VII. *Memoir on the Theory of the Partitions of Numbers.—Part IV. On the Probability that the Successful Candidate at an Election by Ballot may never at any time have Fewer Votes than the One who is Unsuccessful; on a Generalization of this Question; and on its Connexion with other Questions of Partition, Permutation, and Combination.*

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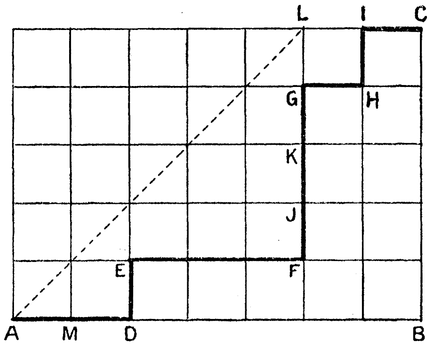
Section 1.

1. Consider a lattice in two dimensions, taking, for instance, one in which AB, BC are 7 and 5 segments in length respectively. It may be utilised for the study of permutations, combinations, and partitions in various ways, also for the study of certain questions in the theory of probabilities.



2.* A "line of route" through the lattice from A to C may be traced by moving over horizontal segments (α segments) in the direction AB, and over vertical segments (β segments) in the direction BC in any order. Thus one line of route is ADEFGHIC. The number of such lines of route is

$$\binom{12}{7},$$

or, in general, if AB, BC contain m, n segments respectively,

$$\binom{m+n}{m}.$$

* See "Memoir on the Theory of the Compositions of Numbers," 'Phil. Trans.,' A, 1893.

3. The line of route above depicted denotes a "principal composition" of the bipartite number $(\overline{75})$, viz.,

$$(\overline{21},\ \overline{33},\ \overline{11},\ \overline{10}),$$

and, in general, some principal composition of the bipartite number $(\overline{mn})$.

4. It also denotes the permutation

$$\alpha^2\beta\alpha^3\beta^3\alpha\beta\alpha$$

of the letters in the product $\alpha^7\beta^5$; and, in general, some permutation of the letters in the product $\alpha^m\beta^n$.

5. The line of route divides the lattice into two portions, each of which denotes the Sylvester-Ferrers graph of a partition of a unipartite number.

Consider, for example, the portion of the lattice bounded by ADEFGHICB. We obtain the graph of a partition in two ways:—

(i) By placing a unit (or node) in the centre of each square contained in the bounded area; thus

```
    1
   11
   11
   11
11111
```

denotes the partition 54111, or its conjugate 52221 of the number 12.

In general, we thus obtain a partition of some numbers into n, or fewer, parts, the part magnitude being limited not to exceed m, and its conjugate a partition of some numbers into m, or fewer, parts, the part magnitude being limited not to exceed n.

Similarly the remaining portion of the lattice denotes some partition and its conjugate.

(ii) By placing a unit (or node) at the centre of each segment to the right hand of the points A, E, J, K, G, I respectively; thus

```
      1
     11
     11
     11
  11111
1111111
```

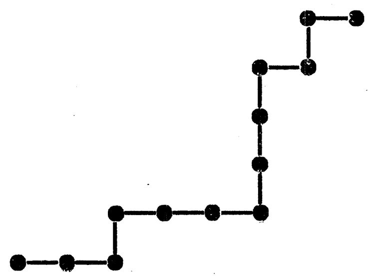
denotes the partition 6522211, or its conjugate 752221 of the number 19.

In general, we thus obtain a partition of some numbers into m parts, the part magnitude being limited not to exceed $n+1$, and its conjugate, a partition having m for the highest part, the number of parts being limited not to exceed $n+1$.

The remaining portion of the lattice may be similarly interpreted.

6. The line of route also denotes the zig-zag graph of a composition of a unipartite number.

For placing nodes at all points passed over by the line of route we obtain



the graph of the composition 341122, and also of three other compositions

221143
12411211
11211421

of the number 13. (See 'Phil. Trans.,' Series A, vol. 207, pp. 65–134.)

In general, we thus obtain four compositions of the unipartite number $m+n+1$.

It will thus be noted that these four compositions of a unipartite number define two pairs of partitions of unipartite numbers, and clearly every theorem in partitions can be made to give a corresponding theorem of compositions.

