\frac{d\mathrm{E}_n}{dr} + \frac{3}{r}\frac{d\mathrm{E}_n}{dr}$$

$$= r\frac{d}{dr}\left(\frac{1}{r}\frac{d\mathrm{E}_n}{dr}\right) + \frac{2(n+3)}{r}\frac{d\mathrm{E}_n}{dr};$$

and thus the equation for Q_n becomes

$$\frac{d^2Q_n}{dr^2}+\frac{2(n+2)}{r}\frac{dQ_n}{dr}+k^2Q_n$$

$$=-\frac{1}{2n+1}\left[\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{dE_n}{dr}\right)+\frac{2(n+2)}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{dE_n}{dr}\right)\right]$$

$$+\frac{1}{2n+1}\frac{k^2}{h^2}\left[\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{dE_n}{dr}\right)+\frac{2(n+2)}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{dE_n}{dr}\right)+s^2r\frac{d}{dr}\left(\frac{1}{r}\frac{dE_n}{dr}\right)\right.$$

$$\left.+2(n+3)s^2\left(\frac{1}{r}\frac{dE_n}{dr}\right)\right]$$

$$=-\frac{1}{2n+1}\left[\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{dE_n}{dr}\right)+\frac{2(n+2)}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{dE_n}{dr}\right)+k^2\left(\frac{1}{r}\frac{dE_n}{dr}\right)\right]$$

$$+\frac{1}{2n+1}\left[k^2\left(\frac{1}{r}\frac{dE_n}{dr}\right)-\frac{k^2}{h^2}\{h^2+(n+6)s^2\}\left(\frac{1}{r}\frac{dE_n}{dr}\right)+\frac{k^2}{h^2}2(n+3)s^2\left(\frac{1}{r}\frac{dE_n}{dr}\right)\right]$$

by the equation (40). Hence we have

$$Q_n=-\frac{1}{2n+1}\frac{1}{r}\frac{dE_n}{dr}+Q'_n, \qquad (49)$$

where Q'_n satisfies the equation

$$\frac{d^2Q'_n}{dr^2}+\frac{2(n+2)}{r}\frac{dQ'_n}{dr}+k^2Q'_n=\frac{n}{2n+1}\frac{k^2}{h^2}s^2\frac{1}{r}\frac{dE_n}{dr}. \qquad (50)$$

11. To find the form of P'_n, we assume

$$P'_n=\frac{n+1}{2n+1}\frac{k^2}{h^2}s^2A(p_0+p_2r^2+p_4r^4+\ldots). \qquad (51)$$

Then equations (48) and (37) give

$$k^2p_0+2(2n+1)p_2=-\frac{2n+1}{h^2+(n+4)s^2},$$

$$k^2p_2+4(2n+3)p_4=\tfrac{1}{2},$$

$$k^2p_4+6(2n+5)p_6=-\frac{h^2+(n+6)s^2}{2.4.(2n+3)},$$

$$k^2p_6+8(2n+7)p_8=\frac{\{h^2+(n+6)s^2\}\{h^2+(n+8)s^2\}}{2.4.6(2n+3)(2n+5)},$$

$$\ldots,$$

$$k^2p_{2\kappa}+(2\kappa+2)(2n+2\kappa+1)p_{2\kappa+2}$$

$$=(-1)^{\kappa+1}\frac{\{h^2+(n+6)s^2\}\{h^2+(n+8)s^2\}\ldots\{h^2+(n+2\kappa+2)s^2\}}{2.4\ldots2\kappa(2n+3)(2n+5)\ldots(2n+2\kappa-1)},$$

$$\ldots.$$

By these equations p_0 is left arbitrary, and p_2, $p_4 \ldots$ are determined when p_0 is chosen. As we need a particular integral only of the equation for P'_n, we may choose p_0 in any way that may be convenient. We shall put $p_0 = 0$. Then

$$\left.\begin{aligned}
p_2 &= -\frac{1}{2\{h^2+(n+4)s^2\}}, \quad p_4 = \frac{1}{2.4\,(2n+3)} + \frac{1}{2.4.(2n+3)}\frac{k^2}{h^2+(n+4)s^2},\\
p_6 &= -\frac{1}{2.4.6\,(2n+3)(2n+5)}\left\{h^2+(n+6)s^2+k^2+\frac{k^4}{h^2+(n+4)s^2}\right\},\\
p_8 &= \frac{1}{2.4.6.8\,(2n+3)(2n+5)(2n+7)}\Big[\{h^2+(n+6)s^2\}\{h^2+(n+8)s^2\}\\
&\qquad\qquad +k^2\{h^2+(n+6)s^2\}+k^4+\frac{k^6}{h^2+(n+4)s^2}\Big],\\
&\ldots
\end{aligned}\right\} \quad (52)$$

To find the form of Q'_n, we assume

$$Q'_n = \frac{n}{2n+1}\frac{k^2}{h^2}s^2 A\,(q_0+q_2r^2+q_4r^4+\ldots). \quad \ldots \ldots \quad (53)$$

Then equations (50) and (38) give

$$k^2q_0+2\,(2n+5)\,q_2 = \frac{1}{2n+3},$$

$$k^2q_2+4\,(2n+7)\,q_4 = -\frac{h^2+(n+6)s^2}{2.(2n+3)(2n+5)},$$

$$k^2q_4+6\,(2n+9)\,q_6 = \frac{\{h^2+(n+6)s^2\}\{h^2+(n+8)s^2\}}{2.4\,(2n+3)(2n+5)(2n+7)},$$

$$k^2q_{2\kappa}+(2\kappa+2)(2n+2\kappa+5)\,q_{2\kappa+2} = (-)^\kappa\frac{\{h^2+(n+6)s^2\}\{h^2+(n+8)s^2\}\ldots\{h^2+(n+2\kappa+4)s^2\}}{2.4\ldots(2\kappa+2)\,(2n+3)\,(2n+5)\ldots(2n+2\kappa+3)},$$

$$\ldots.$$

As before, q_0 can be chosen arbitrarily, and then q_2, $q_4, \ldots$ are known. We observe that if we put

$$q_0 = \frac{2p_2}{2n+3}, \quad q_2 = \frac{4p_4}{2n+5}, \quad \ldots, \quad q_{2\kappa} = \frac{(2\kappa+2)\,p_{2\kappa+2}}{2n+2\kappa+3}, \ldots$$

the sequence equations for the q's are transformed into the sequence equations for the p's, beginning with the equation containing p_2, p_4. We shall therefore choose q_0 to be $(2n+3)^{-1}2p_2$, and then

$$Q'_n = \frac{n}{2n+1}\frac{k^2}{h^2}s^2A\left\{\frac{2p_2}{2n+3}+\frac{4p_4}{2n+5}r^2+\frac{6p_6}{2n+7}r^4+\ldots\right\}. \quad \ldots \quad (54)$$

This choice of the q's amounts to subjecting the functions P'_n and Q'_n to the equation

$$\frac{d}{dr}(r^{2n+3}Q'_n) = \frac{n}{n+1} r^{2n+1} \frac{dP'_n}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

To see that this equation is compatible with the differential equations (48) and (50) for P'_n and Q'_n, we observe that

$$\frac{d\theta_n}{dr} = \frac{1}{r^{2n+1}} \frac{d}{dr}\left\{r^{2n+3}\left(\frac{1}{r}\frac{dE_n}{dr}\right)\right\},$$

and that from the equation (48) we can form the equation

$$\left\{\frac{d^2}{dr^2} - \frac{2(n+1)}{r}\frac{d}{dr} + k^2 + \frac{2(n+1)}{r^2}\right\}\left(r^{2n+1}\frac{dP'_n}{dr}\right) = \frac{n+1}{2n+1}\frac{k^2}{h^2}s^2\frac{d}{dr}\left\{r^{2n+3}\left(\frac{1}{r}\frac{dE_n}{dr}\right)\right\},$$

while from (50) we can form the equation

$$\left\{\frac{d^2}{dr^2} - \frac{2(n+1)}{r}\frac{d}{dr} + k^2 + \frac{2(n+1)}{r^2}\right\}\left\{\frac{d}{dr}(r^{2n+3}Q'_n)\right\} = \frac{n}{2n+1}\frac{k^2}{h^2}s^2\frac{d}{dr}\left\{r^{2n+3}\left(\frac{1}{r}\frac{dE_n}{dr}\right)\right\}.$$

12. We have still to satisfy the condition

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = f_n\omega_n . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

If we form the expression in the left-hand member from the expressions of the type (44) for u_1, v_1, w_1, we have

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = \left\{\frac{n}{r}\frac{dP_n}{dr} - \frac{n+1}{r^{2n+2}}\frac{d}{dr}(r^{2n+3}Q_n)\right\}\omega_n.$$

By means of the formulæ (47) and (49) for P_n and Q_n, the coefficient of ω_n in the right-hand member of this equation is transformed into

$$\frac{1}{2n+1}\left[\frac{n}{r}\frac{d\theta_n}{dr} + \frac{n+1}{r^{2n+2}}\frac{d}{dr}\left(r^{2n+2}\frac{dE_n}{dr}\right)\right] + \frac{n+1}{r^{2n+2}}\left[\frac{n}{n+1}r^{2n+1}\frac{dP'_n}{dr} - \frac{d}{dr}(r^{2n+3}Q'_n)\right].$$

The first term is equal to

$$\frac{1}{2n+1}[nf_n + (n+1)f_n] \quad \text{or} \quad f_n,$$

and the second term vanishes identically in virtue of equation (55). It follows that, with our choice of p_0 and q_0, the equation (56) is satisfied identically.

13. We have now completed the determination of the forms of u, v, w in terms of the spherical solid harmonics ω_n, F_n, ϕ_n, and of certain functions of r, viz.: f_n, θ_n, E_n,

P'_n, Q'_n, ψ_n. Various relations between these functions have been noted incidentally. It will be convenient hereafter to have noted the following properties of $\psi_n(kr)$:—

$$\psi_n(kr) = \frac{(-1)^n}{1.3.5\ldots(2n+1)}\left\{1 - \frac{k^2r^2}{2(2n+3)} + \frac{k^4r^4}{2.4(2n+3)(2n+5)} - \ldots\right\}, \quad (57)$$

$$\left\{\frac{d^2}{d(kr)^2} + \frac{2(n+1)}{kr}\frac{d}{d(kr)} + 1\right\}\psi_n(kr) = 0, \quad (58)$$

$$kr\frac{d\psi_n(kr)}{d(kr)} = k^2r^2\psi_{n+1}(kr) = -\{\psi_{n-1}(kr) + (2n+1)\psi_n(kr)\}. \quad (59)$$

Adjustment of the Harmonics to Satisfy the Boundary Conditions.

14. In order to express F_n in terms of ω_n and ϕ_n we use the condition that $4\pi\gamma\rho_0(E_n\omega_n + F_n)$ is the potential at points within the sphere $r = a$ of a distribution of volume density within the sphere and of surface density on the sphere. The corresponding external potential is

$$4\pi\gamma\rho_0(a/r)^{2n+1}\{E_n(a)\omega_n + F_n\},$$

where $E_n(a)$ is the value of E_n at $r = a$. The surface density is $\rho_0 U_a$, where U_a is the value at $r = a$ of the radial displacement U. Hence we have the equation

$$\frac{\partial}{\partial r}\left[\left(\frac{a}{r}\right)^{2n+1}\{E_n(a)\omega_n + F_n\}\right] - \frac{\partial}{\partial r}(E_n\omega_n + F_n) = -U,$$

which holds at the surface $r = a$; it gives

$$\left[\left(\frac{2n+1}{r}E_n + \frac{dE_n}{dr}\right)\omega_n + \frac{2n+1}{r}F_n\right]_{r=a} = U_a.$$

Now

$$U = \left\{\frac{n}{r}P_n - (n+1)rQ_n\right\}\omega_n - \frac{3ns^2}{rh^2}F_n + \frac{n}{r}\{\psi_{n-1}(kr) + k^2r^2\psi_{n+1}(kr)\}\phi_n.$$

It follows that the equation

$$\left\{\frac{2n+1}{a} + \frac{3ns^2}{ah^2}\right\}F_n = \omega_n\left[\frac{n}{r}P_n - (n+1)rQ_n - \frac{dE_n}{dr} - \frac{2n+1}{r}E_n\right]_{r=a}$$
$$+ \phi_n\frac{n}{a}\{\psi_{n-1}(ka) + k^2a^2\psi_{n+1}(ka)\}$$

holds at the surface $r = a$. Since this equation connects the values at $r = a$ of three spherical solid harmonics of the same degree n, it holds for all values of r, and gives

a generally valid expression of F_n in terms of ω_n and ϕ_n. By means of equations (35) and (59) the equation becomes

$$F_n = -\frac{h^2}{3ns^2+(2n+1)h^2}[\{\theta_n - nP_n + (n+1)a^2Q_n\}_{r=a}\,\omega_n + n(2n+1)\psi_n(ka)\phi_n]. \quad . \quad (60)$$

15. The three remaining conditions which hold at the surface $r = a$ are expressed by equations of the type

$$\left(\frac{k^2}{h^2}-2\right)x\Delta + \frac{\partial}{\partial x}(rU) + r\frac{\partial u}{\partial r} - u + \frac{k^2}{h^2}s^2 xrU = 0. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Every term of the left-hand member can be expressed in terms of the spherical solid harmonics

$$\frac{\partial \omega_n}{\partial x},\quad r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right),\quad \frac{\partial \phi_n}{\partial x},\quad r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right).$$

We have

$$x\Delta = x\omega_n \,.\, f_n = \frac{r^2}{2n+1}f_n\frac{\partial \omega_n}{\partial x} - \frac{r^{2n+3}}{2n+1}f_n\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right).$$

Also

$$\begin{aligned} rU = {} & \{nP_n - (n+1)r^2Q_n\}\,\omega_n - n(2n+1)\psi_n(kr)\phi_n \\ & + \frac{3ns^2}{3ns^2+(2n+1)h^2}[\{\theta_n - nP_n + (n+1)a^2Q_n\}_{r=a}\,.\,\omega_n + n(2n+1)\psi_n(ka)\phi_n], \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\partial}{\partial x}(rU) = {} & \left[nP_n - (n+1)r^2Q_n + \frac{3ns^2}{3ns^2+(2n+1)h^2}\{\theta_n - nP_n + (n+1)a^2Q_n\}_{r=a}\right]\frac{\partial \omega_n}{\partial x} \\ & - \left[n(2n+1)\psi_n(kr) - \frac{3ns^2}{3ns^2+(2n+1)h^2}\psi_n(ka)\right]\frac{\partial \phi_n}{\partial x} \\ & + \frac{d}{dr}\{nP_n - (n+1)r^2Q_n\}\frac{r}{2n+1}\left\{\frac{\partial \omega_n}{\partial x} - r^{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\} \\ & - nr\frac{d\psi_n(kr)}{dr}\left\{\frac{\partial \phi_n}{\partial x} - r^{2n+1}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)\right\}. \end{aligned}$$

Again,

$$\begin{aligned} r\frac{\partial u}{\partial r} - u = {} & \left\{r\frac{dP_n}{dr} + (n-2)P_n\right\}\frac{\partial \omega_n}{\partial x} + \left\{r\frac{d}{dr}(r^{2n+3}Q_n) - (n+3)r^{2n+3}Q_n\right\}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right) \\ & + \frac{3(n-2)s^2}{3ns^2+(2n+1)h^2}\left[\{\theta_n - nP_n + (n+1)a^2Q_n\}_{r=a}\frac{\partial \omega_n}{\partial x} + n(2n+1)\psi_n(ka)\frac{\partial \phi_n}{\partial x}\right] \\ & + \left\{r\frac{d\psi_{n-1}(kr)}{dr} + (n-2)\psi_{n-1}(kr)\right\}\frac{\partial \phi_n}{\partial x} \\ & - \frac{n}{n+1}k^2\left[r\frac{d}{dr}\{r^{2n+3}\psi_{n+1}(kr)\} - (n+3)r^{2n+3}\psi_{n+1}(kr)\right]\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right). \end{aligned}$$

Further,

$$xr\mathrm{U} = \left[n\mathrm{P}_n-(n+1)r^2\mathrm{Q}_n + \frac{3ns^2}{3ns^2+(2n+1)h^2}\{\theta_n-n\mathrm{P}_n+(n+1)a^2\mathrm{Q}_n\}_{r=a}\right]\frac{r^2}{2n+1}\left\{\frac{\partial\omega_n}{\partial x}-r^{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\}$$
$$-n(2n+1)\left\{\psi_n(kr)-\frac{3ns^2}{3ns^2+(2n+1)h^2}\psi_n(ka)\right\}\frac{r^2}{2n+1}\left\{\frac{\partial\phi_n}{\partial x}-r^{2n+1}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)\right\}.$$

16. The equations which hold at the surface $r = a$ can be arranged in such forms as

$$\mathrm{A}_n\frac{\partial\omega_n}{\partial x}+\mathrm{B}_n r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)+\mathrm{C}_n\frac{\partial\phi_n}{\partial x}+\mathrm{D}_n r^{2n+3}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)=0, \quad . \quad . \quad . \quad (62)$$

in which A_n, B_n, C_n, D_n are certain functions of a, viz.:

$$\mathrm{A}_n = \left(\frac{k^2}{h^2}-2\right)\frac{a^2}{2n+1}f_n+\frac{3ns^2}{3ns^2+(2n+1)h^2}\theta_n+\frac{(2n+1)h^2}{3ns^2+(2n+1)h^2}\{n\mathrm{P}_n-(n+1)a^2\mathrm{Q}_n\}$$
$$+\frac{a}{2n+1}\frac{d}{da}\{n\mathrm{P}_n-(n+1)a^2\mathrm{Q}_n\}+a\frac{d\mathrm{P}_n}{da}+(n-2)\mathrm{P}_n$$
$$+\frac{3(n-2)s^2}{3ns^2+(2n+1)h^2}[\theta_n-n\mathrm{P}_n+(n+1)a^2\mathrm{Q}_n]$$
$$+\frac{s^2k^2a^2}{(2n+1)h^2}\left[\frac{3ns^2}{3ns^2+(2n+1)h^2}\theta_n+\frac{(2n+1)h^2}{3ns^2+(2n+1)h^2}\{n\mathrm{P}_n-(n+1)a^2\mathrm{Q}_n\}\right], \quad . \quad (63)$$

$$\mathrm{B}_n = -\left(\frac{k^2}{h^2}-2\right)\frac{1}{2n+1}f_n-\frac{1}{2n+1}\frac{1}{a}\frac{d}{da}\{n\mathrm{P}_n-(n+1)a^2\mathrm{Q}_n\}+a\frac{d\mathrm{Q}_n}{da}+n\mathrm{Q}_n$$
$$-\frac{s^2k^2}{(2n+1)h^2}\left[\frac{3ns^2}{3ns^2+(2n+1)h^2}\theta_n+\frac{(2n+1)h^2}{3ns^2+(2n+1)h^2}\{n\mathrm{P}_n-(n+1)a^2\mathrm{Q}_n\}\right], \quad . \quad (64)$$

$$\mathrm{C}_n = -n\left[(2n+1)\psi_n(ka)+a\frac{d}{da}\psi_n(ka)-\frac{3ns^2(2n+1)}{3ns^2+(2n+1)h^2}\psi_n(ka)\right]$$
$$+\frac{3(n-2)s^2n(2n+1)}{3ns^2+(2n+1)h^2}\psi_n(ka)+a\frac{d}{da}\psi_{n-1}(ka)+(n-2)\psi_{n-1}(ka)$$
$$-\frac{n(2n+1)s^2k^2a^2}{3ns^2+(2n+1)h^2}\psi_n(ka), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

$$\mathrm{D}_n = \frac{n}{a}\frac{d}{da}\psi_n(ka)-\frac{n}{n+1}k^2\left\{a\frac{d}{da}\psi_{n+1}(ka)+n\psi_{n+1}(ka)\right\}$$
$$+\frac{n(2n+1)s^2k^2}{3ns^2+(2n+1)h^2}\psi_n(ka). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

In these formulæ a is supposed to be written instead of r in the expressions for f_n, θ_n, P_n, Q_n.