This manifold interpretation of the line of route through a lattice must be borne in mind throughout the following investigation.

7. My object now is to show how certain questions of probability can be treated by means of the lattice.

BERTRAND and DESIRÉ ANDRÉ* have discussed a question which they have stated in the following terms :—

"Pierre et Paul sont soumis à un scrutin de ballottage ; l'urne contient m bulletins favorables à Pierre, n favorables à Paul ; m est plus grand que n, Pierre sera élu. Quelle est la probabilité pour que, pendant le dépouillement du scrutin, les bulletins sortent dans un ordre tel que Pierre ne cesse pas un seul instant d'avoir l'avantage ?"

The probability is found by an ingenious method to be

$$\frac{m-n}{m+n}.$$

8. I discuss the question by drawing in the lattice the line AL,† making an angle of 45° with the line AB. The problem of BERTRAND and ANDRÉ is seen to be

* 'Calcul des Probabilités,' par J. BERTRAND, Paris, 1888. J. BERTRAND et D. ANDRÉ, 'Comptes Rendus de l'Académie des Sciences,' tome cv., p. 369 et 436, Paris, 1887.

† "Théories des Nombres," tome 1, par EDOUARD LUCAS, 'Le Scrutin de ballottage' (pp. 83, 84, 164).

identical with that of enumerating the lines of route which neither cross nor touch the line AL, for each such line of route gives a permutation of the letters in $\alpha^m\beta^n$ which is required by the conditions.

I prefer in the first instance to alter the conditions of the problem so as to determine the probability that Pierre never at any instant has fewer votes than Paul. The lines of route to be enumerated are then those which do not cross, but which may touch the line AL.

Owing to the different interpretations that may be given to the line of route many courses of procedure are open. I select that one which in the special graphs gives the partition

$$752221$$

and, in general, a partition having m for the highest part and a number of parts not exceeding $n+1$.

Regarding zero as an admissible part, let the parts of such a partition be (in descending order)

$$\alpha_1, \alpha_2, \ldots \alpha_{n+1}.$$

These parts are subject to the conditions

$$\begin{array}{ll} \alpha_1 \geq \alpha_2+1 & (a_1), \\ \alpha_2 \geq \alpha_3 & (a_2), \\ \alpha_1 \geq \alpha_3+2 & (a_3), \\ \alpha_3 \geq \alpha_4 & (a_4), \\ \alpha_1 \geq \alpha_4+3 & (a_5), \\ \vdots & \\ \alpha_{n'-1} \geq \alpha_{n'} & (a_{2n'-4}), \\ \alpha_1 \geq \alpha_{n'}+n' \quad 1 & (a_{2n'-3}), \end{array}$$

where $n' = n+1$.

We can perform the summation

$$\Sigma x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_{n'}^{\alpha_{n'}}$$

for all numbers satisfying the above $2n'-3$ conditions.

9. Suppose $n = 3$, the sum in question is

$$\underset{\geq}{\Omega} \frac{a_1^{-1}a_3^{-2}}{1-a_1a_3x_1 \,.\, 1-\dfrac{a_2}{a_1}x_2 \,.\, 1-\dfrac{1}{a_2a_3}x_3},*$$

the meaning of the symbol $\underset{\geq}{\Omega}$ being that, after expansion in ascending powers of x_1, x_2, x_3, all terms involving negative powers of a_1, a_2, a_3 are to be rejected, and that in the surviving terms a_1, a_2, a_3 are, each of them, to be put equal to unity.

* See "Memoir on the Theory of the Partitions of Numbers.—Part II," 'Phil. Trans.,' A, vol. 192, 1899.