17. We may express A_n and B_n in more convenient forms by the use of the identities

$$nP_n-(n+1)\,a^2Q_n = \theta_n-(n+1)\,E_n+nP'_n-(n+1)\,a^2Q'_n,$$

$$\frac{d\theta_n}{da} = af_n,$$

$$\frac{d^2E_n}{da^2}+\frac{2(n+1)}{a}\frac{dE_n}{da} = f_n,$$

$$a\frac{dE_n}{da} = \theta_n-(2n+1)\,E_n,$$

$$a\frac{dQ'_n}{da}+(2n+3)\,Q'_n = \frac{n}{n+1}\frac{1}{a}\frac{dP'_n}{da}.$$

We find

$$A_n = \frac{k^2}{h^2}\frac{a^2}{2n+1}(f_n+s^2\theta_n)+a\frac{dP'_n}{da}+2(n-1)\left(\frac{\theta_n}{2n+1}+P'_n\right)$$
$$+\frac{6(n-1)\,s^2-k^2s^2a^2}{3ns^2+(2n+1)h^2}\{(n+1)\,E_n-nP'_n+(n+1)\,a^2Q'_n\}, \quad . \quad . \quad (67)$$

$$B_n = -\frac{k^2}{h^2}\frac{1}{2n+1}(f_n+s^2\theta_n)+\frac{2n+4}{2n+1}\frac{1}{a}\frac{dE_n}{da}+a\frac{dQ'_n}{da}-Q'_n$$
$$+\frac{k^2s^2}{3ns^2+(2n+1)\,h^2}\{(n+1)\,E_n-nP'_n+(n+1)\,a^2Q'_n\}. \quad . \quad . \quad . \quad (68)$$

We may also express C_n and D_n in simpler forms by the use of the equations connecting the ψ functions. We find

$$C_n = -2(n-1)\,k^2a^2\left[\psi_{n+1}(ka)+\frac{(2n+1)^2}{3ns^2+(2n+1)\,h^2}\frac{h^2}{k^2}\frac{1}{a^2}\psi_n(ka)+\frac{ns^2}{3ns^2+(2n+1)\,h^2}\psi_n(ka)\right]$$
$$+\frac{(2n+1)\,h^2k^2a^2}{3ns^2+(2n+1)\,h^2}\psi_n(ka), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

$$D_n = k^2\left[\frac{n}{n+1}\{\psi_n(ka)+(2n+4)\,\psi_{n+1}(ka)\}+\frac{(2n+1)\,ns^2}{3ns^2+(2n+1)\,h^2}\psi_n(ka)\right]. \quad . \quad . \quad . \quad . \quad (70)$$

The Frequency Equation and the Condition of Gravitational Instability.

18. Exactly as in the problem of the vibrating sphere which is free from gravitation, it follows from the equations of type (62) that we must have at once

$$A_n\omega_n+C_n\phi_n = 0, \quad \text{and} \quad B_n\omega_n+D_n\phi_n = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

and the frequency equation is of the form

$$A_nD_n-B_nC_n = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

The forms of all the functions which enter into the expressions of A_n, B_n, C_n, D_n have been determined.

To investigate gravitational instability, we have to determine the conditions which

must hold in order that the frequency equation may be satisfied by $p^2 = 0$. When p^2 vanishes, h^2 and k^2 also vanish, but the quantity k^2/h^2, which is $(\lambda+2\mu)/\mu$, has a determinate value. We may not, however, obtain the result which we seek by first replacing k^2/h^2, wherever it occurs, by $(\lambda+2\mu)/\mu$, and then putting h^2 and k^2 equal to zero wherever they occur, otherwise than in the ratio k^2/h^2. This precautionary statement is necessary because it appears from the formulæ (69) and (70) of § 17 that C_n and D_n both vanish if h^2 and k^2 vanish. Thus we ought to regard the equations (71) as being equivalent to the equations

$$A_n\omega_n + (C_n k^{-2})\, k^2\phi_n = 0, \quad B_n\omega_n + (D_n k^{-2})\, k^2\phi_n = 0\,;$$

in other words, we ought to remove a factor k^2 from the equation $A_nD_n - B_nC_n = 0$ before putting $h^2 = 0$ and $k^2 = 0$. An exceptional case occurs when $n = 1$. In this case $C_n k^{-2}$ vanishes when h^2 vanishes, and it will appear that A_n also vanishes with h^2, and the equations (71) ought to be regarded as equivalent to

$$(A_1 h^{-2})\, \omega_1 + (C_1 h^{-2} k^{-2})\, (k^2\phi_1) = 0, \quad B_1\omega_1 + D_1 k^{-2}\, (k^2\phi_1) = 0,$$

and we must remove a factor h^2k^2 from the equation $A_1D_1 - B_1C_1 = 0$ before putting $h^2 = 0$ and $k^2 = 0$. When we proceed in this way, the equation $A_nD_n - B_nC_n = 0$, with the appropriate factors removed, and with h^2 and k^2 put equal to zero after their removal, becomes an equation to determine s^2a^2, or $\frac{4}{3}\pi\gamma\rho_0{}^2 a^2/(\lambda+2\mu)$. If the equation has a real root, the value so determined for s^2a^2 gives a value of $\lambda+2\mu$ for which instability can occur. Since $k^2\phi_n$ is a finite multiple of ω_n when $\lambda+2\mu$ has any such value, it is certain that the homogeneous spherical configuration really is unstable for such values of $\lambda+2\mu$. If the value of $\lambda+2\mu$ belonging to the body is but little greater than the critical value, the equilibrium is practically unstable; for a large displacement takes place if the sphere begins to vibrate according to the type specified by the degree n of the corresponding spherical harmonic function. For practical stability it is necessary that the value of $\lambda+2\mu$ should be well above any critical value. The equation which yields the critical values contains the constant $(\lambda+2\mu)/\mu$ as well as s^2a^2. It will be convenient to write

$$\nu = \frac{\mu}{\lambda+2\mu} = \frac{h^2}{k^2}. \qquad (73)$$

The value of ν cannot be negative, nor can it be greater than $\frac{3}{4}$. If the POISSON'S ratio $\{\lambda/2(\lambda+\mu)\}$ of the material is positive, ν cannot exceed $\frac{1}{2}$. If the modulus of rigidity μ were very small in comparison with the modulus of compression $\lambda+\frac{2}{3}\mu$, ν would be very small. If the velocity of propagation of waves of dilatation were twice that of waves of distortion, ν would be $\frac{1}{4}$. This appears to be the most appropriate value to assume in the case of the Earth (see § 40 below). Since it is improbable that the ratio of the rigidity to the modulus of compression of the Earth has diminished since the date of consolidation, it will be sufficient for our purpose to

examine the two cases in which $\nu = 0$ and $\nu = \frac{1}{4}$. We have now to discuss the conditions of gravitational instability in respect of the values 0, 1, 2, ... of the number n which specifies the type of vibration.

Instability in Respect of Radial Displacements.

19. The case in which $n = 0$ is the case of a sphere vibrating radially. This case is not very easily included in the foregoing analysis, and it is very easy to investigate it independently. Let U denote the radial displacement. Then U is a function of r, and we have

$$u = \frac{x}{r}\mathrm{U}, \quad v = \frac{y}{r}\mathrm{U}, \quad w = \frac{z}{r}\mathrm{U}, \quad \Delta = \frac{d\mathrm{U}}{dr} + \frac{2\mathrm{U}}{r}.$$

We go back to the equations (15) of the type

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x} + \mu\nabla^2 u + \rho_0 p^2 u + \tfrac{4}{3}\pi\gamma\rho_0{}^2 x\Delta + \rho_0\frac{\partial \mathrm{W}}{\partial x} = 0,$$

where W, the additional potential, is a function of r. This equation is

$$(\lambda+\mu)\frac{x}{r}\frac{d}{dr}\left(\frac{d\mathrm{U}}{dr} + \frac{2\mathrm{U}}{r}\right) + \mu\left\{x\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\left(\frac{\mathrm{U}}{r}\right) + 2\frac{x}{r}\frac{d}{dr}\left(\frac{\mathrm{U}}{r}\right)\right\} + \rho_0 p^2\frac{x}{r}\mathrm{U}$$
$$+ s^2(\lambda+2\mu)\,x\left(\frac{d\mathrm{U}}{dr} + \frac{2\mathrm{U}}{r}\right) + \rho_0\frac{x}{r}\frac{d\mathrm{W}}{dr} = 0. \quad . \quad (74)$$

Now

$$\frac{d^2\mathrm{W}}{dr^2} + \frac{2}{r}\frac{d\mathrm{W}}{dr} = 4\pi\gamma\rho_0\left(\frac{d\mathrm{U}}{dr} + \frac{2\mathrm{U}}{r}\right),$$

and therefore we may write

$$\frac{d\mathrm{W}}{dr} = 4\pi\gamma\rho_0\mathrm{U} + \mathrm{R},$$

where

$$\frac{d\mathrm{R}}{dr} + \frac{2}{r}\mathrm{R} = 0, \quad \text{or} \quad \mathrm{R}r^2 = \text{const.}$$

Since $d\mathrm{W}/dr$ is finite at $r = 0$, we must have $\mathrm{R} = 0$, and thus equation (74) becomes after division by $(\lambda+2\mu)\,x/r$

$$\frac{d^2\mathrm{U}}{dr^2} + \frac{2}{r}\frac{d\mathrm{U}}{dr} - \frac{2\mathrm{U}}{r^2} + s^2\left(r\frac{d\mathrm{U}}{dr} + 2\mathrm{U}\right) + 3s^2\mathrm{U} + h^2\mathrm{U} = 0,. \quad . \quad . \quad . \quad (75)$$

where $s^2 = \frac{4}{3}\pi\gamma\rho_0{}^2/(\lambda+2\mu)$ and $h^2 = p^2\rho_0/(\lambda+2\mu)$. This equation can be solved by means of a series, which is convergent for all finite values of r, in the form

$$\mathrm{U} = \mathrm{A}\left[\frac{r}{3} - \frac{h^2+6s^2}{2\,.\,3\,.\,5}r^3 + \frac{(h^2+6s^2)(h^2+8s^2)}{2\,.\,4\,.\,3\,.\,5\,.\,7}r^5 - \dots\right.$$
$$\left.(-)^\kappa\frac{(h^2+6s^2)(h^2+8s^2)\dots\{h^2+(2\kappa+4)s^2\}}{2\,.\,4\dots2\kappa\,.\,3\,.\,5\dots(2\kappa+3)}r^{2\kappa+1}\dots\right], \quad . \quad (76)$$

where A is an arbitrary constant. The second solution of the differential equation for U becomes infinite at $r = 0$, and thus the above is the most general form for U.

20. The condition that the surface $r = a+U_a$ is free from traction can easily be shown to be the condition that

$$\tfrac{4}{3}\pi\gamma\rho_0^2 r\mathrm{U}+(\lambda+2\mu)\frac{d\mathrm{U}}{dr}+2\lambda\frac{\mathrm{U}}{r} = 0$$

at $r = a$. Hence we have

$$\left(\frac{d\mathrm{U}}{dr}\right)_a+(2-4\nu+s^2a^2)\frac{\mathrm{U}_a}{a} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

where $\nu = \mu/(\lambda+2\mu)$, so that $2\lambda/(\lambda+2\mu) = 2-4\nu$. The frequency equation is therefore

$$\tfrac{1}{3}(1+2-4\nu+s^2a^2)-\frac{h^2+6s^2}{2\,.\,3\,.\,5}a^2(3+2-4\nu+s^2a^2)+\ldots$$
$$(-)^\kappa\frac{(h^2+6s^2)(h^2+8s^2)\ldots\{h^2+(2\kappa+4)s^2\}}{2\,.\,4\ldots2\kappa\,.\,3\,.\,5\ldots(2\kappa+3)}a^{2\kappa}(2\kappa+1+2-4\nu+s^2a^2)\ldots = 0. \quad . \quad (78)$$

The condition of gravitational instability is obtained by putting $h^2=0$. It is

$$\tfrac{1}{3}(3-4\nu+s^2a^2)-\frac{6s^2a^2}{2\,.\,3\,.\,5}(5-4\nu+s^2a^2)+\ldots$$
$$(-)^\kappa\frac{6\,.\,8\ldots(2\kappa+4)\,s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa\,.\,3\,.\,5\ldots(2\kappa+3)}(2\kappa+3-4\nu+s^2a^2)\ldots = 0. \quad . \quad (79)$$

21. The coefficient of $s^{2\kappa}a^{2\kappa}$ in the left-hand member of (79) is

$$\frac{(-)^\kappa}{8}\left\{\frac{(2\kappa+2)(2\kappa+4)}{3\,.\,5\ldots(2\kappa+3)}(2\kappa+3-4\nu)-\frac{2\kappa(2\kappa+2)}{3\,.\,5\ldots(2\kappa+1)}\right\},$$

or

$$\frac{(-)^\kappa}{2}\left[\left\{\frac{1}{3\,.\,5\ldots(2\kappa-1)}+\frac{1}{3\,.\,5\ldots(2\kappa+1)}\right\}\right.$$
$$\left.-\nu\left\{\frac{1}{3\,.\,5\ldots(2\kappa-1)}+\frac{2}{3\,.\,5\ldots(2\kappa+1)}-\frac{1}{3\,.\,5\ldots(2\kappa+3)}\right\}\right],$$

and the equation (79) can be written

$$\tfrac{1}{2}\left[\left(1-x^2+\frac{x^4}{3}-\frac{x^6}{3\,.\,5}+\ldots\right)+\left(1-\frac{x^2}{3}+\frac{x^4}{3\,.\,5}-\ldots\right)\right]$$
$$-\frac{\nu}{2}\left[\left(1-x^2+\frac{x^4}{3}-\ldots\right)+2\left(1-\frac{x^2}{3}+\frac{x^4}{3\,.\,5}-\ldots\right)-\left(\tfrac{1}{3}-\frac{x^2}{3\,.\,5}+\frac{x^4}{3\,.\,5\,.\,7}-\ldots\right)\right] = 0,$$

where x is written for sa. Now we have

$$\int_0^x e^{\frac{1}{2}x^2}dx = e^{\frac{1}{2}x^2}\left(x-\frac{x^3}{3}+\frac{x^5}{3\,.\,5}-\ldots\right),$$

and therefore

$$1-\frac{x^2}{3}+\frac{x^4}{3\,.\,5}-\ldots = x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\,;$$

and the equation may be written

$$1-\nu\left(1-\frac{1}{x^2}\right)-\left[\left(1-\frac{1}{x^2}\right)-\nu\left(1-\frac{2}{x^2}-\frac{1}{x^4}\right)\right]xe^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx = 0. \quad . \quad . \quad . \quad (80)$$

22. The left-hand member of equation (80), being equal to

$$2\left(1-\tfrac{4}{3}\nu\right)+2\left(\tfrac{4}{5}\nu-\tfrac{2}{3}\right)x^2+\ldots,$$

is positive when $x = 0$. To determine its sign for large values of x we observe that

$$\begin{aligned}\int_0^x e^{\frac{1}{2}x^2}dx &= \int_0^1 e^{\frac{1}{2}x^2}dx+\int_1^x x^{-1}.\,e^{\frac{1}{2}x^2}x\,dx \\ &= \int_0^1 e^{\frac{1}{2}x^2}dx+x^{-1}e^{\frac{1}{2}x^2}-e^{\frac{1}{2}}+\int_1^x x^{-2}e^{\frac{1}{2}x^2}dx \\ &= \int_0^1 e^{\frac{1}{2}x^2}dx+x^{-1}e^{\frac{1}{2}x^2}-e^{\frac{1}{2}}+x^{-3}e^{\frac{1}{2}x^2}-e^{\frac{1}{2}}+3\int_1^x x^{-4}e^{\frac{1}{2}x^2}dx \\ &\ldots,\end{aligned}$$

and therefore there is an asymptotic expansion* for $xe^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx$ when x is large in the form

$$1+x^{-2}+3x^{-4}+3\,.\,5x^{-6}+3\,.\,5\,.\,7x^{-8}+\ldots.$$

Hence the expression in the left-hand member of (80) is asymptotically equal to

$$1-\nu\left(1-\frac{1}{x^2}\right)-\left\{1-\frac{1}{x^2}-\nu\left(1-\frac{2}{x^2}-\frac{1}{x^4}\right)\right\}\left[1+\frac{1}{x^2}+\frac{3}{x^4}+\frac{3\,.\,5}{x^6}+\ldots\right].$$

The term of highest degree independent of ν is $-2x^{-4}$; the term of highest degree containing ν is $8\nu x^{-6}$. It follows that the expression is always negative when x is sufficiently great. The expression therefore changes sign for some positive value of x, and the equation (80) has at least one positive root.

23. When $\nu = 0$ the equation (80) becomes

$$1-\frac{x^2-1}{x}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx = 0.$$

If $x^2 < 1$ the left-hand member is necessarily positive. We shall take $x^2 > 1$ and write the equation

$$\frac{x}{x^2-1}e^{\frac{1}{2}x^2}-\int_0^x e^{\frac{1}{2}x^2}dx = 0.$$

Let y denote the left-hand member of this equation. Then we have

$$\frac{dy}{dx} = e^{\frac{1}{2}x^2}\left\{\frac{1}{x^2-1}+\frac{x^2}{x^2-1}-\frac{2x^2}{(x^2-1)^2}-1\right\} = -\frac{2}{(x^2-1)^2}e^{\frac{1}{2}x^2}.$$

Since this expression cannot vanish, the equation cannot have more than one positive root.

* For the suggestion that this step might prove useful in demonstrating the existence of a root of equation (80) I am indebted to Mr. G. H. Hardy, Fellow of Trinity College, Cambridge.

Again, when $\nu = \frac{1}{4}$, the equation can be written

$$3+\frac{1}{x^2}-\left(3-\frac{2}{x^2}+\frac{1}{x^4}\right)\left\{x^2-xe^{-\frac{1}{2}x^2}\int_0^x x^2 e^{\frac{1}{2}x^2}dx\right\} = 0,$$

or

$$\frac{(5-3x^2)\,x^3}{3x^4-2x^2+1}\,e^{\frac{1}{2}x^2}+\int_0^x x^2 e^{\frac{1}{2}x^2}dx = 0,$$

where the left-hand member is certainly positive when $x^2 < 1$; also the differential coefficient of the left-hand member with respect to x is

$$-\frac{8x^2\,(3x^2-2)}{(3x^4-2x^2+1)^2}\,e^{\frac{1}{2}x^2},$$

and this expression cannot vanish for any value of x which is greater than unity. Hence the equation (80) cannot have more than one positive root.

24. Now take $\nu = 0$, and write the equation (80)

$$\frac{1}{x^2-1}-\left(1-\frac{x^2}{3}+\frac{x^4}{3\,.\,5}-\ldots\right) = 0. \quad \text{(81)}$$

When $x^2 = 4$, we have

$$1-\frac{x^2}{3}+\frac{x^4}{3\,.\,5}-\ldots = \left(1-\frac{4}{3}+\frac{4^2}{3\,.\,5}-\ldots\right)$$

$$= \frac{44123}{3\,.\,5\ldots 13}-\frac{4^7}{3\,.\,5\ldots 15}\left(1-\frac{4}{17}\right)-\frac{4^9}{3\,.\,5\ldots 19}\left(1-\frac{4}{21}\right)-\ldots,$$

and

$$\frac{1}{x^2-1} = \frac{1}{3} = \frac{45045}{3\,.\,5\ldots 13}.$$

Hence, when $x^2 = 4$, the sign of the left-hand member of the equation (81) is plus.

When $x^2 = 5$, we have

$$1-\frac{x^2}{3}+\frac{x^4}{3\,.\,5}-\ldots = 1-\frac{5}{3}+\frac{5^2}{3\,.\,5}-\ldots$$

$$= \frac{20056}{3\,.\,7\,.\,9\,.\,11\,.\,13\,.\,3}+\frac{15625}{3\,.\,7\,.\,9\,.\,11\,.\,13\,.\,3\,.\,17}\left(1-\frac{5}{19}\right)$$

$$+\frac{5^8}{3\,.\,7\,.\,9\,.\,11\,.\,13\,.\,3\,.\,17\,.\,19\,.\,21}\left(1-\frac{5}{23}\right)+\ldots.$$

Also

$$\frac{15625}{17}\,\frac{14}{19} > 677,$$

and therefore

$$1-\frac{5}{3}+\frac{5^2}{3\,.\,5}-\ldots > \frac{20733}{3\,.\,7\ldots 13\,.\,3},$$

but

$$\tfrac{1}{4}(3\,.\,7\ldots 13\,.\,3) = 20270+\tfrac{1}{4} < 20733.$$

Hence, when $x^2 = 5$, the sign of the left-hand member of equation (81) is minus. It follows that the value of x^2, or s^2a^2, which satisfies the equation is between 4 and 5.

25. Again, when $\nu = \frac{1}{4}$, equation (80) can be written

$$\frac{3x^2+1}{3x^4-2x^2+1} - \left(1 - \frac{x^2}{3} + \frac{x^4}{3 \, . \, 5} - \ldots\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (82)$$

If we put $x^2 = 4$, the left-hand member becomes

$$\frac{13}{41} - \left(1 - \frac{4}{3} + \frac{4^2}{3 \, . \, 5} - \ldots\right).$$

Now

$$1 - \frac{4}{3} + \frac{4^2}{3 \, . \, 5} - \ldots = \frac{645461}{3 \, . \, 5 \ldots 15} + \frac{4^8}{3 \, . \, 5 \ldots 17}\left(1 - \frac{4}{19}\right) + \ldots,$$

and

$$\frac{13}{41}(3 \, . \, 5 \ldots 15) = 642715 + \frac{10}{41}.$$

Hence $1 - \frac{4}{3} + \frac{4^2}{3 \, . \, 5} - \ldots > \frac{13}{41}$, and the sign of the left-hand member of (82) is minus when $x^2 = 4$. When we put $x^2 = 3$, the left-hand member of (82) becomes

$$\frac{5}{11} - \left(1 - \frac{3}{3} + \frac{3^2}{3 \, . \, 5} - \ldots\right);$$

but

$$1 - \frac{3}{3} + \frac{3^2}{3 \, . \, 5} - \ldots = \frac{3}{5} - \frac{9}{5 \, . \, 7} + \frac{3}{5 \, . \, 7} - \frac{9}{5 \, . \, 7 \, . \, 11}\left(1 - \frac{3}{13} + \ldots\right)$$

$$= \frac{3}{7} - \frac{9}{5 \, . \, 7 \, . \, 11}\left(1 - \frac{3}{13} + \ldots\right),$$

and

$$\frac{5}{11} > \frac{3}{7} > \left(1 - \frac{3}{3} + \frac{3^2}{3 \, . \, 5} - \ldots\right),$$

or the sign of the left-hand member of (82) is plus when $x^2 = 3$. It follows that the root of the equation (82) for x^2 lies between 3 and 4.