The quantity a_2 is at once eliminated and we obtain

$$\underset{\geqq}{\Omega}\frac{a_1^{-1}a_3^{-2}}{1-a_1a_3x_1\,.\,1-\dfrac{1}{a_1}x_2\,.\,1-\dfrac{1}{a_1a_3}x_2x_3};$$

a_3 is now easily eliminated and we obtain

$$\underset{\geqq}{\Omega}\frac{a_1x_1^2}{1-a_1x_1\,.\,1-\dfrac{1}{a_1}x_2\,.\,1-x_1x_2x_3}.$$

To eliminate a_1 we require the easily established theorem

$$\underset{\geqq}{\Omega}\frac{c^s}{1-cx\,.\,1-\dfrac{1}{c}y}=\frac{1+y+\ldots+y^s-xy\,(1+y+\ldots+y^{s-1})}{1-x\,.\,1-xy}.$$

Thence we reach the final result

$$\frac{x_1^2+x_1^2x_2-x_1^3x_2}{1-x_1\,.\,1-x_1x_2\,.\,1-x_1x_2x_3},$$

which shows in the clearest possible manner how the partitions are constructed. The denominator factors indicate that we may write down any partition composed of three parts; the numerator terms x_1^2, $x_1^2x_2$ show that we may add either 2 to the first part or simultaneously 2 to the first and 1 to the second part. We in that manner obtain every partition satisfying the conditions, but the numerator term $-x_1^3x_2$ shows that certain partitions are in this manner obtained twice over.

10. As we are only concerned with the magnitude of the highest part and not at all with the weight of the partition, we may for the present purpose put $x_1=x$, $x_2=x_3=1$, and consider the result

$$\frac{2x^2-x^3}{(1-x)^3}$$

as the one to generalise.

11. I write down the expression

$$\underset{\geqq}{\Omega}\frac{a_1^{-1}a_3^{-2}a_5^{-3}\ldots a_{2n'-5}^{-(n'-2)}a_{2n'-3}^{-(n'-1)}}{1-a_1a_3a_5\ldots a_{2n-3}x\,.\,1-\dfrac{a_2}{a_1}\,.\,1-\dfrac{a_4}{a_2a_3}\,.\,1-\dfrac{a_6}{a_4a_5}\,.\,\ldots\,1-\dfrac{a_{2n'-4}}{a_{2n'-6}a_{2n'-5}}\,.\,1-\dfrac{1}{a_{2n'-4}a_{2n'-3}}}$$

as the crude expression for the sum.

We can immediately eliminate all the auxiliaries a which have an even suffix and reach the expression

$$\underset{\geqq}{\Omega}\frac{a_1^{-1}a_3^{-2}a_5^{-3}\ldots a_{2n'-5}^{-(n'-2)}a_{2n'-3}^{-(n'-1)}}{1-a_1a_3a_5\ldots a_{2n-3}x\,.\,1-\dfrac{1}{a_1}\,.\,1-\dfrac{1}{a_1a_3}\,.\,1-\dfrac{1}{a_1a_3a_5}\,.\,\ldots\,1-\dfrac{1}{a_1a_3\ldots a_{2n'-3}}}.$$

We now require the auxiliary theorem

$$\underset{\geqq}{\Omega} \frac{c^{-s}}{1-cx_1 \,.\, 1-\frac{1}{c}x_2 \,.\, 1-\frac{1}{c}x_3 \ldots 1-\frac{1}{c}x_p}$$

$$= \frac{x_1^{\,s}}{1-x_1 \,.\, 1-x_1x_2 \,.\, 1-x_1x_3 \,, \ldots 1-x_1x_p};$$

so that, eliminating α_1, we reach

$$\underset{\geqq}{\Omega} \frac{\alpha_3^{-1}\alpha_5^{-2}\ldots\alpha_{2n'-5}^{-(n'-3)}\alpha_{2n'-3}^{-(n'-2)}x}{(1-\alpha_3\alpha_5\ldots\alpha_{2n'-3}x)^2(1-\alpha_5\ldots\alpha_{2n'-3}x)(1-\alpha_7\ldots\alpha_{2n'-3}x)\ldots(1-\alpha_{2n'-3}x)(1-x)};$$

and, eliminating α_3,

$$\underset{\geqq}{\Omega} \frac{2\alpha_5^{-1}\alpha_7^{-2}\ldots\alpha_{2n'-5}^{-(n'-4)}\alpha_{2n'-3}^{-(n'-3)}x^2-\alpha_7^{-1}\ldots\alpha_{2n'-5}^{-(n'-5)}\alpha_{2n'-3}^{-(n'-4)}x^3}{(1-\alpha_5\alpha_7\ldots\alpha_{2n'-3}x)^3(1-\alpha_7\ldots\alpha_{2n'-3}x)\ldots(1-\alpha_{2n'-3}x)(1-x)}.$$

Note that for $n' = 3$ or $n = 2$ this becomes the before obtained expression

$$\frac{2x^2-x^3}{(1-x)^3}.$$

To eliminate α_5, we have to substitute for

$$\frac{\alpha_5^{-1}}{(1-\alpha_5\alpha_7\ldots\alpha_{2n'-3}x)^3}$$

the expression

$$\frac{\alpha_5^{-1}}{(1-\alpha_5\alpha_7\ldots\alpha_{2n'-3}x)^3}-1,$$

and then put $\alpha_5 = 1$, thus getting

$$2\alpha_7^{-2}\alpha_9^{-3}\ldots\alpha_{2n'-5}^{-(n'-4)}\alpha_{2n'-3}^{-(n'-3)}(3\alpha_7\ldots\alpha_{2n'-3}x-3\alpha_7^{\,2}\ldots\alpha^2{}_{2n'-3}x^2+\alpha_7^{\,3}\ldots\alpha^3{}_{2n'-3}x^3)\,x^2$$

as the expression to be substituted for

$$2\alpha_5^{-1}\alpha_7^{-2}\ldots\alpha_{2n'-3}^{-(n'-3)}x^2$$

in the numerator, which thus becomes

$$\underset{\geqq}{\Omega} \frac{5\alpha_7^{-1}\alpha_9^{-2}\ldots\alpha_{2n'-3}^{-(n'-4)}x^3-6\alpha_9^{-1}\alpha_{11}^{-2}\ldots\alpha_{2n'-3}^{-(n'-5)}x^4+2\alpha_7\alpha_{11}^{-1}\alpha_{13}^{-2}\ldots\alpha_{2n'-3}^{-(n'-6)}x^5}{(1-\alpha_7\alpha_9\ldots\alpha_{2n'-3}x)^4(1-\alpha_9\ldots\alpha_{2n'-3}x)\ldots(1-\alpha_{2n'-3}x)(1-x)}.$$

This, for $n' = 4$ or $n = 3$, is

$$\frac{5x^3-6x^4+2x^5}{(1-x)^4},$$

which may be written

$$\frac{2x^2(3x-3x^2+x^3)-x^3}{(1-x)^4},$$

in a form showing its mode of derivation from

$$\frac{2x^2-x^3}{(1-x)^3}.$$

We now see that, when $n' = p+1$, we get a form which may be written

$$\frac{u_{p1}x^p - u_{p2}x^{p+1} + \ldots (-)^{p-1} u_{p,p} x^{2p-1}}{(1-x)^{p+1}};$$

and then, when $n' = p+2$, the form is

$$\frac{u_{p+1,1}x^{p+1} - u_{p+1,2}x^{p+2} + \ldots (-)^{p+1} u_{p+1,p+1} x^{2p+1}}{(1-x)^{p+2}},$$

where the numerator of the function last written is

$$u_{p1}x^p \left\{ \binom{p+1}{1} x - \binom{p+1}{2} x^2 + \ldots (-)^p x^{p+1} \right\} - u_{p2}x^{p+1} + u_{p3}x^{p+2} - \ldots (-)^p u_{pp} x^{2p-1}.$$

Hence

$$u_{p+1,1} = \binom{p+1}{1} u_{p1} - u_{p2},$$

$$u_{p+1,2} = \binom{p+1}{2} u_{p1} - u_{p3},$$

$$u_{p+1,3} = \binom{p+1}{3} u_{p1} - u_{p4},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$u_{p+1,p-1} = \binom{p+1}{p-1} u_{p1} - u_{pp},$$

$$u_{p+1,p} = \binom{p+1}{p} u_{p1},$$

$$u_{p+1,p+1} = \binom{p+1}{p+1} u_{p1}.$$

These relations are satisfied by

$$u_{pq} = \frac{1}{p} \binom{p+q-2}{q-1} \binom{2p}{p-q},$$

so that the result of the summation is

$$\frac{\frac{1}{n}\binom{n-1}{0}\binom{2n}{n-1}x^n - \frac{1}{n}\binom{n}{1}\binom{2n}{n-2}x^{n+1} + \frac{1}{n}\binom{n+1}{2}\binom{2n}{n-3}x^{n+2} - \ldots (-)^n \frac{1}{n}\binom{2n-2}{n-1}\binom{2n}{0}x^{2n-1}}{(1-x)^{n+1}};$$

and there is no difficulty in showing that this in fact is equal to

$$\sum_{s=n}^{s=\infty} \left\{ \binom{n+s-1}{n} - \binom{n+s-1}{n-2} \right\} x.$$