Instability in respect of Displacements specified by Harmonics of the First Degree.

26. When $n > 0$, we have to calculate expressions for A_n, B_n, C_n, D_n from the formulæ of § 17. If $n = 1$, we have

$$A_n = A_1 = \frac{1}{\nu}\frac{a^2}{3}(f_1 + s^2\theta_1) + a\frac{dP'_1}{da} - k^2\frac{s^2a^2}{3s^2+h^2}(2E_1 - P'_1 + 2a^2Q'_1). \quad . \quad . \qquad (83)$$

Now if we put $h^2 = 0$ and $k^2 = 0$, we find

$$f_1 = A\left[1 - \frac{7s^2a^2}{2 \, . \, 5} + \frac{9s^4a^4}{2 \, . \, 4 \, . \, 5} - \ldots (-)^\kappa \frac{(2\kappa+5)\, s^{2\kappa}a^{2\kappa}}{2 \, . \, 4 \ldots 2\kappa \, . \, 5} \ldots\right],$$

$$s^2\theta_1 = A\left[-\frac{3}{5} + \frac{s^2a^2}{2} - \frac{7s^4a^4}{2 \, . \, 4 \, . \, 5} + \ldots (-)^{\kappa+1}\frac{(2\kappa+3)\, s^{2\kappa}a^{2\kappa}}{2 \, . \, 4 \ldots 2\kappa \, . \, 5} \ldots\right],$$

$$P'_1 = \frac{2}{3}\frac{1}{\nu}A\left[-\frac{a^2}{2 \, . \, 5} + \frac{s^2a^4}{2 \, . \, 4 \, . \, 5} - \ldots (-)^\kappa \frac{s^{2\kappa-2}a^{2\kappa}}{2 \, . \, 4 \ldots 2\kappa \, . \, 5} \ldots\right],$$

and therefore, if we put $h^2 = 0$ and $k^2 = 0$ in A_1, we get

$$A_1 = \frac{a^2}{3\nu}\frac{2A}{5}\left[1-\frac{s^2a^2}{2}+\frac{s^4a^4}{2.4}-\ldots(-)^{\kappa}\frac{s^{2\kappa}a^{2\kappa}}{2.4\ldots2\kappa}\ldots\right]$$
$$+\frac{2}{3\nu}\frac{A}{5}\left[-a^2+\frac{s^2a^4}{2}-\ldots(-)^{\kappa}\frac{s^{2\kappa-2}a^{2\kappa}}{2.4\ldots(2\kappa-2)}\ldots\right]$$
$$= 0.$$

It follows that A_1 vanishes to the first order in h^2 and k^2, and therefore, as has been explained, we must evaluate the limit of A_1h^{-2} when h^2 and k^2 vanish. We have to expand the terms of f_1, $s^2\theta_1$ and adP'_1/da correctly as far as h^2; in calculating the remaining terms of A_1, we may put h^2 and k^2 equal to zero in E_1, P'_1 and Q'_1.

The terms of f_1 which are of the first order in h^2 are

$$Ah^2\left[-\frac{a^2}{2.5}+\frac{7.9s^2a^4}{2.4.5.7}\left(\frac{1}{7}+\frac{1}{9}\right)-\ldots(-)^{\kappa}\frac{7.9\ldots(2\kappa+5)\,s^{2\kappa-2}a^{2\kappa}}{2.4\ldots2\kappa.5.7\ldots(2\kappa+3)}\left(\frac{1}{7}+\frac{1}{9}+\ldots+\frac{1}{2\kappa+5}\right)\ldots\right].$$

The terms of $s^2\theta_1$ which are of the first order in h^2 are

$$Ah^2\left[\frac{3}{5}\frac{1}{5s^2}-\frac{s^2a^4}{2.4.5}+\ldots(-)^{\kappa+1}\frac{7.9\ldots(2\kappa+3)\,s^{2\kappa-2}a^{2\kappa}}{2.4\ldots2\kappa.5.7\ldots(2\kappa+1)}\left(\frac{1}{7}+\frac{1}{9}+\ldots+\frac{1}{2\kappa+3}\right)\ldots\right].$$

Hence the terms of $(f_1+s^2\theta_1)\,a^2/3\nu$ which are of first order in h^2 are

$$\frac{Ah^2a^2}{15\nu}\left[\frac{3}{5s^2}-\frac{a^2}{2}+\frac{9s^2a^4}{2.4.7}-\ldots(-)^{\kappa}\frac{s^{2\kappa-2}a^{2\kappa}}{2.4\ldots2\kappa}\left\{1+2\left(\frac{1}{7}+\frac{1}{9}+\ldots+\frac{1}{2\kappa+3}\right)\right\}\ldots\right].$$

Again, when we keep those terms only which are of the first order in h^2 and k^2, we find from (52) of § 11,

$$p_2 = \frac{1}{2.5s^2}\frac{h^2}{5s^2},$$

$$p_4 = \frac{1}{2.4.5}\frac{k^2}{5s^2},$$

$$p_6 = -\frac{1}{2.4.6.5.7}(h^2+k^2),$$

$$p_8 = \frac{1}{2.4.6.8.5.7.9}\left\{7.9s^2h^2\left(\frac{1}{7}+\frac{1}{9}\right)+k^2\,7s^2\right\},$$

$$p_{10} = -\frac{1}{2.4.6.8.10.5.7.9.11}\left\{7.9.11s^4h^2\left(\frac{1}{7}+\frac{1}{9}+\frac{1}{11}\right)+k^2\,7.9.s^2\right\},$$

$$\ldots,$$

$$p_{2\kappa} = (-)^{\kappa}\frac{1}{2.4\ldots2\kappa.5.7\ldots(2\kappa+1)}\left\{7.9\ldots(2\kappa+1)\,s^{2\kappa-6}h^2\left(\frac{1}{7}+\frac{1}{9}+\ldots+\frac{1}{2\kappa+1}\right)\right.$$
$$\left.+k^2.7.9\ldots(2\kappa-1)\,s^2\right\};$$

and therefore the terms of adP'_1/da which are of the first order in h^2 and k^2 are

$$\frac{2}{3\nu}s^2\mathrm{A}\left[2p_2a^2+4p_4a^4+\ldots+2\kappa p_{2\kappa}a^{2\kappa}+\ldots\right],$$

where $p_2,\ldots$ have the above values, that is to say, these terms are

$$\frac{2\mathrm{A}h^2}{15\nu}\left[\frac{a^2}{5s^2}-\frac{s^2a^6}{2\,.\,4\,.\,7}+\ldots(-)^\kappa\frac{s^{2\kappa-4}a^{2\kappa}}{2\,.\,4\ldots(2\kappa-2)}\left(\frac{1}{7}+\frac{1}{9}+\ldots+\frac{1}{2\kappa+1}\right)\ldots\right]$$
$$+\frac{2\mathrm{A}k^2}{15\nu}\left[\frac{a^4}{2\,.\,5}-\frac{s^2a^6}{2\,.\,4\,.\,7}+\ldots(-)^\kappa\frac{s^{2\kappa-4}a^{2\kappa}}{2\,.\,4\ldots(2\kappa-2)(2\kappa+1)}\ldots\right].$$

It follows that the terms of the first order in h^2 and k^2 in

$$\frac{a^2}{3\nu}(f_1+s^2\theta_1)+a\frac{dP'_1}{da}$$

are

$$\frac{\mathrm{A}h^2a^2}{15\nu}\left[\frac{1}{s^2}-\frac{a^2}{2}+\frac{s^2a^4}{2\,.\,4}-\ldots(-)^\kappa\frac{s^{2\kappa-2}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\ldots\right]$$

$$+\frac{2\mathrm{A}k^2a^2}{15\nu}\left[\frac{1}{5}\frac{a^2}{2}-\frac{1}{7}\frac{s^2a^4}{2\,.\,4}+\ldots(-)^{\kappa+1}\frac{1}{2\kappa+3}\frac{s^{2\kappa-2}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\ldots\right],$$

or they are

$$\frac{\mathrm{A}h^2a^2}{15s^2}e^{-\frac{1}{2}s^2a^2}+\frac{2\mathrm{A}k^2a^2}{15s^2\nu}\left(\frac{1}{5}\frac{s^2a^2}{2}-\frac{1}{7}\frac{s^4a^4}{2\,.\,4}+\ldots\right).$$

Again, when we put h^2 and k^2 equal to zero, we find

$$2\mathrm{E}_1=\frac{2\mathrm{A}}{5s^2}\left[-1+\frac{s^2a^2}{2}-\frac{s^4a^4}{2\,.\,4}+\ldots(-)^{\kappa+1}\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\ldots\right],$$
$$-\mathrm{P}'_1=\frac{2\mathrm{A}s^2}{15\nu}\left[\frac{a^2}{2s^2}-\frac{a^4}{2\,.\,4}+\frac{s^2a^6}{2\,.\,4\,.\,6}-\ldots(-)^{\kappa+1}\frac{s^{2\kappa-4}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\ldots\right],$$
$$2a^2\mathrm{Q}'_1=\frac{2\mathrm{A}s^2}{15\nu}\left[-\frac{a^2}{5s^2}+\frac{1}{7}\frac{a^4}{2}-\frac{1}{9}\frac{s^2a^6}{2\,.\,4}\ldots(-)^\kappa\frac{s^{2\kappa-4}a^{2\kappa}}{(2\kappa+3)\,2\,.\,4\ldots(2\kappa-2)}\ldots\right],$$

and therefore the terms of the first order in h^2 and k^2 in

$$-\frac{k^2s^2a^2}{3s^2+h^2}(2\mathrm{E}_1-\mathrm{P}'_1+2a^2\mathrm{Q}'_1)$$

are

$$-\frac{1}{3}k^2a^2\left[-\frac{2\mathrm{A}}{5s^2}e^{-\frac{1}{2}s^2a^2}+\frac{2\mathrm{A}}{5s^2\nu}\left\{\frac{s^2a^2}{2\,.\,5}-\frac{s^4a^4}{2\,.\,4\,.\,7}+\ldots(-)^{\kappa+1}\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa(2\kappa+3)}\ldots\right\}\right].$$

Hence the terms of A_1 which are of the first order in h^2 and k^2 are

$$\frac{1}{5}\frac{\mathrm{A}k^2a^2}{s^2}e^{-\frac{1}{2}s^2a^2},$$

and

$$\lim_{h=0}\frac{\mathrm{A}_1}{h^2}=\frac{\mathrm{A}a^2}{5s^2\nu}e^{-\frac{1}{2}s^2a^2}\quad\ldots\ldots\ldots\ldots\quad(84)$$

27. Again, when we put $h^2 = 0$ and $k^2 = 0$, we find that

$$B_1 = -\frac{1}{3\nu}(f_1+s^2\theta_1) + \frac{2}{a}\frac{dE_1}{da} + a\frac{dQ'_1}{da} - Q'_1,$$

where

$$-\frac{1}{3\nu}(f_1+s^2\theta_1) = -\frac{2A}{15\nu}\left[1-\frac{s^2a^2}{2}+\frac{s^4a^4}{2\,.\,4}-\ldots(-)^\kappa\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\ldots\right],$$

$$\frac{2}{a}\frac{dE_1}{da} = \frac{2A}{5}\left[1-\frac{s^2a^2}{2}+\frac{s^4a^4}{2\,.\,4}-\ldots(-)^\kappa\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\ldots\right],$$

$$a\frac{dQ'_1}{da} - Q'_1 = \frac{A}{15\nu}\left[\frac{1}{5}+\frac{s^2a^2}{2\,.\,7}-\frac{3s^4a^4}{2\,.\,4\,.\,9}+\ldots(-)^{\kappa+1}\frac{(2\kappa-1)\,s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa\,.\,(2\kappa+5)}\ldots\right]$$

$$= \frac{A}{15\nu}\left[\left(\frac{6}{5}-1\right)-\frac{s^2a^2}{2}\left(\frac{6}{7}-1\right)+\frac{s^4a^4}{2}\left(\frac{6}{9}-1\right)-\ldots\right.$$

$$\left.(-)^\kappa\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\left(\frac{6}{2\kappa+5}-1\right)\ldots\right]$$

$$= -\frac{A}{15\nu}e^{-\frac{1}{2}s^2a^2}+\frac{2A}{5\nu}\left(\frac{1}{5}-\frac{1}{7}\frac{s^2a^2}{2}+\frac{1}{9}\frac{s^4a^4}{2\,.\,4}-\ldots\right).$$

Hence we have, when $h^2 = 0$ and $k^2 = 0$,

$$B_1 = \frac{2A}{5}\left(1-\frac{1}{2\nu}\right)e^{-\frac{1}{2}s^2a^2}+\frac{2A}{5\nu}\left\{\frac{1}{5}-\frac{1}{7}\frac{s^2a^2}{2}+\frac{1}{9}\frac{s^4a^4}{2\,.\,4}-\ldots\right\}$$

$$= -\frac{A}{5}\left(2-\frac{1}{\nu}\right)e^{-\frac{1}{2}s^2a^2}+\frac{2A}{5\nu}(sa)^{-5}\int_0^{sa}x^4e^{-\frac{1}{2}x^2}dx \quad \ldots\ldots \quad (85)$$

Also when $n = 1$ we have

$$C_n = C_1 = \frac{h^2k^2a^2}{s^2}\psi_n(ka),$$

and therefore

$$\lim_{k=0}\frac{C_1}{h^2k^2} = -\frac{a^2}{3s^2}; \quad \ldots\ldots \quad (86)$$

and we have also

$$D_n = D_1 = k^2[\tfrac{1}{2}\psi_1(ka)+3\psi_2(ka)+(1+h^2/s^2)^{-1}\psi_1(ka)],$$

and therefore

$$\lim_{k=0}\frac{D_1}{k^2} = \frac{3}{2}\left(\frac{-1}{3}\right)+3\frac{1}{3\,.\,5} = -\frac{3}{10}. \quad \ldots\ldots \quad (87)$$

28. In the case where $n = 1$ the condition of gravitational instability, viz., $\lim_{k=0}(A_1h^{-2}D_1k^{-2}-B_1C_1h^{-2}k^{-2}) = 0$, becomes

$$-\frac{3}{10}\frac{Aa^2}{5s^2\nu}e^{-\frac{1}{2}s^2a^2}+\frac{a^2}{3s^2}\left\{\frac{A}{5}\left(2-\frac{1}{\nu}\right)e^{-\frac{1}{2}s^2a^2}+\frac{2A}{5\nu}(sa)^{-5}\int_0^{sa}x^4e^{-\frac{1}{2}x^2}dx\right\} = 0. \quad . \quad (88)$$

But we have

$$\int_0^{sa}x^4e^{-\frac{1}{2}x^2}dx = -(s^3a^3+3sa)e^{-\frac{1}{2}s^2a^2}+\int_0^{sa}3e^{-\frac{1}{2}x^2}dx,$$

and therefore the condition of gravitational instability becomes

$$3\int_0^{sa}e^{-\frac{1}{2}x^2}dx-e^{-\frac{1}{2}s^2a^2}\left\{3sa+(sa)^3+\left(\frac{19}{20}-\nu\right)(sa)^5\right\} = 0.$$

If now we put

$$s^2a^2 = x^2 = 2z^2,$$

the equation becomes

$$\frac{3\sqrt{\pi}}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt\right]-e^{-z^2}\left\{3z+2z^3+\left(\frac{19}{5}-4\nu\right)z^5\right\} = 0, \quad . \quad . \quad . \quad (89)$$

where the factor $2\pi^{-\frac{1}{2}}$ has been inserted because the expression in the square brackets is tabulated in many easily accessible books.

Let y denote the left-hand member of the equation (89). When z is small, y is small of order z^5. In fact, we have

$$\int_0^z e^{-t^2}dt = e^{-z^2}\left[z+\frac{2z^3}{3}+\frac{4z^5}{3\,.\,5}+\ldots+\frac{2^\kappa z^{2\kappa+1}}{3\,.\,5\ldots(2\kappa+1)}+\ldots\right];$$

and when z is small, the first approximation to y is $-(3-4\nu)z^5$. When z is great, $\int_0^z e^{-t^2}dt$ is approximately equal to $\frac{1}{2}\sqrt{\pi}$; and thus y is positive when z is great. The equation (89) has a root zero and at least one positive root. The zero root is irrelevant to our problem; it is introduced in transforming equation (88) into equation (89). Now we have

$$\frac{dy}{dz} = -z^4e^{-z^2}\left\{(15-20\nu)-\frac{2}{5}(19-20\nu)z^2\right\};$$

and, since $15-20\nu$ and $19-20\nu$ are positive when $\nu<\frac{3}{4}$, the expression last written vanishes for one positive value of z. Hence it follows that the equation (89) has only one positive root, and there is one and only one positive value of s^2a^2 which satisfies equation (88).

By means of the tables it can be shown that, when $\nu = 0$, the root z lies between 1·9 and 2, so that s^2a^2 lies between 7·22 and 8. When $\nu = \frac{1}{4}$, the root z lies between 1·8 and 1·9, so that s^2a^2 lies between 6·48 and 7·22.

Stability in Respect of Displacements Specified by Harmonics of the Second and Third Degrees.

29. When $n = 2$ and $h^2 = 0$, $k^2 = 0$, we have

$$A_n = A_2 = \frac{a^2}{5\nu}(f_2+s^2\theta_2)+a\frac{dP'_2}{da}+\frac{2}{5}\theta_2+3E_2+3a^2Q'_2,$$

where

$$f_2 = A\left[1-\frac{8}{2.7}s^2a^2+\frac{8.10}{2.4.7.9}s^4a^4-\ldots(-)^\kappa\frac{8.10\ldots(2\kappa+6)\,s^{2\kappa}a^{2\kappa}}{2.4\ldots2\kappa.7.9\ldots(2\kappa+5)}\ldots\right],$$

$$\theta_2 = A\left[-\frac{5}{6s^2}+\tfrac{1}{2}a^2-\frac{8s^2a^4}{2.4.7}+\ldots(-)^{\kappa+1}\frac{8.10\ldots(2\kappa+4)\,s^{2\kappa-2}a^{2\kappa}}{2.4\ldots2\kappa.7.9\ldots(2\kappa+3)}\ldots\right],$$

$$E_2 = A\left[-\frac{1}{6s^2}+\frac{a^2}{2.7}-\frac{8s^2a^4}{2.4.7.9}+\ldots(-)^{\kappa+1}\frac{8.10\ldots(2\kappa+4)\,s^{2\kappa-2}a^{2\kappa}}{2.4\ldots2\kappa.7.9\ldots(2\kappa+5)}\ldots\right],$$

$$P'_2 = \frac{3s^2A}{5\nu}\left[-\frac{a^2}{12s^2}+\frac{a^4}{2.4.7}-\frac{8s^2a^6}{2.4.6.7.9}+\ldots(-)^\kappa\frac{8.10\ldots(2\kappa+2)\,s^{2\kappa-4}a^{2\kappa}}{2.4\ldots2\kappa.7.9\ldots(2\kappa+3)}\ldots\right],$$

$$Q'_2 = \frac{2As^2}{5\nu}\left[-\frac{1}{6.7.s^2}+\frac{a^2}{2.7.9}-\frac{8s^2a^4}{2.4.7.9.11}+\ldots(-)^{\kappa+1}\frac{8.10\ldots(2\kappa+4)\,s^{2\kappa-2}a^{2\kappa}}{2.4\ldots2\kappa.7.9\ldots(2\kappa+7)}\ldots\right].$$

From these we find

$$\begin{aligned}A_2 = &-\frac{2Aa^2}{5\nu}\left[\frac{1}{6}-\frac{s^2a^2}{2.7}+\frac{8s^4a^4}{2.4.7.9}-\ldots(-)^\kappa\frac{(2\kappa+2)(2\kappa+4)\,s^{2\kappa}a^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+5)}\ldots\right]\\
&-\frac{6Aa^2}{5\nu}\left[\frac{1}{6.7}-\frac{s^2a^2}{2.7.9}+\frac{8s^4a^4}{2.4.7.9.11}-\ldots(-)^\kappa\frac{(2\kappa+2)(2\kappa+4)\,s^{2\kappa}a^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+7)}\ldots\right]\\
&-\frac{3A}{s^2}\left[\frac{1}{6}-\frac{s^2a^2}{2.7}+\frac{8s^4a^4}{2.4.7.9}-\ldots(-)^\kappa\frac{(2\kappa+2)(2\kappa+4)\,s^{2\kappa}a^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+5)}\ldots\right]\\
&-\frac{2A}{s^2}\left[\frac{1}{6}-\frac{s^2a^2}{2.5}+\frac{8s^4a^4}{2.4.5.7}-\ldots(-)^\kappa\frac{(2\kappa+2)(2\kappa+4)\,s^{2\kappa}a^{2\kappa}}{2.4.6.5.7\ldots(2\kappa+3)}\ldots\right].\end{aligned}$$