12. Hence the number of partitions having a highest part m and $n+1$ parts, zero being included as a part, subject to the given conditions as regards magnitude is

$$\binom{n+m-1}{n}-\binom{n+m-1}{n-2},$$

which may be also written

$$\frac{m-n+1}{m+1}\binom{m+n}{m}.$$

This, therefore, is the number of lines of route which do not cross the line AL.

Hence the probability that Pierre is never in a minority is

$$\frac{m-n+1}{m+1}.$$

13. From this probability, which call F (m, n), is immediately derivable the probability discussed by BERTRAND and ANDRÉ, which call P (m, n).

For in the lattice $\frac{m}{m+n}$ is the probability that the line ot route passes through the point M, and thence we find

$$\mathrm{P}(m, n)=\frac{m}{m+n} \mathrm{F}(m-1, n)=\frac{m-n}{m+n}.$$

Other probability questions may be discussed in a similar manner, with the advantage that light is at the same time thrown upon several other problems of partitions, compositions, and combinations of unipartite and bipartite numbers. In the above investigation we have had before us partitions of unipartite numbers which have a given number of parts, a given highest part and parts which in addition satisfy certain inequalities.

14. If we had had before us the parallel theory of the compositions of unipartite numbers there would have been the composition

$$\beta_1\beta_2\beta_3\ldots\beta_n$$

in correspondence with the partition

$$\alpha_1\alpha_2\alpha_3\ldots\alpha_n,$$

the weight $\Sigma\beta$ and the number of parts n would have been given and the parts $\beta_1, \beta_2, \ldots$ would have been subject to the inequalities

$$\begin{gathered}\beta_1 \geqq 2,\\ \beta_1+\beta_2 \geqq 4,\\ \beta_1+\beta_2+\beta_3 \geqq 6,\\ \vdots\\ \beta_1+\beta_2+\ldots+\beta_{n-1} \geqq 2n-2.\end{gathered}$$

In the former case the partitions of highest part m and n parts (zero not excluded) are enumerated by

$$\frac{m-n+2}{m+1}\binom{m+n-1}{m}.$$

In the latter the compositions of the number w, having n parts (zero excluded because $\beta_n = \alpha_n+1$), are enumerated by

$$\frac{w-2n+2}{w-n+1}\binom{w-1}{n-1}.$$

Ex. gr. for $w = 6$, $n = 3$, we have the five compositions

411
321
312
231
222

which satisfy the given inequalities.

In this Section I have shown the connexion between a well-known question in probabilities and various other combinatorial questions in preparation for the generalization to which I now proceed in Section 2.

Section 2.

15. In the Second Memoir on the Partitions of Numbers I broached the subject of the two-dimensional partitions of numbers. I start with any Sylvester-Ferrers graph of an ordinary one-dimensional partition—say

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● ● ● ●
● ● ● ●
● ●
●

and I consider the parts of the partition to be placed at the nodes in suchwise that the numbers in the rows, read from West to East, and also in the columns read from North to South, are in descending order of magnitude. Thus

433222
3222
2111
21
2

is a two-dimensional partition of the number 35.

The Memoir referred to contained some striking results in the theory, but the general result as conjectured and verified in numerous instances remained unproved.

The present paper is mainly concerned with the partitions into different parts placed at the nodes of any graph, and with the associated question in probabilities, a generalization of that of Section 1.

Taking any graph of n nodes and any n different integers, the inquiry is as to the number of ways of placing the numbers at the nodes so that the descending orders in the rows and columns, as above defined, are in evidence.