By means of the identities

$$\frac{(2\kappa+2)(2\kappa+4)}{(2\kappa+3)(2\kappa+5)} = 1-\frac{2}{2\kappa+3}+\frac{3}{(2\kappa+3)(2\kappa+5)},$$

$$\frac{(2\kappa+2)(2\kappa+4)}{(2\kappa+5)(2\kappa+7)} = 1-\frac{6}{2\kappa+5}+\frac{15}{(2\kappa+5)(2\kappa+7)},$$

$$\frac{(2\kappa+2)(2\kappa+4)}{(2\kappa+1)(2\kappa+3)} = 1+\frac{2}{2\kappa+1}-\frac{1}{(2\kappa+1)(2\kappa+3)},$$

we transform the series

$$\frac{1}{5}\left[\frac{1}{6}-\frac{x^2}{2.7}+\frac{8x^4}{2.4.7.9}-\ldots(-)^\kappa\frac{(2\kappa+2)(2\kappa+4)x^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+5)}\ldots\right],$$

$$\frac{1}{5}\left[\frac{1}{6.7}-\frac{x^2}{2.7.9}+\frac{8x^4}{2.4.7.9.11}-\ldots(-)^\kappa\frac{(2\kappa+2)(2\kappa+4)x^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+7)}\ldots\right],$$

$$\left[\frac{1}{6}-\frac{x^2}{2.5}+\frac{8x^4}{2.4.5.7}-\ldots(-)^\kappa\frac{(2\kappa+2)(2\kappa+4)x^{2\kappa}}{2.4.6.5.7\ldots(2\kappa+3)}\ldots\right],$$

respectively, into the forms

$$\frac{1}{16}\left[\left(1-\frac{2}{3}+\frac{3}{3.5}\right)-x^2\left(\frac{1}{3}-\frac{2}{3.5}+\frac{3}{3.5.7}\right)+x^4\left(\frac{1}{3.5}-\frac{2}{3.5.7}+\frac{3}{3.5.7.9}\right)-\ldots\right],$$

$$\frac{1}{16}\left[\left(\frac{1}{3}-\frac{6}{3.5}+\frac{15}{3.5.7}\right)-x^2\left(\frac{1}{3.5}-\frac{6}{3.5.7}+\frac{15}{3.5.7.9}\right)\right.$$
$$\left.+x^4\left(\frac{1}{3.5.7}-\frac{6}{3.5.7.9}+\frac{15}{3.5.7.9.11}\right)-\ldots\right],$$

$$\frac{1}{16}\left[\left(1+2-\frac{1}{3}\right)-x^2\left(1+\frac{2}{3}-\frac{1}{3.5}\right)+x^4\left(\frac{1}{3}+\frac{2}{3.5}-\frac{1}{3.5.7}\right)-\ldots\right],$$

or

$$\frac{1}{16}\left[\left(1-\frac{x^2}{3}+\frac{x^4}{3.5}-\ldots\right)-2\left(\frac{1}{3}-\frac{x^2}{3.5}+\frac{x^4}{3.5.7}-\ldots\right)+3\left(\frac{1}{3.5}-\frac{x^2}{3.5.7}+\frac{x^4}{3.5.7.9}-\ldots\right)\right],$$

$$\frac{1}{16}\left[\left(\frac{1}{3}-\frac{x^2}{3.5}+\frac{x^4}{3.5.7}-\ldots\right)-6\left(\frac{1}{3.5}-\frac{x^2}{3.5.7}+\frac{x^4}{3.5.7.9}-\ldots\right)\right.$$
$$\left.+15\left(\frac{1}{3.5.7}-\frac{x^2}{3.5.7.9}+\frac{x^4}{3.5.7.9.11}-\ldots\right)\right],$$

$$\frac{1}{16}\left[\left(1-x^2+\frac{x^4}{3}-\frac{x^6}{3.5}+\ldots\right)+2\left(1-\frac{x^2}{3}+\frac{x^4}{3.5}-\ldots\right)-\left(\frac{1}{3}-\frac{x^2}{3.5}+\frac{x^4}{3.5.7}-\ldots\right)\right].$$

Now we have

$$1-\frac{x^2}{3}+\frac{x^4}{3.5}-\ldots=x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx,$$

$$\frac{1}{3}-\frac{x^2}{3.5}+\frac{x^4}{3.5.7}-\ldots=\frac{1}{x^2}-x^{-3}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx,$$

$$\ldots,$$

and therefore the three series are respectively equal to

$$\frac{1}{16}\left[\left(1+\frac{2}{x^2}+\frac{3}{x^4}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx-\left(\frac{1}{x^2}+\frac{3}{x^4}\right)\right],$$

$$\frac{1}{16}\left[\frac{1}{x^4}+\frac{15}{x^6}-\left(\frac{1}{x^2}+\frac{6}{x^4}+\frac{15}{x^6}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\right],$$

$$\frac{1}{16}\left[1-\frac{1}{x^2}-\left(x^2-2-\frac{1}{x^2}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\right].$$

On substituting for the three series in the expression for A_2 we find

$$A_2 = \frac{Aa^2}{8\nu}\left[\frac{1}{s^2a^2} - \frac{45}{s^6a^6} - \left(1 - \frac{1}{s^2a^2} - \frac{15}{s^4a^4} - \frac{45}{s^6a^6}\right)(sa)^{-1}e^{-\frac{1}{2}s^2a^2}\int_0^{sa} e^{\frac{1}{2}x^2}dx\right]$$

$$- \frac{A}{16s^2}\left[2 - \frac{17}{s^2a^2} - \frac{45}{s^4a^4} - \left(2s^2a^2 - 19 - \frac{32}{s^4a^4} - \frac{45}{s^6a^6}\right)(sa)^{-1}e^{-\frac{1}{2}s^2a^2}\int_0^{sa} e^{\frac{1}{2}x^2}dx\right]. \quad (90)$$

30. Again we have, when $h^2 = 0$ and $k^2 = 0$,

$$B_2 = -\frac{1}{5\nu}(f_2 + s^2\theta_2) + \frac{8}{5a}\frac{dE_2}{da} + a\frac{dQ'_2}{da} - Q'_2,$$

and with the values already used for $f_2, \ldots$ this gives

$$B_2 = -\frac{A}{5\nu}\left[\frac{1}{6} - \frac{s^2a^2}{2.7} + \frac{8s^4a^4}{2.4.7.9} - \ldots (-)^\kappa \frac{(2\kappa+2)(2\kappa+4)s^{2\kappa}a^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+5)}\ldots\right]$$

$$+ \frac{8A}{5}\left[\frac{1}{7} - \frac{8s^2a^2}{2.7.9} + \frac{8.10s^4a^4}{2.4.7.9.11} - \ldots (-)^\kappa \frac{(2\kappa+2)(2\kappa+4)(2\kappa+6)s^{2\kappa}a^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+7)}\ldots\right]$$

$$- \frac{2A}{5\nu}\left[-\frac{1}{6.7} - \frac{s^2a^2}{2.7.9} + \frac{3.8s^4a^4}{2.4.7.9.11} - \ldots (-)^\kappa \frac{(2\kappa-1)(2\kappa+2)(2\kappa+4)s^{2\kappa}a^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+7)}\ldots\right].$$

The first of these series has already been transformed into

$$-\frac{A}{16\nu}\left[\left(1 + \frac{2}{s^2a^2} + \frac{3}{s^4a^4}\right)(sa)^{-1}e^{-\frac{1}{2}s^2a^2}\int_0^{sa} e^{\frac{1}{2}x^2}dx - \left(\frac{1}{s^2a^2} + \frac{3}{s^4a^4}\right)\right].$$

By means of the identity

$$\frac{(2\kappa+2)(2\kappa+4)(2\kappa+6)}{(2\kappa+3)(2\kappa+5)(2\kappa+7)} = 1 - \frac{3}{2\kappa+3} + \frac{9}{(2\kappa+3)(2\kappa+5)} - \frac{15}{(2\kappa+3)(2\kappa+5)(2\kappa+7)}$$

we transform the series

$$\frac{1}{5}\left[\frac{1}{7} - \frac{8x^2}{2.7.9} + \ldots (-)^\kappa \frac{(2\kappa+2)(2\kappa+4)x^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+7)}\ldots\right]$$

into

$$\frac{1}{16}\left[\left(1 - \frac{x^2}{3} + \frac{x^4}{3.5} - \ldots\right) - 3\left(\frac{1}{3} - \frac{x^2}{3.5} + \frac{x^4}{3.5.7} - \ldots\right) + 9\left(\frac{1}{3.5} - \frac{x^2}{3.5.7} + \frac{x^4}{3.5.7.9} - \ldots\right)\right.$$

$$\left. -15\left(\frac{1}{3.5.7} - \frac{x^2}{3.5.7.9} + \frac{x^4}{3.5.7.9.11} - \ldots\right)\right],$$

which is the same as

$$\frac{1}{16}\left[\left(1 + \frac{3}{x^2} + \frac{9}{x^4} + \frac{15}{x^6}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx - \left(\frac{1}{x^2} + \frac{4}{x^4} + \frac{15}{x^6}\right)\right],$$

and by means of the identity

$$\frac{(2\kappa-1)(2\kappa+2)(2\kappa+4)}{(2\kappa+3)(2\kappa+5)(2\kappa+7)} = 1-\frac{10}{2\kappa+3}+\frac{51}{(2\kappa+3)(2\kappa+5)}-\frac{120}{(2\kappa+3)(2\kappa+5)(2\kappa+7)}$$

we transform the series

$$\frac{1}{5}\left[-\frac{1}{6.7}-\frac{x^2}{2.7.9}+\frac{3.8x^4}{2.4.7.9.11}-\ldots(-)^\kappa\frac{(2\kappa-1)(2\kappa+2)(2\kappa+4)x^{2\kappa}}{2.4.6.7.9\ldots(2\kappa+7)}\ldots\right]$$

into

$$\frac{1}{16}\left[\left(1-\frac{x^2}{3}+\frac{x^4}{3.5}-\ldots\right)-10\left(\frac{1}{3}-\frac{x^2}{3.5}+\frac{x^4}{3.5.7}-\ldots\right)+51\left(\frac{1}{3.5}-\frac{x^2}{3.5.7}+\frac{x^4}{3.5.7.9}-\ldots\right)\right.$$
$$\left.-120\left(\frac{1}{3.5.7}-\frac{x^2}{3.5.7.9}+\frac{x^4}{3.5.7.9.11}-\ldots\right)\right],$$

which is the same as

$$\frac{1}{16}\left[\left(1+\frac{10}{x^2}+\frac{51}{x^4}+\frac{120}{x^6}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}\,dx-\left(\frac{1}{x^2}+\frac{11}{x^4}+\frac{120}{x^6}\right)\right].$$

Hence we have

$$\mathrm{B}_2=\frac{\mathrm{A}}{16\nu}\left[\frac{3}{s^2a^2}+\frac{25}{s^4a^4}+\frac{240}{s^6a^6}-\left(3+\frac{22}{s^2a^2}+\frac{105}{s^4a^4}+\frac{240}{s^6a^6}\right)(sa)^{-1}e^{-\frac{1}{2}s^2a^2}\int_0^{sa}e^{\frac{1}{2}x^2}\,dx\right]$$
$$-\frac{\mathrm{A}}{2}\left[\frac{1}{s^2a^2}+\frac{4}{s^4a^4}+\frac{15}{s^6a^6}-\left(1+\frac{3}{s^2a^2}+\frac{9}{s^4a^4}+\frac{15}{s^6a^6}\right)(sa)^{-1}e^{-\frac{1}{2}s^2a^2}\int_0^{sa}e^{\frac{1}{2}x^2}\,dx\right]. \quad . \quad (91)$$

Again we find

$$\lim_{h=0}\frac{\mathrm{C}_2}{k^2}=-\frac{1}{5.7.9.s^2}(8s^2a^2+175\nu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

and

$$\lim_{h=0}\frac{\mathrm{D}_2}{k^2}=\frac{33}{5.7.9}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

Hence the equation $\lim_{h=0}(\mathrm{A}_2\mathrm{D}_2k^{-2}-\mathrm{B}_2\mathrm{C}_2k^{-2})=0$ is

$$(8x^2+175\nu)\left[\left(\frac{3}{x^2}+\frac{25}{x^4}+\frac{240}{x^6}\right)-\left(3+\frac{22}{x^2}+\frac{105}{x^4}+\frac{240}{x^6}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\right.$$
$$\left.-8\nu\left\{\left(\frac{1}{x^2}+\frac{4}{x^4}+\frac{15}{x^6}\right)-\left(1+\frac{3}{x^2}+\frac{9}{x^4}+\frac{15}{x^6}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\right\}\right]$$
$$+33\left[\left(2-\frac{90}{x^4}\right)-\left(2x^2-2-\frac{30}{x^2}-\frac{90}{x^4}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\right.$$
$$\left.-\nu\left\{\left(2-\frac{17}{x^2}-\frac{45}{x^4}\right)-\left(2x^2-19-\frac{32}{x^2}-\frac{45}{x^4}\right)x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\right\}\right]=0, \quad . \quad (94)$$

where x is written for sa. A factor $\nu^{-1}s^{-2}$ has been omitted in forming the equation, as neither ν nor s is supposed to vanish. The terms of lowest degree in the left-hand

member of the equation (94) will be found to be $\frac{1}{3}80\nu(24\nu-19)$, which is negative when $\frac{3}{4} > \nu > 0$. Hence the left-hand member of this equation is negative when $x = 0$. Also it will be found by the method of asymptotic expansion (*cf.* § 22 above) that, when x is great, the left-hand member is approximately equal to $-1280x^{-4}$ for all values of ν. After the previous cases, in which the corresponding equations have a single root, we are led to expect that in this case there is no root, for it is unlikely that there is more than one. We proceed to verify this expectation in the cases where $\nu = 0$ and $\nu = \frac{1}{4}$.

31. Multiply the left-hand member of (94) by x^4 and put $\nu = 0$. We get

$$8\left[(3x^4+25x^2+240)-(3x^6+22x^4+105x^2+240)\,x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\right]$$
$$+33\left[(2x^4-90)-(2x^6-2x^4-30x^2-90)\,x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx\right] = 0,$$

or, since 10 is a factor,

$$9x^4+20x^2-105-(9x^6+11x^4-15x^2-105)\,x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx = 0.$$

The term of lowest degree in the left-hand member, when expanded in powers of x, is $-\frac{16}{3}x^6$; when x is great, the left-hand member approximates to -128. Now multiply by $xe^{\frac{1}{2}x^2}$, and put

$$y = (9x^5+20x^3-105x)\,e^{\frac{1}{2}x^2}-(9x^6+11x^4-15x^2-105)\int_0^x e^{\frac{1}{2}x^2}dx.$$

We know that when x is small y is small and negative of the order $-\frac{16}{3}x^7$, and that when x is great y is great and negative of the order $-128xe^{\frac{1}{2}x^2}$. Now

$$\frac{dy}{dx} = (54x^4-30x^2)\,e^{\frac{1}{2}x^2}-(54x^5+44x^3-30x)\int_0^x e^{\frac{1}{2}x^2}dx,$$

and, if we put z for $x^{-1}\,dy/dx$,

$$z = (54x^3-30x)\,e^{\frac{1}{2}x^2}-(54x^4+44x^2-30)\int_0^x e^{\frac{1}{2}x^2}dx,$$

where z is negative both when x is small and when x is great; also

$$\frac{dz}{dx} = 88x^2e^{\frac{1}{2}x^2}-(216x^3+88x)\int_0^x e^{\frac{1}{2}x^2}dx,$$

and if we put w for $x^{-1}\,dz/dx$,

$$w = 88xe^{\frac{1}{2}x^2}-(216x^2+88)\int_0^x e^{\frac{1}{2}x^2}dx,$$

where w is negative both when x is small and when x is great; and now

$$\frac{dw}{dx} = -128x^2e^{\frac{1}{2}x^2}-432x\int_0^x e^{\frac{1}{2}x^2}dx,$$

which is always negative. Hence w is always negative, and therefore dz/dx is always negative, therefore also z is always negative and dy/dx is always negative, and therefore y is always negative. Thus the equation $y = 0$ has no real root other than the irrelevant root $x = 0$, which was introduced in the process of forming the equation.

Again, when $\nu = \frac{1}{4}$, we multiply the left-hand member of (94) by x^6 and obtain the equation

$$(23x^6+128x^4-70x^2+3675)-(23x^8+105x^6+268x^4+1155x^2+3675)\,x^{-1}e^{-\frac{1}{2}x^2}\int_0^x e^{\frac{1}{2}x^2}dx = 0,$$

of which the left-hand member is of the order $-x^6$ when x is small and $-x^2$ when x is great. We put

$$y = (23x^7+128x^5-70x^3+3675x)\,e^{\frac{1}{2}x^2}-(23x^8+105x^6+268x^4+1155x^2+3675)\int_0^x e^{\frac{1}{2}x^2}dx,$$

and then we put

$$z = \frac{1}{x}\frac{dy}{dx}, \quad w = \frac{1}{x}\frac{dz}{dx}, \quad u = \frac{1}{x}\frac{dw}{dx},$$

and find

$$\frac{du}{dx} = -(512x^4+4552x^2+3640)\,e^{\frac{1}{2}x^2}-8832\int_0^x e^{\frac{1}{2}x^2}dx,$$

which is always negative. Just as before, we deduce that y is always negative.

It is therefore proved that the equation

$$\lim_{h=0}(\mathrm{A}_2\mathrm{D}_2k^{-2}-\mathrm{B}_2\mathrm{C}_2k^{-2}) = 0$$

has no real root.

32. When $n = 3$, and $h^2 = 0$ and $k^2 = 0$, we have

$$\mathrm{A}_3 = \frac{a^2}{7\nu}(f_3+s^2\theta_3)+\frac{4}{7}\theta_3+a\frac{d\mathrm{P}'_3}{da}+\frac{4}{3}(4\mathrm{E}_3+4a^2\mathrm{Q}'_3),$$

where

$$f_3 = \mathrm{A}\left(1-\frac{s^2a^2}{2}+\frac{s^4a^4}{2\,.\,4}-\ldots(-)^\kappa\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\ldots\right),$$

$$\theta_3 = \frac{-\mathrm{A}}{s^2}\left(1-\frac{s^2a^2}{2}+\frac{s^4a^4}{2\,.\,4}-\ldots(-)^\kappa\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa}\ldots\right),$$

$$\mathrm{P}'_3 = \frac{4\mathrm{A}}{7\nu}\left(-\frac{a^2}{7}+\frac{s^2a^4}{2\,.\,9}-\frac{s^4a^6}{2\,.\,4\,.\,11}+\ldots(-)^\kappa\frac{s^{2\kappa-2}a^{2\kappa}}{2\,.\,4\ldots2\kappa\,.\,(2\kappa+5)}\ldots\right),$$

$$\mathrm{E}_3 = \frac{\mathrm{A}}{s^2}\left(-\frac{1}{7}+\frac{s^2a^2}{2\,.\,9}-\frac{s^4a^4}{2\,.\,4\,.\,11}+\ldots(-)^{\kappa+1}\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa\,(2\kappa+7)}\ldots\right),$$

$$\mathrm{Q}'_3 = \frac{3\mathrm{A}}{7\nu}\left(-\frac{1}{7\,.\,9}+\frac{s^2a^2}{2\,.\,9\,.\,11}-\frac{s^4a^4}{2\,.\,4\,.\,11\,.\,13}+\ldots(-)^{\kappa+1}\frac{s^{2\kappa}a^{2\kappa}}{2\,.\,4\ldots2\kappa(2\kappa+5)(2\kappa+7)}\ldots\right).$$

We use the identity

$$\frac{4}{(2\kappa+5)(2\kappa+7)} = \frac{2}{2\kappa+5} - \frac{2}{2\kappa+7}$$

to transform Q'_3 and then we use the results

$$1 - \frac{s^2a^2}{2} + \frac{s^4a^4}{2.4} - \dots = e^{-\frac{1}{2}s^2a^2},$$

$$\frac{1}{7} - \frac{s^2a^2}{2.9} + \frac{s^4a^4}{2\ 4.11} - \dots = (sa)^{-7}\int_0^{sa} x^6 e^{-\frac{1}{2}x^2}\,dx,$$

$$\frac{1}{9} - \frac{s^2a^2}{2.11} + \frac{s^4a^4}{2.4.13} - \dots = (sa)^{-9}\int_0^{sa} x^8 e^{-\frac{1}{2}x^2}\,dx.$$

We thus find

$$A_3 = -\frac{4A}{7\nu s^2}\Big[3(sa)^{-5}\int_0^{sa} x^6 e^{-\frac{1}{2}x^2}\,dx - 2(sa)^{-7}\int_0^{sa} x^8 e^{-\frac{1}{2}x^2}\,dx$$
$$+\nu e^{-\frac{1}{2}s^2a^2} + \frac{28}{3}\nu(sa)^{-7}\int_0^{sa} x^6 e^{-\frac{1}{2}x^2}\,dx\Big]. \quad . \quad (95)$$

Again we have, when $h^2 = 0$ and $k^2 = 0$,

$$B_3 = -\frac{1}{7\nu}(f_3 + s^2\theta_3) + \frac{10}{7}\frac{1}{a}\frac{dE_3}{da} + a\frac{dQ'_3}{da} - Q'_3,$$

and here we have

$$a\frac{dQ'_3}{da} - Q'_3 = -\frac{3A}{7\nu}\Big[-\frac{1}{7.9} - \frac{s^2a^2}{2.9.11} + \frac{3s^4a^4}{2.4.11.13} - \dots(-)^\kappa \frac{(2\kappa-1)s^{2\kappa}a^{2\kappa}}{2.4\dots2\kappa(2\kappa+7)(2\kappa+9)}\dots\Big],$$

while the other series can be obtained from those written above. We use the identity

$$\frac{2\kappa-1}{(2\kappa+7)(2\kappa+9)} = -\frac{4}{2\kappa+7} + \frac{5}{2\kappa+9}$$

to transform the series last written, and we use also the results which we used in obtaining the expression (95) for A_3, and find

$$B_3 = \frac{A}{7\nu}\Big[12(sa)^{-7}\int_0^{sa} x^6 e^{-\frac{1}{2}x^2}\,dx - 15(sa)^{-9}\int_0^{sa} x^8 e^{-\frac{1}{2}x^2}\,dx + 10\nu(sa)^{-9}\int_0^{sa} x^8 e^{-\frac{1}{2}x^2}\,dx\Big]. \quad . \quad (96)$$

We find also the results

$$\lim_{h=0}\frac{C_3}{k^2} = \frac{4}{3.5.7.9.s^2}(2s^2a^2 + 49\nu), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

$$\lim_{h=0}\frac{D_3}{k^2} = \frac{-3}{4.5.7}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

Hence the equation

$$\lim_{h=0}(A_3D_3k^{-2}-B_3C_3k^{-2}) = 0$$

becomes

$$3\left[\left(3x^2+\frac{28}{3}\nu\right)x^{-7}\int_0^x x^6e^{-\frac{1}{2}x^2}dx+\nu e^{-\frac{1}{2}x^2}-2x^{-7}\int_0^x x^8e^{-\frac{1}{2}x^2}dx\right]$$

$$-\frac{4}{27}(2x^2+49\nu)\left[12x^{-7}\int_0^x x^6e^{-\frac{1}{2}x^2}dx-(15-10\nu)x^{-9}\int_0^x x^8e^{-\frac{1}{2}x^2}dx\right] = 0, \quad . \quad (99)$$

where x is written for sa.

33. An irrelevant factor ν has been introduced into the left-hand member of (99). We find that when $x = 0$ this expression becomes

$$\frac{\nu}{81}\left(539-\frac{1960}{3}\nu\right),$$

which is positive for all admissible values of ν. We find also, by the method of asymptotic expansion, that the expression is positive, when x is great, for all values of ν. We proceed to show that, in the important cases $\nu = 0$ and $\nu = \frac{1}{4}$, the equation has no real root. The left-hand member of (99) being an even function of x, we may treat x as positive.

When $\nu = 0$, the left-hand member of equation (99) is

$$\frac{7}{9}x^{-7}\left[7x^2\int_0^x x^6e^{-\frac{1}{2}x^2}\,dx-2\int_0^x x^8e^{-\frac{1}{2}x^2}\,dx\right],$$

which is positive for small values of x. The differential coefficient of the expression within the square brackets is

$$14x\int_0^x x^6e^{-\frac{1}{2}x^2}\,dx+5x^8e^{-\frac{1}{2}x^2},$$

which is positive for all real values of x. Hence the left-hand member of equation (99) cannot be negative if x is positive, or the equation has no real root.

When $\nu = \frac{1}{4}$, the left-hand member of equation (99) becomes

$$x^{-9}\left[\left(\frac{49}{9}x^4-\frac{133}{9}x^2\right)\int_0^x x^6e^{-\frac{1}{2}x^2}\,dx-\left(\frac{62}{27}x^2-\frac{1225}{54}\right)\int_0^x x^8e^{-\frac{1}{2}x^2}\,dx\right]+\frac{3}{4}e^{-\frac{1}{2}x^2} = 0. \quad (100)$$

The expression within the square brackets is greater than

$$5\,(x^4-3x^2)\int_0^x x^6e^{-\frac{1}{2}x^2}\,dx-(3x^2-22)\int_0^x x^8e^{-\frac{1}{2}x^2}\,dx, \quad . \quad . \quad . \quad . \quad . \quad (101)$$

where this expression is obtained from the other by replacing every positive coefficient by the next smaller integer and every negative coefficient by the next greater integer.

Since

$$\int_0^x x^8 e^{-\frac{1}{2}x^2}\,dx = -x^7 e^{-\frac{1}{2}x^2}+7\int_0^x x^6 e^{-\frac{1}{2}x^2}\,dx,$$

the expression (101), when multiplied by 7, is

$$(5x^4-36x^2+154)\int_0^x x^8 e^{-\frac{1}{2}x^2}\,dx+5\,(x^2-3)\,x^9 e^{-\frac{1}{2}x^2}. \quad . \quad . \quad . \quad . \quad (102)$$

Also we have

$$\int_0^x x^8 e^{-\frac{1}{2}x^2}\,dx = x^9 e^{-\frac{1}{2}x^2}\left(\frac{1}{9}+\frac{x^2}{9\,.\,11}+\frac{x^4}{9\,.\,11\,.\,13}+\ldots\right);$$

and therefore the coefficient of $x^9 e^{-\frac{1}{2}x^2}$ in the expansion of (102) is $\frac{1}{9}(154-9\times 15)$, the coefficient of $x^{11}e^{-\frac{1}{2}x^2}$ is $\frac{1}{99}(154-36\times 11+5\times 99)$, and the coefficient of $x^{9+2\kappa}e^{-\frac{1}{2}x^2}$ for all values of κ which are greater than 1 is

$$\frac{1}{9\,.\,11\ldots(2\kappa+5)}\left(\frac{154}{(2\kappa+7)\,(2\kappa+9)}-\frac{36}{2\kappa+7}+5\right),$$

or

$$\frac{20\kappa^2+88\kappa+145}{9\,.\,11\ldots(2\kappa+5)}.$$

Hence all the coefficients are positive, the expression (102) is positive, and the left-hand member of (100) is positive for all positive values of x. Thus, in this case also, the equation (99) has no real root.

Summary of the Solution of the Mathematical Problem.

34. We have now solved in essentials the mathematical problem of the vibrations of a gravitating sphere, initially homogeneous and in a state of hydrostatic pressure, and have found the conditions of gravitational instability. We have shown that, when any normal, or principal, vibration is taking place, the dilatation at a distance r from the centre is specified by the product of a certain function of r and a spherical harmonic of positive integral degree. We have shown further that, in each such mode of vibration, the components of displacement can be expressed in terms of the same spherical harmonic, and that the radial displacement at a point distant r from the centre is the product of a function of r and the same harmonic. We obtained the form of the frequency equation, and the forms of all the functions which enter into its expression.

We proceeded to investigate the conditions which must hold in order that the frequency equation may be satisfied by a zero value of the frequency. We showed that, when such a value is not introduced irrelevantly in the process of forming the equation, its occurrence points to genuine gravitational instability. We found that the condition of such instability is the condition that a certain equation, containing the variable quantity s^2a^2, or $\frac{4}{3}\pi\gamma\rho_0^2/(\lambda+2\mu)$, may be satisfied by a real positive value

of this quantity. The constant μ denotes the rigidity and $\lambda+\frac{2}{3}\mu$ the modulus of compression. When the harmonic specifying the vibrations is of zero degree, that is to say, when the vibrations are radial, we found that the critical value of s^2a^2 lies between 4 and 5 if ν, or $\mu/(\lambda+2\mu)$, is zero, and that it lies between 3 and 4 if ν is $\frac{1}{4}$. In the case of vibrations specified by spherical harmonics of the first degree, we found that the critical value of s^2a^2 lies between 7·22 and 8 if $\nu = 0$, and it lies between 6·48 and 7·22 if $\nu = \frac{1}{4}$. In the cases of vibrations specified by spherical harmonics of the second and third degrees we found that there is no critical value of s^2a^2, or that the sphere is stable, in respect of the corresponding types of displacement, for all values of $\lambda+2\mu$. It was to be expected that the critical values of s^2a^2 would increase rapidly as the complexity of the type of vibration, specified by the degree of the appropriate harmonic, increases; and we appear to be justified in concluding that instability cannot occur in respect of displacements specified by spherical harmonics of any degree higher than the first.

35. The result that the critical values of s^2a^2 are lower when $\nu = \frac{1}{4}$ than when $\nu = 0$ means that a higher value of the constant $\lambda+2\mu$ is required, to secure stability, when there is considerable rigidity than when there is very little rigidity. This result accords with general dynamical principles; for it is a known result, and one which has been shown to be in accordance with such principles, that the frequency of any mode of vibration, involving compression, of a sphere free from gravitation diminishes as $(\lambda+2\mu)/\mu$ diminishes, that is, as ν increases.* Consequently, for a given value of $\gamma\rho_0{}^2a^2$, the value of $\gamma\rho_0{}^2a^2/(\lambda+2\mu)$ which would be required, in order to reduce the frequency to zero, diminishes as ν increases, or the critical value of $\lambda+2\mu$ increases as ν increases.

36. The result that the critical value of s^2a^2 is lower when $n = 0$ than when $n = 1$ means that a higher value of $\lambda+2\mu$ would be required, to secure stability, in the case of radial displacements than in the case of displacements specified by spherical harmonics of the first degree. The spherical body of uniform density could be stable in respect of all types of displacement except radial displacements. If the value of s^2a^2 were intermediate between the critical values corresponding to $n = 0$ and $n = 1$, this would be the case; and the body would tend to take up a different configuration, in which the density would be more concentrated towards the centre. The result that, in the case where $n = 1$ also, there exists a critical value of s^2a^2, which is not more than twice as great as the value associated with $n = 0$, the initial state in both cases being one of uniform density, suggests very strongly that there would be a critical value of $\lambda+2\mu$, in respect of the case $n = 1$, even if the configuration were such that the body was stable as regards radial displacements. We should then have a body with a spherically symmetrical distribution of density, but with elasticity too small for this configuration to be stable in respect of displacements specified by spherical harmonics of the first degree; and it may be inferred that the critical mean

* *Cf.* H. LAMB, *loc. cit.*, *ante*, p. 173.

value of $\lambda+2\mu$ for such a body would not be very different from the critical value obtained for $\lambda+2\mu$ by treating the body as homogeneous, and paying attention to those types of displacement only which are specified by spherical harmonics of the first degree.

37. If this conclusion is admitted, as I think it must be, it would follow that a spherical planet with a spherically symmetrical distribution of density, and stable as regards radial displacements, might be unstable as regards displacements of the type in question; and then it would tend to be displaced in such a way that the boundary, or any concentric sphere, moves to a position in which its centre no longer coincides with the centre of gravity, while the matter in a thin spherical layer becomes condensed in one hemisphere and rarefied in the other. The density being in excess in one hemisphere and in defect in the other, and the excess or defect at any point, at a stated distance from the centre, being proportional to the distance of the point from the bounding plane of the two hemispheres, the distribution of density may be aptly described as "hemispherical," and the state of the body may be described as one of "lateral disturbance." The concentration of density towards one radius, on which the centre of gravity lies, has the effect of diminishing the potential energy of gravitation, and this diminution may more than counterbalance the increment of potential energy due to strain. The proved existence of a critical value for $\lambda+2\mu$ (in the case of a homogeneous body) indicates that this state of things really can occur. An illustration of the nature of a hemispherical distribution of density will be found in §§ 47, 48 below.

38. The results found by JEANS (1903) in the solution of the problem of the gravitating sphere subjected to an external field of force, which balances gravitation throughout the sphere when it is at rest, may be compared with those obtained above in the case where the gravitation is balanced by initial pressure. In JEANS' solution, just as here, the modes of vibration are specified by the spherical harmonics which enter into the expression for the dilatation; and, in any normal mode, the formula for the dilatation contains a single spherical harmonic, and the radial displacement at any stated distance from the centre is proportional to the same harmonic. If the degree of the harmonic exceeds zero, instability can occur for a sufficiently small value of the resistance to compression, whatever the degree of the harmonic may be. It is not restricted to the case where the degree is unity, as it is in our problem of initial stress; but the value of the resistance to compression required for instability diminishes rapidly as the degree of the harmonic increases. Instability enters first when the harmonic is of the first degree,* that is to say, for lateral disturbances. The critical values of s^2a^2 are $6{\cdot}72$ when $\nu = 0$ and $5{\cdot}33$ when $\nu = \frac{1}{4}$, the degree of the harmonic being unity. Since these values are a little less than the critical values found in the solution of the problem of initial stress, it may be concluded that the effect of initial stress, as compared with that of an external field of force, is to

* The question of radial instability was not considered by JEANS.

increase slightly the stability of the body in respect of disturbances specified by harmonics of the first degree, and to increase it enormously in respect of disturbances specified by harmonics of higher degrees.

Application to the Problem of the Gravitational Stability of the Earth.

39. For a body of the same size and mass as the Earth, the values of a and ρ_0 in C.G.S. units are $6\cdot37\times10^8$ and $5\cdot53$; the value of γ being $6\cdot65\times10^{-8}$, the value of $\frac{4}{3}\pi\gamma\rho_0^2a^2$ is $3\cdot46\times10^{12}$. In the following table the first column gives a value of s^2a^2, the second column gives the corresponding value of $\lambda+2\mu$ (the body being of the same size and mass as the Earth), the third and fourth columns give the values of the corresponding moduluses of compression in the cases where $\nu = 0$ and $\nu = \frac{1}{4}$, irrelevant entries being omitted. These moduluses are denoted by k_0 and k_1. The quantities given in the fifth, sixth, and seventh columns are the moduluses of compression of steel, glass, and mercury (denoted by k_s, k_g, k_m).

s^2a^2.	$\lambda+2\mu$.	k_0.	k_1.	k_s.	k_g.	k_m.
—	—	—	—	$1\cdot43\times10^{12}$	—	—
3	$1\cdot15\times10^{12}$	—	$7\cdot68\times10^{11}$	—	—	—
4	$8\cdot64\times10^{11}$	$8\cdot64\times10^{11}$	$5\cdot76\times10^{11}$	—	—	—
5	$6\cdot91\times10^{11}$	$6\cdot91\times10^{11}$	—	—	—	—
—	—	—	—	—	$4\cdot54\times10^{11}$	—
$6\cdot48$	$5\cdot33\times10^{11}$	—	$3\cdot57\times10^{11}$	—	—	—
$7\cdot22$	$4\cdot79\times10^{11}$	$4\cdot79\times10^{11}$	$3\cdot19\times10^{11}$	—	—	—
8	$4\cdot32\times10^{11}$	$4\cdot32\times10^{11}$	—	—	—	—
—	—	—	—	—	—	$2\cdot60\times10^{11}$

According to these results, a homogeneous solid body of the same size and mass as the Earth, with a modulus of compression as great as that of steel, would have complete gravitational stability. If the modulus of compression were equal to, or less than that of glass, the planet would be unstable as regards radial disturbances, and a concentration of density towards the centre would take place. If the critical value of

$\lambda+2\mu$, which was found in the case of lateral disturbances, is assumed to be the critical *mean* value of $\lambda+2\mu$ for a planet in which the mass is condensed towards the centre, then we may say that, if the mean modulus of compression were about equal to that of glass, and there were very little rigidity, the planet would be unstable as regards lateral disturbances; but, if there were considerable rigidity, it would be stable. If, on the other hand, the mean modulus of compression were decidedly less than that of glass, though not so small as that of mercury, the planet would be unstable as regards lateral disturbances, even though it possessed a considerable mean rigidity.

40. In order to settle the question of the gravitational stability or instability of the Earth, we must assign the appropriate values to the constants λ and μ. Lord KELVIN's theory of elastic tides in a solid sphere led to the result that the tidal effective rigidity of the Earth is not less than that of steel. This result suggests that μ should not be taken to be less than $8 \cdot 19 \times 10^{11}$ C.G.S. units; but, since it was obtained by treating the Earth as incompressible, it affords no means of determining the value of λ. JEANS (1903) proposed to deduce the values of λ and μ from the observed velocities of propagation of earthquake shocks. In a homogeneous elastic solid body, free from gravitation and initial stress, irrotational waves of dilatation are propagated with the velocity $[(\lambda+2\mu)/\rho_0]^{\frac{1}{2}}$, where ρ_0 is the density, and equivoluminal waves of distortion are propagated with the velocity $[\mu/\rho_0]^{\frac{1}{2}}$, while waves of a third type are propagated over the surface with a velocity approximately equal to $(0 \cdot 9)[\mu/\rho_0]^{\frac{1}{2}}$. When a great earthquake takes place, the disturbance received at a distance from the source consists of three sets of disturbances: two sets of "preliminary tremors," and the "main shock." The first set of preliminary tremors is received at distant places at such times as it would be if it travelled directly through the Earth with a velocity of about 10 kiloms. per second. The second set of tremors is propagated apparently in a rather less regular fashion, but the times at which it can be observed at distant stations are nearly the same as they would be if it travelled directly through the Earth with a velocity of about 5 kiloms. per second. The main shock is received at distant places at such times as it would be if it travelled over the surface of the Earth with a velocity of about 3 kiloms. per second.* The identification of the three sets of disturbances with the three sets of waves which are theoretically known seems to be inevitable, and the discrepancy between the ratio of velocities of equivoluminal and superficial waves and the ratio of velocities of the second set of tremors and the main shock may be explained by the supposition that, while the velocity of transmission of these tremors depends upon the mean rigidity of the Earth as a whole, the velocity of transmission of the main shock depends upon the average

* Reference may be made to a Memoir by R. D. OLDHAM, "On the Propagation of Earthquake Motion to great distances," London, 'Phil. Trans. Roy. Soc.,' ser. A, 194, 1900, and to the Reports of the Seismological Committee of the British Association, in particular that published in 'Brit. Assoc. Rep.,' 1902.

rigidity of surface rock. Assuming this explanation, we are led to attribute to surface rocks an average rigidity approximately equal to 6×10^{11} C.G.S. units, and to the Earth as a whole the much higher mean rigidity $1{\cdot}38 \times 10^{12}$ C.G.S. units; further, since the ratio of velocities of the first and second set of tremors is approximately 2 : 1, we are led to assume for $\lambda+2\mu$ the value $5{\cdot}53 \times 10^{12}$ C.G.S. units, and for ν, or $\mu/(\lambda+2\mu)$, the value $\frac{1}{4}$. By analogy to the "tidal effective rigidity" we may introduce the phrases "seismic effective rigidity" and "seismic effective modulus of compression"; and the values of these quantities would be $1{\cdot}38 \times 10^{12}$ and $3{\cdot}69 \times 10^{12}$ C.G.S. units respectively. When the value of $\lambda+2\mu$ for the Earth is taken to be $5{\cdot}53 \times 10^{12}$, the corresponding value of s^2a^2 is $0{\cdot}625$. The results of § 39 appear to warrant the conclusion that the moduluses of elasticity of the Earth in its present state are sufficiently great to render a spherically symmetrical configuration completely stable.

41. In obtaining the above values for $\lambda+2\mu$ and μ no account is taken of gravitation or initial stress, and it is possible that the most appropriate values would be a little different from those found above if gravitation and initial stress, to say nothing of heterogeneity of density, could be taken into account. For this reason, although a complete solution of the problem of wave-propagation in a gravitating planet, even when it is regarded as homogeneous, cannot be obtained, the following argument may not be without value:—The equations of vibratory motion of a gravitating sphere in a state of initial pressure have been obtained in § 3 above. From equations (10) and (11) of § 3 we can deduce the equation

$$\frac{\partial^2 \Delta}{\partial t^2} = \frac{\lambda+2\mu}{\rho_0}\left\{\nabla^2\Delta + 6s^2\Delta + s^2 r\frac{\partial \Delta}{\partial r}\right\}, \quad \ldots \ldots \quad (103)$$

and the three equations of the type

$$\frac{\partial^2 \varpi_x}{\partial t^2} = \frac{\mu}{\rho_0}\left\{\nabla^2 \varpi_x - \frac{s^2}{2\nu}\left(y\frac{\partial \Delta}{\partial z} - z\frac{\partial \Delta}{\partial y}\right)\right\}, \quad \ldots \ldots \quad (104)$$

where ϖ_x, ϖ_y, ϖ_z denote the components of rotation, so that

$$2\varpi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \ldots \quad \ldots \ldots \quad (105)$$

In a general way we can see that the terms which contain s^2 in these equations are small compared with the remaining terms; for, if waves of length L are propagated, $\nabla^2\Delta$ is of the order $L^{-2}\Delta$, and $s^2\Delta$ is small in comparison with this in the order s^2L^2, which is comparable with L^2/a^2, since s^2a^2 is comparable with unity. It would thus appear that the velocities of propagation of the waves are not much affected by gravitation and initial stress when the wave-length is small compared with the radius of the sphere; and the conclusion would be applicable to superficial waves as well as to waves of dilatation and waves of distortion, because such waves are, in any case, to be investigated by means of equations of the types (103) and (104).

42. In the case of waves of dilatation the argument can be put in a more definite shape. Let us suppose that, near a place, the waves are plane, so that Δ is a function of x and t, and let us write

$$V_1^2 = (\lambda+2\mu)/\rho_0, \quad \ldots \ldots \ldots \ldots \quad (106)$$

so that V_1 is the velocity of waves of dilatation when gravitation and initial stress are disregarded. We have the equation

$$\frac{\partial^2\Delta}{\partial x^2}+s^2x\frac{\partial\Delta}{\partial x}+6s^2\Delta = \frac{1}{V_1^2}\frac{\partial^2\Delta}{\partial t^2},$$

or

$$\frac{\partial^2}{\partial x^2}(e^{\frac{1}{4}s^2x^2}\Delta)+s^2\left(\frac{11}{2}-\frac{1}{4}s^2x^2\right)(e^{\frac{1}{4}s^2x^2}\Delta) = \frac{1}{V_1^2}\frac{\partial^2}{\partial t^2}(e^{\frac{1}{4}s^2x^2}\Delta).$$

In considering the passage of waves near a place, we may treat the term $-\frac{1}{4}s^2x^2$ in the coefficient of $e^{\frac{1}{4}s^2x^2}\Delta$ as a constant; and then the equation is satisfied by putting

$$e^{\frac{1}{4}s^2x^2}\Delta = B\cos\{2\pi L^{-1}(x-x_0-V'_1t)\},$$

provided that

$$V_1'^2 = V_1^2\left\{1-\frac{s^2L^2}{4\pi^2}\left(\frac{11}{2}-\frac{1}{4}s^2x^2\right)\right\}. \quad \ldots \ldots \quad (107)$$

Since the greatest values of s^2x^2 are comparable with unity, the value of V'_1, the local velocity of transmission, is a little less than V_1, or the actual value of $\lambda+2\mu$ is a little greater than the seismic effective value. The result (107) may be accepted as being not far from the truth in a region large compared with the wave-length, and small compared with the radius, and situated at a considerable distance from the source of disturbance.

43. Since the equations of type (104) contain the dilatation as well as the components of rotation, it appears that the customary law of independence of waves of dilatation and waves of distortion ceases to hold when gravitation and initial stress are taken into account. It appears also that the velocities of propagation, both of those waves which are mainly dilatational and of those which are mainly distortional, depend on the wave-lengths, and, for the same wave-length, they vary from place to place. When the theory can be developed further, these results may possibly prove to be useful in explaining the observed irregularities in the propagation of the tremors which are recorded in the case of great earthquakes. The high values which seismic observations lead us to attribute to the elastic constants of the earth as a whole are in accord with Lord RAYLEIGH's view* that great initial stress increases the effective values both of resistance to compression and of rigidity.

* *Loc. cit.*, *ante*, p. 173.

PART II.

A Past State of Gravitational Instability as a Reason for the existing Distribution of Land and Water.

44. Although the conclusion reached by JEANS (1903), that a spherical planet of the same size, mass and elasticity as the Earth, in its present state, would be in a condition of gravitational stability, is confirmed and strengthened by the present investigation, it by no means follows that the Earth has always been in such a state as it is now. The fact that the mean density of the Earth as a whole is greater than the average density of surface rocks points to a concentration of mass towards the centre, and suggests that such a concentration may have come about through the elasticity having once been too small for a homogeneous state to be stable. We have seen that this would have been the case if the modulus of compression was once as small as, or smaller than, that of glass. But we also saw reason to think that, if the mean modulus of compression was once decidedly less than that of glass, spherically symmetrical states of aggregation would also have been unstable, and the body would have existed in some other state. Further, we saw that, if the body was at rest, the state in which it would have existed is that which we have described as a state of lateral disturbance with a hemispherical distribution of density. The excess of density in one hemisphere and defect in the antipodal hemisphere would have existed alongside of the concentration of mass towards the centre.

45. In the paper already cited JEANS (1903) struck out the idea that the distribution of land and water on the surface of the globe is associated with a past state of gravitational instability. He had found that such instability would manifest itself in what has been called above a hemispherical distribution of density. When the square of the irregularity is neglected, the figure of a planet at rest, with such a distribution of density, is a sphere, but the centre of figure does not coincide with the centre of gravity. On taking account of the square of the irregularity, JEANS found that the surface of the planet, still supposed to be at rest, would be such as can be described roughly as a nearly spherical ellipsoid of revolution, with one half slightly flattened at the middle, and the other slightly tapered in the antipodal direction. The figure was described as "pear-shaped," the "pear" having a blunt end, a sharper end, and a waist. The waters of the ocean would presumably collect in the hollow of the waist, and JEANS pointed out that there is some resemblance of the shape of the Earth to this figure, although the "stalk" end of the "pear" was difficult to discover.

In the same year a paper was published by W. J. SOLLAS,* in which it was concluded from a discussion of the geographical facts that the shape of the Earth

* "The Figure of the Earth," 'Quart. J. Geol. Soc.,' 59 (1903), p. 180.

resembles that of a "pear"; but SOLLAS' and JEANS' "pears" have little in common beyond the name. JEANS' ideal distribution would consist of a hemisphere which is nearly all land, and an antipodal hemisphere which is nearly all ocean, with a central island in the middle of this ocean. SOLLAS' account of the actual distribution is that in one hemisphere there is a central continent (Africa) nearly surrounded by a belt of seas, while in the antipodal hemisphere there is a central ocean (the Pacific) nearly surrounded by a ring of land, the belt and ring being broken at three places, which are distributed nearly symmetrically around the centres of the two hemispheres. This description suggests very strongly a mathematical account expressed in terms of surface harmonics of the third degree.

If we neglect the rotation of the planet, and regard it as at rest under no external forces, we can reach no other result than that reached by JEANS, viz., that, if the modulus of compression was once so small that a spherically symmetrical state of aggregation would have been unstable, the state of the planet would have been one of lateral disturbance with a hemispherical distribution of density. We should not be in a position to account at all for the geographical facts as presented by SOLLAS.

46. The Earth is a rotating globe, and it is now generally believed to be the larger of two fragments into which a single body has been broken up; the other fragment is the Moon. In the early history of the Earth-Moon system the two fragments rotated, nearly as a single rigid body; the period of revolution of the Moon was nearly the same as the period of rotation of the Earth. We wish to trace the consequences of supposing that the average elasticity of the material was once much smaller than it is at present—that the average modulus of compression was more of the order of that of mercury, or even water, than of that of glass or steel, and the average rigidity was smaller in comparison with the modulus of compression than it is to-day. We have the problem of determining the distribution of density within the planet, and the consequent shape of its surface. The problem cannot be solved completely, but we can make some progress with it; and we can then attempt to discover the extent to which our results accord with geographical observation. In so far as the accord is good we may regard geography as supporting the hypothesis as to the past state of the Earth.

Illustration of the Nature of a Hemispherical Distribution of Density.

47. We have reason to think that, in the absence of rotation and external forces, the planet, if of sufficiently small elasticity, would have been in the state which we have described as a state of lateral disturbance with a hemispherical distribution of density. Before proceeding to take account of the rotation and external attractions, we consider further the nature of such a disturbance. For this purpose we take the problem of a spherical body, homogeneous when unstrained, and devoid of all rigidity, and suppose that in the initial state the self-attraction of the body is balanced by

hydrostatic pressure. We suppose also that the law of elasticity of the body is that the increment of pressure is proportional to the increment of density. We show that equilibrium is possible in strained states, in which the excess of density at any assigned distance from the centre is proportional to a spherical surface harmonic of the first degree.

In the initial state the pressure p_0 and potential V_0 are given by the formulæ

$$p_0 = \tfrac{2}{3}\pi\gamma\rho_0^2(a^2-r^2), \quad V_0 = \tfrac{2}{3}\pi\gamma\rho_0(3a^2-r^2).$$

In the strained state the pressure P, density ρ, and potential V are expressed by the formulæ

$$P = p_0+\lambda\xi, \quad \rho = \rho_0(1+\xi), \quad V = V_0+W,$$

where ξ denotes the condensation. The equations of equilibrium are

$$\rho\frac{\partial V}{\partial x}-\frac{\partial P}{\partial x} = 0, \quad \rho\frac{\partial V}{\partial y}-\frac{\partial P}{\partial y} = 0, \quad \rho\frac{\partial V}{\partial z}-\frac{\partial P}{\partial z} = 0,$$

and W is connected with ξ by the equation

$$\nabla^2 W = -4\pi\gamma\rho_0\xi.$$

When terms of the second order in the small quantity ξ are neglected, the equations of equilibrium become three equations of the type

$$\rho_0\frac{\partial W}{\partial x}-\tfrac{4}{3}\pi\gamma\rho_0^2 x\xi-\lambda\frac{\partial\xi}{\partial x} = 0; \quad \ldots \ldots \quad (108)$$

and, on eliminating W, and writing s^2 for $\tfrac{4}{3}\pi\gamma\rho_0^2/\lambda$, we have

$$\nabla^2\xi+s^2 r\frac{\partial\xi}{\partial r}+6s^2\xi = 0. \quad \ldots \ldots \quad (109)$$

This equation is satisfied by putting

$$\xi = A(1-\tfrac{1}{5}s^2r^2)e^{-\frac{1}{2}s^2r^2}\omega_1,$$

where A is an arbitrary constant and ω_1 is a spherical solid harmonic of the first degree, and this is the most general form of solution in which ξ is finite at $r = 0$, and is proportional when $r =$ const. to a surface harmonic of the first degree. The additional potential W has the form

$$W = 4\pi\gamma\rho_0\{\tfrac{1}{5}As^{-2}e^{-\frac{1}{2}s^2r^2}\omega_1+F_1\},$$

where F_1 denotes a spherical solid harmonic of the first degree.

Let the bounding surface become

$$r = a+U_a.$$

Since the pressure vanishes at this surface, the expression

$$\tfrac{2}{3}\pi\gamma\rho_0^2\{a^2-(a+U_a)^2\}+\lambda A\left(1-\tfrac{1}{5}s^2a^2\right)e^{-\frac{1}{2}s^2a^2}a\,(\omega_1/r)$$

vanishes, or we have, neglecting U_a^2,

$$U_a s^2 = -\left(\tfrac{1}{5}s^2a^2-1\right)e^{-\frac{1}{2}s^2a^2}A\,(\omega_1/r),$$

so that U_a contains the same surface harmonic as ω_1. The form of F_1 is determined by the condition that W is the potential of a distribution of density $\rho_0\xi$ through the volume of the sphere $r = a$, together with a distribution of density $\rho_0 U_a$ on its surface. Just as in § 14, this condition leads to the equation

$$F_1 = \tfrac{2}{15}As^{-2}e^{-\frac{1}{2}s^2a^2}\omega_1.$$

48. Now let the bounding surface in the strained state be

$$r = a+b\cos\theta,$$

which represents a sphere with its centre at a small distance b from the origin in the direction of the axis of the harmonic. We find

$$\xi = \frac{s^2r^2-5}{s^2a^2-5}e^{\frac{1}{2}s^2(a^2-r^2)}bs^2r\cos\theta,$$

$$W = -\frac{4\pi\gamma\rho_0}{s^2a^2-5}\left(\tfrac{2}{3}+e^{\frac{1}{2}s^2(a^2-r^2)}\right)br\cos\theta.$$

If $s^2a^2 > 5$, the condensation is greatest near the centre, and it is positive on the side remote from that towards which the surface is displaced, so that the centre of gravity is displaced in the opposite sense to the surface. The distance of the centre of gravity from the origin is easily proved to be $5b/(s^2a^2-5)$.

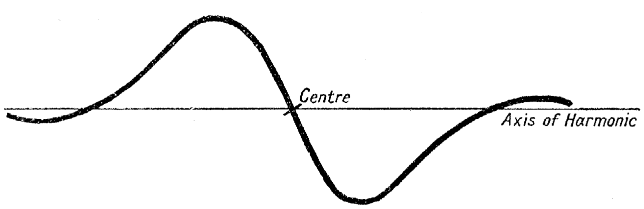

Fig. 1.

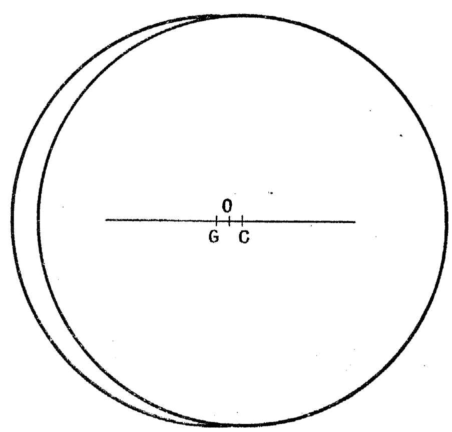

Fig. 2.

The variation of the excess density along the axis of the harmonic is illustrated in fig. 1. The surface $r = a(1+\epsilon\cos\theta)$ can be an equipotential surface if

$$a\epsilon = -5b/(s^2a^2-5),$$

and thus a sphere of radius a with its centre at the displaced centre of gravity is an equipotential surface. The relative situation of the bounding surface and of this equipotential is illustrated in fig. 2, in which O denotes the undisplaced centre, C the centre of the displaced surface, and G the centre of gravity of the strained sphere. The figures are drawn for the case in which $s^2a^2 = 10$.

The type of disturbance which has been called above a lateral disturbance with a hemispherical distribution of density would be the same in a body possessing some degree of rigidity, but the numerical details would be different.

49. If the equipotential surfaces of a nearly spherical body, with a nearly symmetrical distribution of density, are referred to the centre of gravity of the body as origin, their equations take such forms as

$$r = a(1+\epsilon_2S_2+\epsilon_3S_3+\ldots),$$

in which $\epsilon_2, \ldots$ denote small coefficients, and $S_2, \ldots$ denote spherical surface harmonics of degrees indicated by the suffixes. There is no term of the form ϵ_1S_1. In the case of the Earth, the coefficients $\epsilon_2, \ldots$ can be determined by means of pendulum experiments. If we referred to a different origin, near the centre of gravity, a term of the form ϵ_1S_1 would be introduced, but the coefficient ϵ_1 could not be determined by means of pendulum experiments, for it does not affect the formula for the variation of gravity over the surface.* If we choose an origin in accordance with geometrical considerations, *e.g.*, as the centre of that oblate spheroid which most nearly coincides with the surface of the ocean, the results of pendulum experiments cannot tell us whether this origin coincides with the centre of gravity or not.

Effect of Rotation upon a Planet with a Hemispherical Distribution of Density.

50. In all the preceding work the rotation of the Earth has been neglected. We have now to consider the effect of rotation upon a nearly spherical planet which, in the absence of rotation, would have a hemispherical distribution of density. To simplify the analysis, we shall disregard the concentration of mass towards the centre and also the rigidity of the body. We shall take as the "initial" state of the body a state in which the density is uniform and the stress is hydrostatic pressure,

* The result may be inferred from STOKES' investigation of the "Variation of Gravity over the Surface of the Earth," Cambridge, 'Trans. Phil. Soc.,' 8 (1849), or 'Math. and Phys. Papers,' vol. 2, Cambridge, 1883. It is easy to prove it independently.

while the body is rotating, as if rigid, about the axis of z with angular velocity ω; and we shall seek a strained state in which the body could exist without the application of any external forces, this state being such that, in the absence of rotation, the distribution of density would be hemispherical. In the notation of § 47 the equations of steady motion of the body are

$$-\rho\omega^2 x = \rho\frac{\partial V}{\partial x} - \frac{\partial P}{\partial x}, \quad -\rho\omega^2 y = \rho\frac{\partial V}{\partial y} - \frac{\partial P}{\partial y}, \quad 0 = \rho\frac{\partial V}{\partial z} - \frac{\partial P}{\partial z}. \quad . \quad . \quad (110)$$

The initial state is determined by the same equations with ρ_0, V_0, p_0 substituted for ρ, V, P. Now the initial figure is an oblate spheroid, and the initial form of V is

$$V_0 = \text{const.} - \tfrac{1}{2}\{A'(x^2+y^2) + C'z^2\},$$

where A′ and C′ are constants; also the initial form of P is

$$\begin{aligned} p_0 &= \text{const.} + \rho_0\{V_0 + \tfrac{1}{2}\omega^2(x^2+y^2)\}, \\ &= \text{const.} - \tfrac{1}{2}\rho_0\{(A'-\omega^2)(x^2+y^2) + C'z^2\}. \end{aligned}$$

When we write, as in § 47,

$$\rho = \rho_0(1+\xi), \quad P = p_0 + \lambda\xi, \quad V = V_0 + W,$$

and neglect terms which cancel on account of the values of p_0 and V_0, and also neglect terms which are of the second order in the small quantity ξ, the equations (110) become

$$\left.\begin{aligned} \rho_0\omega^2 x\xi &= -\rho_0\xi A'x + \rho_0\frac{\partial W}{\partial x} - \lambda\frac{\partial\xi}{\partial x}, \\ -\rho_0\omega^2 y\xi &= -\rho_0\xi A'y + \rho_0\frac{\partial W}{\partial y} - \lambda\frac{\partial\xi}{\partial y}, \\ 0 &= -\rho_0\xi C'z + \rho_0\frac{\partial W}{\partial z} - \lambda\frac{\partial\xi}{\partial z}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

Now we have the equations

$$2A' + C' = 4\pi\gamma\rho_0, \quad \nabla^2 W = -4\pi\gamma\rho_0\xi,$$

and therefore we can eliminate W and obtain the equation

$$\nabla^2\xi + \left(6s^2 - 2\omega^2\frac{\rho_0}{\lambda}\right)\xi + \frac{\rho_0}{\lambda}\left\{(A'-\omega^2)\left(x\frac{\partial\xi}{\partial x} + y\frac{\partial\xi}{\partial y}\right) + C'z\frac{\partial\xi}{\partial z}\right\} = 0, \quad . \quad (112)$$

where s^2 is written for $\frac{4}{3}\pi\gamma\rho_0^2/\lambda$. If ω were zero, A′ and C′ would both be equal to $\frac{4}{3}\pi\gamma\rho_0$, and we therefore put

$$A' = \tfrac{4}{3}\pi\gamma\rho_0 + A'', \quad C' = \tfrac{4}{3}\pi\gamma\rho_0 + C'';$$

then equation (112) becomes

$$\nabla^2\xi+s^2r\frac{\partial\xi}{\partial r}+6s^2\xi = 2\omega^2\frac{\rho_0}{\lambda}\xi+\frac{\rho_0}{\lambda}\left\{(\mathrm{A}''-\omega^2)\left(x\frac{\partial\xi}{\partial x}+y\frac{\partial\xi}{\partial y}\right)+\mathrm{C}''z\frac{\partial\xi}{\partial z}\right\}. \quad . \quad (113)$$

The left-hand member of this equation is the same as that of equation (107) in § 47 above; and therefore, when ω^2 is neglected, ξ can be of the form ξ_1, where

$$\xi_1 = \mathrm{A}\,(1-\tfrac{1}{5}s^2r^2)\,e^{-\frac{1}{2}s^2r^2}\omega_1,$$

the notation being the same as in § 48. Now we shall suppose that ω^2 is not large, so that we may treat ξ_1 as an approximation to ξ, and substitute ξ_1 for ξ in the right-hand member of equation (113), for all the terms of this member are small of the order $\omega^2\xi$. We are then neglecting ξ^2, but not $\omega^2\xi$. To obtain a second approximation, we put

$$\xi = \xi_1+\xi',$$

and seek a particular integral of the equation

$$\nabla^2\xi'+s^2r\frac{\partial\xi'}{\partial r}+6s^2\xi' = 2\omega^2\frac{\rho_0}{\lambda}\xi_1+\frac{\rho_0}{\lambda}\left\{(\mathrm{A}''-\omega^2)\left(x\frac{\partial\xi_1}{\partial x}+y\frac{\partial\xi_1}{\partial y}\right)+\mathrm{C}''z\frac{\partial\xi_1}{\partial z}\right\}. \quad (114)$$

There would be no special difficulty in obtaining a solution of the equation, but it will be sufficient for our purpose to find the *form* of the solution. The function ξ_1 may be expressed in terms of polar co-ordinates r, θ, ϕ in the form

$$\xi_1 = f(r)\,r\,(\alpha_1\sin\theta\cos\phi+\beta_1\sin\theta\sin\phi+\gamma_1\cos\theta),$$

where α_1, β_1, γ_1 are constants, and $f(r)$ is a certain function of r which has been determined. Hence we have

$$\begin{aligned} x\frac{\partial\xi_1}{\partial x}+y\frac{\partial\xi_1}{\partial y} &= rf(r)(\alpha_1\sin\theta\cos\phi+\beta_1\sin\theta\sin\phi) \\ &\qquad +r^2f'(r)\sin^2\theta\,(\alpha_1\sin\theta\cos\phi+\beta_1\sin\theta\sin\phi+\gamma_1\cos\theta), \\ z\frac{\partial\xi_1}{\partial z} &= rf(r)\gamma_1\cos\theta+r^2f'(r)\cos^2\theta\,(\alpha_1\sin\theta\cos\phi+\beta_1\sin\theta\sin\phi+\gamma_1\cos\theta)\,; \end{aligned}$$

and these can be expressed in the forms

$$\begin{aligned} x\frac{\partial\xi_1}{\partial x}+y\frac{\partial\xi_1}{\partial y} &= r\{f(r)+\tfrac{4}{5}rf'(r)\}\sin\theta\,(\alpha_1\cos\phi+\beta_1\sin\phi)+\tfrac{2}{5}r^2f'(r)\gamma_1\cos\theta \\ &\qquad -r^2f'(r)(\cos^2\theta-\tfrac{1}{5})\sin\theta\,(\alpha_1\cos\phi+\beta_1\sin\phi)-r^2f'(r)\gamma_1(\cos^3\theta-\tfrac{3}{5}\cos\theta), \\ z\frac{\partial\xi_1}{\partial z} &= \tfrac{1}{5}r^2f'(r)\sin\theta\,(\alpha_1\cos\phi+\beta_1\sin\phi)+r\{f(r)+\tfrac{3}{5}rf'(r)\}\gamma_1\cos\theta \\ &\qquad +r^2f'(r)\sin\theta\,(\cos^2\theta-\tfrac{1}{5})(\alpha_1\cos\phi+\beta_1\sin\phi)+r^2f'(r)\gamma_1(\cos^3\theta-\tfrac{3}{5}\cos\theta) \end{aligned}$$

Hence the right-hand member of (114) can be expressed as a sum of terms each of which is the product of a function of r and a spherical surface harmonic, and the

surface harmonics which occur are those of the first degree and the following harmonics of the third degree :—

$$\cos^3\theta - \tfrac{3}{5}\cos\theta, \quad (\cos^2\theta - \tfrac{1}{5})\sin\theta\cos\phi, \quad (\cos^2\theta - \tfrac{1}{5})\sin\theta\sin\phi.$$

To each of these terms there corresponds a term of the same form in ξ', and therefore also in P, or $p_0 + \lambda\xi$; and it follows that the displacement of the bounding surface from its initial form (which is a slightly elliptic oblate spheroid appropriate to the rotation) is expressed by a radial displacement, which consists of a part proportional to a spherical surface harmonic of the first degree, together with parts proportional to the above surface harmonics of the third degree. In like manner all the terms of the additional potential W are the products of functions of r and surface harmonics, which are either of the first degree or are the above harmonics of the third degree; but the coefficients of the various harmonics in W are different from their coefficients in ξ. The equation of the equipotentials under gravity, modified by the rotation, is

$$V_0 + W + \tfrac{1}{2}\omega^2(x^2+y^2) = \text{const.}, \quad \text{or} \quad p_0/\rho_0 + W = \text{const.};$$

and thus the situation of the bounding surface relative to the equipotentials is expressed by a difference of radii at corresponding points, this difference being a sum of terms of the form bS, where b denotes a constant and S denotes a surface harmonic; and the surface harmonics which can occur are those of the first degree and the three of the third degree written above.

51. It appears from this investigation that, if a gravitating body, which is rotating about an axis, has so small a modulus of compression that, if the body were at rest, a spherically symmetrical distribution of density would be unstable, it would tend to take up a state in which the distribution of density would not be exactly hemispherical, but the excess density would also contain terms expressed by spherical harmonics of the third degree. The figure of the body would differ from the oblate spheroidal figure appropriate to the rotation by a radial displacement at each point; and this displacement would be expressed partly by spherical surface harmonics of the first degree, indicating that the centre of gravity does not coincide with the centre of figure, and partly by spherical harmonics of the third degree. If the body were entirely devoid of rigidity, the oblate figure appropriate to the rotation would be the same as that of an equipotential surface under gravity, modified by the rotation; and the *figure* of the body, as determined by difference of level above or below a certain equipotential surface, would be an *harmonic spheroid of the third degree*, and the *situation* of the body would be that of such a spheroid when displaced towards one side. If the body possessed some rigidity, the oblate figure appropriate to the rotation would differ a little from that of a nearly coincident equipotential surface, and the shape of it, determined as before, would be that derived from a certain oblate spheroid of small ellipticity by a displacement proportional to a surface harmonic of the third degree. The surface harmonic would be of a somewhat specialised type.

Effect of certain External Forces.

52. The effect of forces such as the attraction of the Moon at the time when its period of orbital revolution did not differ much from the period of rotation of the Earth would be to draw the planet out into a shape more nearly ellipsoidal, with three unequal axes, than spheroidal. If the planet could have had a symmetrical shape it would have been practically ellipsoidal, and the surfaces of equal density would have been ellipsoids. Whereas the effect of rotation is the same as that of forces derived from a potential of the second degree, symmetrical about the axis of rotation; such forces as we are now considering are derived from a potential, the most important terms of which would be of the second degree, but not symmetrical about the axis of rotation. If the elasticity was too small for an ellipsoidal figure to be stable, the planet would have been in a disturbed state, the nature of which can be inferred from the preceding investigation. We have only to replace in § 50 the initial potential, modified by the rotation, by a general expression of the second degree in the co-ordinates. The only change that would be made in the result would be that those terms in the radial displacement which are expressed by harmonics of the third degree would not be of the specialised type introduced by the rotation, but would be of general type. The figure of the planet would be derived from the ellipsoidal figure appropriate to the rotation, and to the external forces, by a radial inequality expressed by surface harmonics of the first and third degrees. The equipotential surfaces would be obtained from the ellipsoidal equipotentials appropriate to gravity, modified by the rotation and the external forces, by surface harmonics of the same degrees. The result would be that the shape of the planet, as determined by difference of level above or below a certain equipotential, would be a wrinkled ellipsoid, displaced towards one side; and the wrinkle would be expressible by means of a spherical surface harmonic of the third degree.

The Problem of the Shape of the Lithosphere.

53. The problem of determining the form of the equipotentials near the surface of the Earth includes the problem of determining the figure of the surface of the ocean (the "hydrosphere"). The equipotentials which lie outside the nucleus (or "lithosphere") on one side, and sufficiently near to it, cut the surface of the lithosphere towards the other side. Among these equipotential surfaces that one which, outside the lithosphere, coincides with the surface of the ocean is known as the "geoid." The surface of that part of the lithosphere which lies outside the geoid is occupied by land, and can be observed directly; the surface of that part which lies within the geoid can only be observed indirectly by means of soundings. We have no means of investigating the form of the surface of this part of the lithosphere except by estimating its depth at a point below the geoid. The most important deviations from sphericity both of the lithosphere and of the geoid are of such

a nature that these surfaces are nearly oblate spheroids. If the lithosphere were exactly in the form of an oblate spheroid, and its centre of gravity coincided with its centre of figure, it would either lie entirely within the geoid or would protrude from it symmetrically at the North and South Poles. Owing to the rigidity of the lithosphere the ellipticity produced in the geoid by rotation would be slightly greater than that produced in the lithosphere, and thus there is a tendency to lay bare the polar regions; but, since the land of the globe does not consist of two circular islands at the poles, there are other deviations from sphericity, both of lithosphere and geoid, and the relative amounts of these at different places can be expressed by the difference of radii drawn from the centre of gravity. According to the theory which has been here advanced this difference of radii should be, at least in its general features, expressible as a sum of spherical harmonics of the first, second and third degrees.

54. It is easy to verify the presence of some of these harmonics. The effect of a term of the first degree would be to make the lithosphere protrude from the geoid towards one side. If this term were the only one, the land of the globe would form a circular island or continent. It is the fact that most of the land is in one hemisphere. The great circle of the globe which contains most land has a pole situated between Orléans and Le Mans* (latitude 48° N., longitude 30′ E.). Again, the zonal harmonic of the third degree vanishes at three circles, one being a great circle. If this term were the only one, the land of the globe would consist of a circular island surrounded by a belt of ocean in one hemisphere, and in the antipodal hemisphere there would be a circular ocean surrounded by a ring of land. This arrangement corresponds to two features of SOLLAS' description of the Earth's surface. The nearly symmetrical breaking at three places of the belt and three of the ring, which he also noticed, indicates the presence of the sectorial harmonic of the third degree. If we refer to the polar axis, instead of any other morphological axis, the presence of the zonal harmonic of the third degree is indicated by the existence of an Antarctic continent, and by the fact that most of the land of the globe is north of the Equator. The harmonic of the third degree and second rank, referred to the polar axis, vanishes at the Equator and at four meridians symmetrically placed. If this term were the only one, then, in two northern quadrants there would be land, and also in the two alternate southern quadrants, an arrangement which suggests Central Asia and North America as the land quadrants of the northern hemisphere, Australia and South America as those of the southern.

Spherical Harmonic Analysis of the Distribution of Land and Water.

55. By such arguments as the foregoing, and by some trials with small numerical coefficients for the various harmonics, I had convinced myself that many features of the distribution of land and water could be represented by means of harmonics of the third degree, when Professor H. H. TURNER suggested to me the advisability of

* E. BRÜCKNER, 'Die feste Erdrinde und ihre Formen,' Wien, 1897.

adopting a systematic process for the discovery of appropriate coefficients. He very kindly made, and placed at my disposal, a rough preliminary calculation, and the results were sufficiently encouraging to warrant the undertaking of a considerable piece of computation. A professional computer was employed for a time, but eventually I relied upon my own calculations, taking many precautions to ensure accuracy. The systematic process consists in devising a function to represent the "value of land" at any point, and determining, by the method of approximate quadrature, the coefficients of an expansion of the function in spherical harmonics. The results of such a computation clearly depend upon the chosen "value of land," and judgment must be exercised in selecting appropriate values. Little importance can be attached to the heights of mountains, because the highest mountain ranges are, geologically speaking, modern, the ancient mountains being worn down by denudation and erosion. Too much importance is not to be attributed to the actual coast-line, because this line is subject to many causes of change. The coast-line is but one of the contour-lines of the continental block (the geoid being the level of reckoning), and the shape of the block at considerable depths differs a good deal from that at the surface. At mean-sphere-level (8400 feet below sea-level) the continents, with the exception of the Antarctic continent, form a continuous block.* The Arctic Ocean is reduced, so far as is known, to a trough running nearly along the meridian of Greenwich, from about latitude 65° N. to about latitude 80° N. It may extend to the North Pole and surround it. The polar block spreads southwards in two great masses—America and Eurasia. These are joined through the British Isles, Iceland and Greenland on the one side, and across Behring's Strait on the other; the contour-line at mean-sphere-level runs practically along the 60th parallel between America and Europe and along the 50th parallel between America and Asia. The Eurasian division of the block forks near the Persian Gulf, and tapers southwards in two branches, one containing Africa and the other the Malay Peninsula, adjacent islands, Australia and New Zealand. The Red Sea does not go down to mean-sphere-level, and the Mediterranean does so only in two small patches. The American division of the block is continuous across the Gulf of Mexico, the West Indies and the Caribbean Sea, which, at this depth, equally with Mexico, Central America, and the Isthmus of Panama, form part of the ridge joining North and South America. The ridge has some local depressions. The block tapers towards Cape Horn, in the neighbourhood of which, however, it has a great eastward extension, and this extension turns westward and nearly joins the northern continental block to the Antarctic continental block through the South Shetland Islands.† The Antarctic block also shows a

* The information here detailed in regard to the distribution of the continental blocks and oceanic regions at mean-sphere-level is taken from a map drawn by H. R. MILL in 'The Scottish Geographical Magazine' (Edinburgh), vol. 6 (1890), p. 184. Reference may be made to the rough map on p. 237 below.

† It is now known that the depth of the channel is not so great as it was for a long time supposed to be. See a paper by W. S. BRUCE in 'The Scottish Geographical Magazine' (Edinburgh), vol. 21 (1905), p. 402.

northward extension towards Australasia. The contour-line of the continental blocks at mean-sphere-level is a very important and fairly well ascertained datum of the problem. If, however, we attend exclusively to it, we are liable to emphasise unduly those parts of the block which do not rise above the level of the sea.

56. I calculated the coefficients of a spherical harmonic expansion up to harmonics of the third degree for two different assumptions as to the "value of land." In the first assumption the value -1 was attached to those points of the surface which are below mean-sphere-level and the value 0 to those points which are above it. In the second assumption the value 1 was attached to those points of the surface which are above sea-level and the value 0 to those below it. The coefficients obtained by the two assumptions were then added. The somewhat greater importance of the mean sphere may perhaps be sufficiently represented by the result that the maxima obtained by using the first set of coefficients are larger than those obtained by using the second set. The combined distribution for the two sets of coefficients is shown in the following table, in which θ stands for co-latitude measured from the North Pole, and ϕ for longitude measured eastwards from the meridian of Greenwich :—

TABLE.

θ \ ϕ	0°.	5°.	10°.	15°.	20°.	25°.	30°.	35°.	40°.	45°.	50°.	55°.	60°.	65°.	70°.	75°.	80°.	85°.
°																		
5	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
10	−1																	
15	−1																	
20	−1						1						1	1	1	1	1	1
25	−1			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
30			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
35		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
40	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
45	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
50				1		1				1		1	1	1	1	1	1	1
55										1	1	1	1	1	1	1	1	1
60	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
65	1	1	1	1	1	1	1		1	1			1	1	1	1	1	1
70	1	1	1	1	1	1	1			1	1	1		−1		1	1	1
75	1	1	1	1	1	1	1	1		1	1	1		−1			1	1
80	1	1	1	1	1	1	1	1	1				−1	−1			1	
85			1	1	1	1	1	1	1	1		−1	−1	−1				−1
90			1	1	1	1	1	1	1			−1	−1	−1	−1	−1	−1	−1
95	−1		1	1	1	1	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1
100	−1		1	1	1	1	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1
105	−1		1	1	1	1	1	1	1		1	−1	−1	−1	−1	−1	−1	−1
110	−1			1	1	1	1	1		1		−1	−1	−1	−1	−1	−1	−1
115	−1	−1		1	1	1	1			1		−1	−1	−1	−1	−1	−1	−1
120	−1	−1	−1		1	1			−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
125	−1	−1	−1	−1	1			−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
130	−1	−1	−1	−1	−1		−1	−1	−1	−1	−1	1	−1	−1	−1	−1	−1	−1
135	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
140	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
145	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
150	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
155	−1	−1	−1	−1	−1													
160																		
165													1	1	1	1	1	1
170										1	1	1	1	1	1	1	1	1
175	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

TABLE (continued).

θ \ ϕ	90°.	95°.	100°.	105°.	110°.	115°.	120°.	125°.	130°.	135°.	140°.	145°.	150°.	155°.	160°.	165°.	170°.	175°.
°																		
5	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
10																		
15			1	1														
20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
30	1	1	1	1	1	1	1	1	1	1						1		
35	1	1	1	1	1	1	1	1	1	1								
40	1	1	1	1	1	1	1	1	1	1					−1	−1	−1	−1
45	1	1	1	1	1	1	1	1	1	1				−1	−1	−1	−1	−1
50	1	1	1	1	1	1	1	1					−1	−1	−1	−1	−1	−1
55	1	1	1	1	1	1	1					−1	−1	−1	−1	−1	−1	−1
60	1	1	1	1	1	1	1					−1	−1	−1	−1	−1	−1	−1
65	1	1	1	1	1	1	1			−1	−1	−1	−1	−1	−1	−1	−1	−1
70		1	1	1					−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
75			1	1					−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
80			1							−1	−1	−1	−1	−1	−1	−1	−1	−1
85	−1		1	1						−1	−1	−1	−1	−1	−1	−1	−1	−1
90	−1		1		1						−1	−1	−1	−1	−1	−1	−1	−1
95	−1	−1		1						1	1		−1	−1	−1	−1	−1	−1
100	−1	−1													−1	−1	−1	−1
105	−1	−1	−1	−1	−1	−1				1		1		−1	−1	−1	−1	−1
110	−1	−1	−1	−1	−1			1	1	1	1	1					−1	−1
115	−1	−1	−1	−1		1	1	1	1	1	1	1	1		−1			−1
120	−1	−1	−1	−1		1	1	1	1	1	1	1	1		−1			
125	−1	−1	−1	−1		1						1	1		−1	−1		
130	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1				−1	−1	−1		1
135	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1			−1	−1	−1			
140	−1	−1	−1	−1	−1	−1	−1	−1	−1		−1	−1	−1	−1	−1	−1	−1	−1
145	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
150	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
155			1	1	1	1	1	1	1	1	1	1	1	1	1			
160			1	1	1	1	1	1	1	1	1	1	1	1	1			
165	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
170	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
175	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

TABLE (continued).

θ \ ϕ	180°.	185°.	190°.	195°.	200°.	205°.	210°.	215°.	220°.	225°.	230°.	235°.	240°.	245°.	250°.	255°.	260°.	265°.
°																		
5	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
10																		
15													1	1	1	1	1	1
20	1			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
30					1	1	1	1	1	1	1	1	1	1	1	1	1	1
35						−1	−1	−1			1	1	1	1	1	1	1	1
40	−1	−1	−1	−1	−1	−1	−1	−1	−1				1	1	1	1	1	1
45	−1	−1	−1	−1	−1	−1	−1	−1	−1				1	1	1	1	1	1
50	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1			1	1	1	1	1	1
55	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1			1	1	1	1	1
60	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1			1	1	1	1
65	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1				1	1
70	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1			1	
75	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1		1
80	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
85	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
90	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
95	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
100	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
105	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
110	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
115	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
120	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
125		−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
130		−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
135	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
140	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
145	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
150	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
155																		
160																		
165																		
170	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
175	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

TABLE (continued).

θ \ ϕ	270°.	275°.	280°.	285°.	290°.	295°.	300°.	305°.	310°.	315°.	320°.	325°.	330°.	335°.	340°.	345°.	350°.	355°.
°																		
5	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
10				1					1	1	1	1	1					
15	1	1	1		1	1	1	1	1	1	1	1	1					
20	1	1	1	1						1	1	1	1					
25	1	1	1	1						1								
30				1					−1				−1	−1				
35				1	1	1			−1	−1	−1	−1	−1	−1	−1			
40	1	1	1	1	1	1	1			−1	−1	−1	−1	−1	−1			1
45	1	1	1	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1	−1		1
50	1	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1
55	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1		1
60	1				−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1		1	1
65						−1	−1	−1	−1	−1	−1	−1	−1	−1			1	1
70							−1	−1	−1	−1	−1	−1	−1	−1		1	1	1
75	1						−1	−1	−1	−1	−1	−1	−1	−1		1	1	1
80		1			1			−1	−1	−1	−1	−1	−1	−1				1
85	−1	−1		1	1				−1	−1	−1	−1	−1	−1	−1	−1		
90	−1	−1		1	1	1	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1
95	−1		1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1
100	−1	−1		1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1
105	−1	−1			1	1	1	1	1	1		−1	−1	−1	−1	−1	−1	−1
110	−1	−1	−1		1	1	1	1	1	1		−1	−1	−1	−1	−1	−1	−1
115	−1	−1	−1		1	1	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1
120	−1	−1	−1		1	1	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1
125	−1	−1	−1		1	1	1	1		−1	−1	−1	−1	−1	−1	−1	−1	−1
130	−1	−1	−1		1	1				−1	−1	−1	−1	−1	−1	−1	−1	−1
135	−1	−1			1	1		−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
140	−1	−1			1			−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
145	−1	−1											−1	−1	−1	−1	−1	−1
150	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
155									−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
160	1	1	1	1											−1	−1	−1	−1
165	1	1	1	1	1	1	1	1	1	1								
170	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
175	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

57. The surface harmonics of the first degree expressed in ordinary spherical polar co-ordinates θ, ϕ are

$$\sin\theta\cos\phi,\quad \sin\theta\sin\phi,\quad \cos\theta;$$

and any spherical surface harmonic of the first degree can be expressed in the form

$$(p\cos\phi+q\sin\phi)\sin\theta+r\cos\theta, \quad \ldots \quad (115)$$

where p, q, r are numbers. The spherical surface harmonics of the second degree are*

$$\sin 2\theta\cos\phi,\quad \sin 2\theta\sin\phi,\quad \sin^2\theta\cos 2\phi,\quad \sin^2\theta\sin 2\phi,\quad 3\cos 2\theta+1;$$

and any spherical surface harmonic of the second degree can be expressed in the form

$$(\alpha\cos\phi+\beta\sin\phi)\sin 2\theta+(\gamma\cos 2\phi+\delta\sin 2\phi)\sin^2\theta+\epsilon(3\cos 2\theta+1). \quad . \quad (116)$$

The spherical surface harmonics of the third degree are

(i.) The zonal harmonic

$$5\cos^3\theta-3\cos\theta;$$

(ii.) The tesseral harmonics of the first rank

$$(5\cos^2\theta-1)\sin\theta\cos\phi,\quad (5\cos^2\theta-1)\sin\theta\sin\phi;$$

(iii.) The tesseral harmonics of the second rank

$$\sin^2\theta\cos\theta\cos 2\phi,\quad \sin^2\theta\cos\theta\sin 2\phi;$$

(iv.) The sectorial harmonics

$$\sin^3\theta\cos 3\phi,\quad \sin^3\theta\sin 3\phi.$$

Since

$$\begin{aligned} 5\cos^3\theta-3\cos\theta &= \tfrac{5}{4}(\cos 3\theta+\tfrac{3}{5}\cos\theta), \\ (5\cos^2\theta-1)\sin\theta &= \tfrac{1}{4}(\sin\theta+5\sin 3\theta), \\ \sin^2\theta\cos\theta &= \tfrac{1}{4}(\cos\theta-\cos 3\theta), \\ \sin^3\theta &= \tfrac{1}{4}(3\sin\theta-\sin 3\theta), \end{aligned}$$

any spherical surface harmonic of the third degree can be expressed in the form

$$a\mathrm{W}+\mathrm{X}(b\cos\phi+c\sin\phi)+\mathrm{Y}(d\cos 2\phi+e\sin 2\phi)+\mathrm{Z}(f\cos 3\phi+g\sin 3\phi), \quad . \quad (117)$$

where a, b, c, d, e, f, g are numbers, and

$$\left.\begin{aligned} \mathrm{W} &= \cos 3\theta+(0\cdot 6)\cos\theta, \\ \mathrm{X} &= \sin\theta+5\sin 3\theta, \\ \mathrm{Y} &= \cos\theta-\cos 3\theta, \\ \mathrm{Z} &= 3\sin\theta-\sin 3\theta. \end{aligned}\right\} \quad \ldots \quad (118)$$

* The form $3\cos 2\theta+1$ for the zonal harmonic is $4(\frac{3}{2}\cos^2\theta-\frac{1}{2})$, and is taken as being more convenient for calculation.

58. Let $F(\theta, \phi)$ denote the function to be expanded. The coefficients are expressed by equations of the type

$$p\int_0^{2\pi} d\phi \int_0^{\pi} d\theta (\cos\phi \sin\theta)^2 \sin\theta = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta F(\theta, \phi)(\cos\phi\sin\theta)\sin\theta. \quad . \quad (119)$$

The factors multiplying the coefficients p, &c. in the left-hand members are the integrals of the squares of the several harmonics over the surface of a unit sphere. The integrals in the right-hand members are the integrals, over the surface of the same sphere, of the product of the function to be expanded and the corresponding harmonics. The values of the integrated squares multiplying p, &c., are recorded in the following table :—

Coefficient.	Value of integrated square.	Reciprocals.
p, q, r	$\frac{4}{3}\pi$	$(\frac{192}{35}) \times \frac{35}{256}\pi^{-1}$
$\alpha, \beta, \gamma, \delta$	$\frac{16}{15}\pi$	$(\frac{48}{7}) \times \frac{35}{256}\pi^{-1}$
ϵ	$\frac{64}{5}\pi$	$(\frac{4}{7}) \times \frac{35}{256}\pi^{-1}$
a	$\frac{256}{175}\pi$	$(5) \times \frac{35}{256}\pi^{-1}$
b, c	$\frac{512}{21}\pi$	$(0 \cdot 3) \times \frac{35}{256}\pi^{-1}$
d, e	$\frac{256}{105}\pi$	$(3) \times \frac{35}{256}\pi^{-1}$
f, g	$\frac{512}{35}\pi$	$(0 \cdot 5) \times \frac{35}{256}\pi^{-1}$

Since the ratios only are relevant, the integrals in the right-hand members of such equations as (119) are to be multiplied by the numbers in brackets in the third column.

59. To evaluate integrals of the type in the right-hand member of (119), when the value of $F(\theta, \phi)$ is given by the table of § 56, or any similar table, we treat the integral as a double sum, *e.g.*,

$$\sum_{m=0}^{m=71}\left(\frac{\pi}{36}\right)\sum_{n=1}^{n=35}\left(\frac{\pi}{36}\right) F\left(\frac{n\pi}{36}, \frac{m\pi}{36}\right)\left(\cos\frac{m\pi}{36}\sin\frac{n\pi}{36}\right)\sin\frac{n\pi}{36};$$

then we have to evaluate such a double sum as

$$\sum_{m=0}^{71}\sum_{n=1}^{35} F\left(\frac{n\pi}{36}, \frac{m\pi}{36}\right)\cos\frac{m\pi}{36}\sin^2\frac{n\pi}{36}.$$

We sum first with respect to m; but in forming the sum we take account of the fact that $\sin^2(n\pi/36)$ does not change when n is replaced by $36-n$. For example, let F be equal to 1 at the points indicated in the table, and zero at other points. Then the contribution to the terms containing any m of the two parallels given by n and $36-n$ is either 0, 1, or 2, according as a 1 occurs on neither parallel (for the particular m in question), on one, or on both. This number 0, 1, or 2 is to be multiplied by the value of $\cos(m\pi/36)$ for the chosen m; but the same value for the cosine occurs at the meridian given by $72-m$, and the same numerical value with the opposite sign occurs at the meridians given by $36-m$ and $36+m$. We condense into one term the contributions of the eight points given by n, $36-n$, m, $72-m$, $36 \pm m$, and take the ranges of m and n to be respectively 0 to 17 and 1 to 18. Thus, as the multiplier of $\cos(m\pi/36)\sin^2(n\pi/36)$, we have an integral number which necessarily lies between -4 and 4, and may be zero, and we have transformed the sum into a double sum of the form

$$\sum_{m=0}^{17} \sum_{n=1}^{18} \mathrm{F}'(n, m) \cos\frac{m\pi}{36} \sin^2\frac{n\pi}{36},$$

where F′ is the number in question. The most troublesome part of the process is the determination of F′. When F′ has been found it is very easy to form the sum of such a series as that written immediately above by summing first with respect to m and then with respect to n. When we are dealing with tesseral harmonics of the second rank, we can thus condense into one term the contributions of 16 points of the table, and, when the tesseral harmonic is of the third rank, those of 24 points. Much labour is saved by going through this process, troublesome though it is, and much greater accuracy can be secured, because in the multiplication of $\cos(m\pi/36)$ by F′, when F′ is, say, 5 or 6, and the value of the cosine to any chosen number of decimal places is used, it is easier to correct the figure in the last place than it is when the same cosine occurs five or six times in a long column of figures which have to be added together.

60. By the use of this method I computed the values of the coefficients p, &c., for the function $\mathrm{F}(\theta, \phi)$ which is given by the -1's in the table of § 56, the 1's being replaced by zeros. Up to the stage of summation with respect to m, inclusive, I kept four decimal figures. Of the terms of the type

$$\sum_{m=0}^{17}\left(\mathrm{F}'(n, m) \cos\frac{m\pi}{36}\right) . \sin^2\frac{n\pi}{36}$$

I then kept two decimal figures, formed the sums with respect to n, and multiplied them by the corresponding numbers placed in brackets in the third column of the table in § 58. This process gave the coefficients in the second column of the annexed table. The integral parts only were retained. I computed the values of the coefficients p, &c., in the same way for the function given by the 1's in the table of § 56, the -1's being replaced by zero. This process gave the coefficients in the third

column of the annexed table. It is to be understood that in both cases the common factors $\frac{35}{256}\pi^{-1}$ and $(\pi/36)^2$ have been omitted.

	$F(\theta, \phi) = 0$ or -1.	$F(\theta, \phi) = 1$ or 0.	Sum.
p	604	557	1161
q	495	329	824
r	777	630	1407
α	350	243	593
β	295	366	661
γ	−443	−223	−666
δ	−291	68	−223
ϵ	185	98	283
a	−213	−134	−347
b	− 73	− 71	−144
c	− 29	38	9
d	−338	−256	−594
e	396	351	747
f	56	26	82
g	203	122	325

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61. I then computed the values of the harmonics expressed by (115), (116), (117), for values of θ and ϕ, which are multiples of 15° (or $\frac{1}{12}\pi$), using first the coefficients

given in the second column of the table in § 60, and then the coefficients which are given in the third column of the same table. Finally I added the values belonging to the same θ and ϕ. The results are shown in the diagram (fig. 3), where the fine

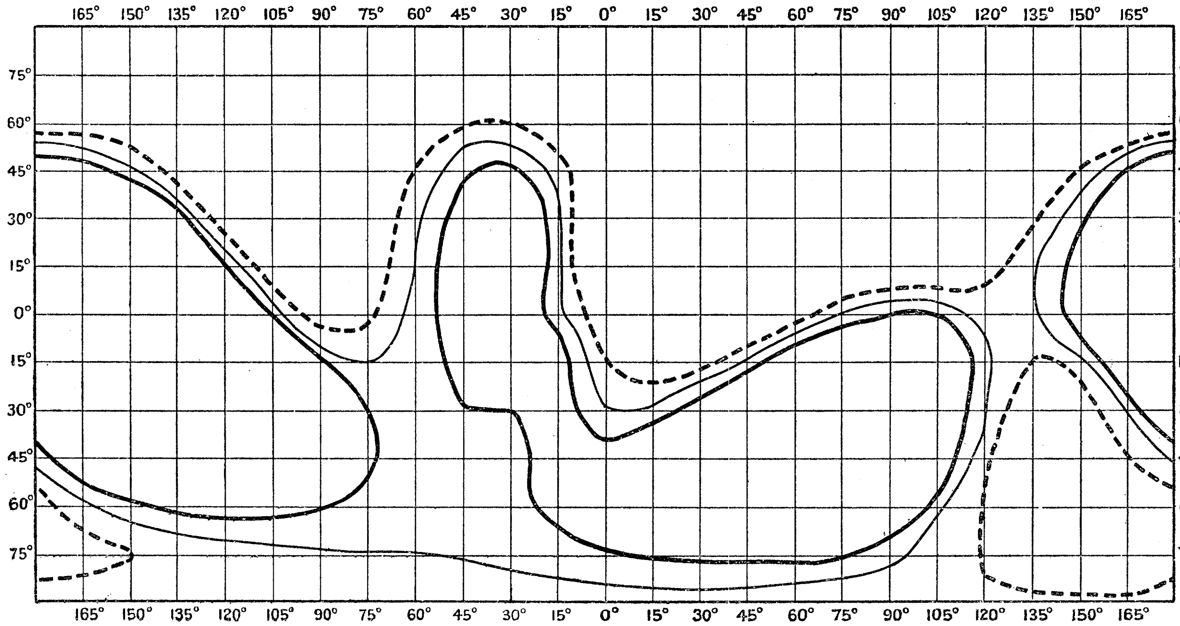

Fig. 3.

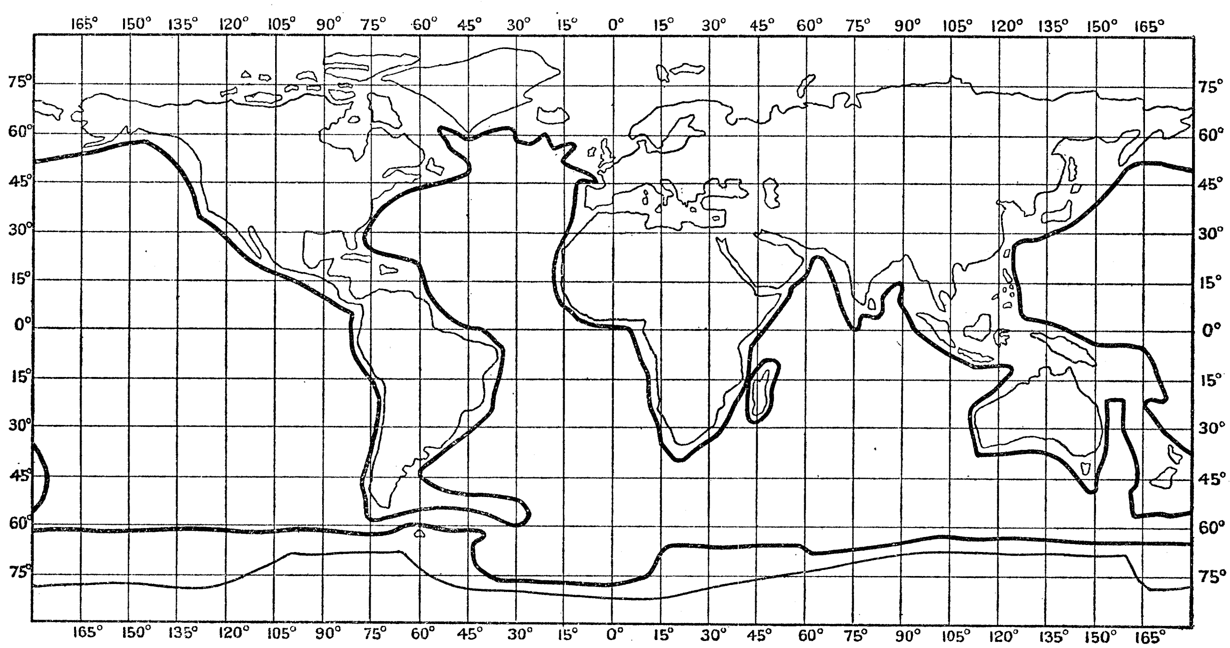

Fig. 4.

continuous line is the contour-line along which the calculated harmonic inequality vanishes, the heavy continuous line is the contour-line along which this inequality is 10 per cent. of its maximum below zero, and the dotted line is the contour-line along

which this inequality is 10 per cent. of its maximum above zero. It is to be observed that an inequality expressed by harmonics of uneven degrees has numerically equal values with opposite signs at antipodal points, and therefore the area on the sphere within which such an inequality is positive is equal to the area within which it is negative. But this equality of positive and negative areas does not hold when the harmonics of the second degree are present. A rough calculation showed that the zero line of the inequality illustrated in fig. 3 divides the surface of the sphere into two unequal areas, and the inequality is negative in the larger area. The excess of the negative area above the positive is nearly 10 per cent. of the whole surface. The heavy line in fig. 3 corresponds more nearly than the other lines to the principle by which geographers construct the contour-line at mean-sphere-level. The diagram in fig. 3 suggests many features of the outline of the continental block, and there can be no doubt that the coefficients could be adjusted so as to secure a better agreement.* It seems best, however, to record the results as they are. For the sake of comparison a rough map of the world is added (fig. 4). The heavy continuous line is the outline of the continental block at mean-sphere-level, and the fine continuous line is the coast-line. I have not attempted to draw the map with minute accuracy, and have omitted many small islands and some small enclosed patches of deep sea, because the object aimed at is a comparison of the general features of the map of the world with those of the diagram in fig. 3. The map is drawn by taking the longitude east of Greenwich and the latitude of any point as the Cartesian co-ordinates of the corresponding point of the map. Fig. 3 is drawn in the same way.

The defects of the arrangement in fig. 3, considered as representing the shape of the continental block, are sufficiently obvious, the chief being the absence of any indication of an Arctic ocean, and the almost complete submersion of South America. On the other hand, the fact that even tolerable agreement in so many respects is obtained from a spherical harmonic analysis of the extremely simple distribution detailed in the table of § 56 may be regarded as a confirmation of the theory which led us to assume that harmonics of the first, second, and third degrees should be predominant.

Geological Implications of the Theory.

62. The results appear to admit of a geological interpretation. We have adduced dynamical reasons for the hypothesis that the lithosphere consolidated in a shape which may be described as an ellipsoid with three unequal axes, with its centre of gravity displaced from its centre of figure, and with a wrinkle upon its surface expressed by spherical surface harmonics of the third degree ; and we have found that the figure of the lithosphere now, as determined by difference of level above or below

* The coefficients r, ϵ, a, b, c are especially sensitive to changes in the assumed distribution in the Arctic and Antarctic regions where the actual distribution is least known.

the geoid, is expressible, at least roughly and approximately, by means of harmonics of the first, second, and third degrees. Now, if the shape of the lithosphere is at all close to that in which it may be presumed to have consolidated, the inference would seem to be that, in respect of general features, as distinguished from local irregularities, the positions of the continental blocks and oceanic regions have not changed much since the date of consolidation. This view has in recent times met with considerable support among geologists.

The theory also enables us to make some attempt to indicate the general nature of those changes which could be expected to take place. In estimating the value of such an attempt some allowance must be made for the fact that the theory of an elastic body in a state of initial stress is very far from complete. We try to follow out certain clues drawn from the scanty knowledge we possess of bodies in states of initial stress. Among these the behaviour of cast iron under tensile tests is perhaps important. It is well known that cast iron which has not previously been tested exhibits a stress-strain curve which is essentially different from that of mild steel, but that, after several tests, its behaviour approaches to that of steel. It has been conjectured that the tests have the effect of gradually removing a state of initial stress, and thus reducing the substance to a "state of ease." That state of a rotating gravitating planet which would correspond to a state of ease in solid bodies at its surface would seem to be a state in which the material would be arranged in concentric spheroidal layers of equal density, and the external surface would be an oblate spheroid, the ellipticity being determined by the speed of rotation and the distribution of density; the state of stress in the planet, when in this state of ease, would be one of hydrostatic pressure, and the surface would be an equipotential surface under gravity modified by the rotation. The partial reduction of the body to the state of ease would be effected by gradual stages, probably of the nature of local fractures. Now the wrinkling of the surface, expressed by harmonics of the third degree, arose as a consequence of the displacement of the centre of gravity and of the ellipsoidal configuration. It would at first be small in comparison with the deviations from spherical symmetry which are expressed by harmonics of the first and second degrees. We should therefore expect that the tendency of secular change in the shape of the lithosphere would be to diminish the coefficients of the harmonics of the first and second degrees. An exception must be made in the case of the coefficient ϵ of the zonal harmonic of the second degree; for this coefficient represents a difference of ellipticity of the meridians of lithosphere and geoid, and these ellipticities depend upon the speed of rotation. When this coefficient is left out of account, the harmonic inequality of the second degree represents ellipticity of the equator* and obliquity of the principal planes; the harmonic inequality of the first degree represents displacement of the centre of gravity from the centre of figure. If the coefficients of

* G. H. DARWIN concluded from his theory of the tidal deformation of a viscous spheroid that an initial ellipticity of the equator would tend to be obliterated. 'Phil. Trans. Roy. Soc.,' vol. 170 (1879), p. 30.

the harmonics of the first degree have ratios anywhere near to those given in the table of § 60, the great circle along which the harmonic inequality of the first degree vanishes has a pole somewhere in south-eastern Europe and the opposite pole in the Pacific Ocean. The inequality is positive in Europe, most of Asia, Africa, North America, the northern and central parts of the Atlantic Ocean, and the Arctic regions. The effect of a gradual diminution of the coefficients of the harmonics of the first degree would be a gradual emptying of the Pacific Ocean, accompanied by a rise of sea-level around the shores of the Atlantic Ocean (except towards the southern parts of Africa and South America) and around the northern and western parts of the Indian Ocean. It has been held that such an effect has taken place and constitutes the reason for the difference between a "Pacific coast" and an "Atlantic coast."* The ratios of the coefficients of the various harmonics of the second degree for the two distributions considered in §§ 56–60 are widely divergent, but they agree in leading to negative values for the harmonic inequality of the second degree in the regions contained within oval curves which lie within the basin of the Pacific, and also in the antipodal regions. In a large part of the Pacific region the harmonics of the first and second degrees reinforce each other; in the antipodal region they are antagonistic. Diminution of the coefficients of the harmonics of the second degree would be manifested by a fall of sea-level in the Pacific, and also in a region antipodal to some part of the Pacific. It may not, perhaps, be altogether fanciful to see in the gradual reduction of area of the "Central Mediterranean Sea" of Mesozoic and Tertiary times† the effect of a continual reduction of those coefficients of harmonic inequalities of the second degree which represent ellipticity of the equator and obliquity of the principal planes. Whether these conjectures as to the particular regions which may have been affected are acceptable or not, it can safely be said that the effects of changes in the harmonic inequality of the first degree, and in those of the second degree which we are now considering, would be progressive in the same sense at the same place. They would be manifested in a tendency of the sea to fall in certain regions and to rise in certain complementary regions and gradually to flood wide areas. The gradual character of the positive movements of the strand-line, by which wide areas have been submerged, has been emphasized by SUESS.‡

The surface of the lithosphere is nearly an oblate spheroid which does not coincide precisely with an equipotential under gravity modified by the rotation; it is less oblate than the geoid. The surface of a shallow ocean covering an oblate spheroidal planet, whose outer surface is not exactly an equipotential surface, is an oblate spheroid, and its ellipticity is a certain multiple of the ellipticity of the surface of the planet. The ratio of the two ellipticities depends partly on the rigidity of the planet,

* E. SUESS, 'The Face of the Earth' (Translation), vol. 2, Oxford, 1906, p. 553.

† *Ibid.*, pp. 258, 299.

‡ *Ibid.*, p. 543.

partly on the ratio of the density of the ocean to the mean density of the planet, and partly on the angular velocity. Owing to tidal friction, the angular velocity of the Earth's rotation is being gradually diminished. The effect of this is that both the ellipticity of the lithosphere and that of the geoid are being diminished, and the difference of these ellipticities is also being diminished. If, therefore, the shape of the lithosphere were continually adjusted to the instantaneous angular velocity, the value of the coefficient ϵ of § 57 would diminish continually, and the adjustment would involve a continually increasing deformation. Eventually the deformation would be so great that the strength of the material would be too small to withstand it, and local fractures would take place.* There is, therefore, a constant tendency for the sea-level to rise in the polar regions and to fall in the equatorial regions, the separation between the regions of rising and falling sea-level being marked by the zero-lines of the zonal harmonic of the second degree, that is, by the parallels of latitude, about 35° N. and 35° S. This rise and fall would be checked at intervals by subsidences, accompanied by series of earthquakes, in equatorial regions.

The effects produced by diminution of the displacement of the centre of gravity, and by changes in the ellipticity of the equator and in the obliquity of the principal planes, appear to be of a different character from the effect of diminishing angular velocity. The former would seem to be spasmodic and occasional, but always in the same sense at the same place; the latter would appear to consist of continuous movements in the same sense, extending over long periods, which are followed by comparatively short periods of spasmodic change in the opposite sense.

These remarks are frankly speculative, and I am well aware that many causes which have contributed to geological changes have been left out of sight. They are put forward as tentative suggestions which, it is hoped, may prove to be of some assistance in the solution of some of the still unsolved problems of geology.

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* According to a "Note" in 'Nature,' vol. 39 (1889), p. 613, this effect of diminishing speed of rotation was noted by M. A. BLYTT. I have not seen the paper referred to in the "Note."