Consider in detail a simple case—that of six different numbers at the nodes of the graph

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We find the 16 arrangements

654	654	653	653	652	652	651	651
32	31	42	41	43	41	43	42
1	2	1	2	1	3	2	3
631	632	641	642	641	643	642	643
52	51	52	51	53	51	53	52
4	4	3	3	2	2	1	1

the second row of eight arrangements being the conjugates of those in the first row because the graph is self-conjugate.

16. The problem is immediately transformable into one concerned with the conditioned permutations of the six numbers in a line.

Take the form

$$\begin{array}{l} 654 \longrightarrow \\ 32 \ \longrightarrow \\ 1 \\ \downarrow\downarrow \end{array}$$

and suppose the six numbers to be written down in a line so that four descending orders

$$\begin{array}{l} 654 \longrightarrow \\ 32 \ \longrightarrow \\ 631 \longrightarrow \\ 52 \ \longrightarrow \end{array}$$

are in evidence; I say that there is a one-to-one correspondence between such permutations and the two-dimensional partitions under investigation.

To see how this is, take any one of the 16 arrangements

632
51
4

and taking each number in succession, in order from the highest to the lowest, write a letter α, β, or γ, according as the number is in the first, second, or third row. Thus beginning with 6 we write down α, then for 5 β, for 4 γ, for 3 α, for 2 α, and, lastly, for 1 β, thus obtaining

$$\alpha,\ \beta,\ \gamma,\ \alpha,\ \alpha,\ \beta.$$

Now underneath the α's write 6, 5, 4 in order, under the β's, 3, 2 in order, and under γ 1, in accordance with the rows of the arrangement

$$\begin{array}{l} 654 \\ 32 \\ 1 \end{array}$$

We thus obtain

$$\begin{array}{cccccc} \alpha & \beta & \gamma & \alpha & \alpha & \beta \\ 6 & 3 & 1 & 5 & 4 & 2. \end{array}$$

I say that

$$631542$$

is a permutation subject to the given conditions as defined by the descending orders in the arrangement

$$\begin{array}{l} 654 \\ 32 \\ 1 \end{array}$$

The 16 permutations corresponding to the 16 graph arrangements are

$\alpha\alpha\alpha\beta\beta\gamma$ 654321	$\alpha\alpha\alpha\beta\gamma\beta$ 654312	$\alpha\alpha\beta\alpha\beta\gamma$ 653421	$\alpha\alpha\beta\alpha\gamma\beta$ 653412	$\alpha\alpha\beta\beta\alpha\gamma$ 653241	$\alpha\alpha\beta\gamma\alpha\beta$ 653142
$\alpha\alpha\beta\beta\gamma\alpha$ 653214	$\alpha\alpha\beta\gamma\beta\alpha$ 653124	$\alpha\beta\gamma\alpha\beta\alpha$ 631524	$\alpha\beta\gamma\alpha\alpha\beta$ 631542	$\alpha\beta\alpha\gamma\beta\alpha$ 635124	$\alpha\beta\alpha\gamma\alpha\beta$ 635142
	$\alpha\beta\alpha\beta\gamma\alpha$ 635214	$\alpha\beta\alpha\alpha\gamma\beta$ 635412	$\alpha\beta\alpha\beta\alpha\gamma$ 635241	$\alpha\beta\alpha\alpha\beta\gamma$ 635421.	

To show that there are no other permutations it is sufficient to prove that one can pass back from a permutation to a graph in a unique manner.

Thus take the permutation

$$635124\,;$$

write α's under 6, 5, 4 ; β's under 3, 2 ; and γ under 1 :—

$$\begin{array}{c} 635124 \\ \alpha\beta\alpha\gamma\beta\alpha\,; \end{array}$$

the succession

$$\alpha\beta\alpha\gamma\beta\alpha$$

indicates that 6 is in the first row corresponding to α,

5	,,	second	,,	,,	,,	β,
4	,,	first	,,	,,	,,	α,
3	,,	third	,,	,,	,,	γ,
2	,,	second	,,	,,	,,	β,
1	,,	first	,,	,,	,,	α.

Hence the arrangement

$$\begin{array}{l} 641 \\ 52 \\ 3 \end{array}$$

The transformation is quite general; thus from

$$\begin{array}{cccc} 12 & 9 & 7 & 4 \\ 11 & 8 & 6 & 2 \\ 10 & 5 & 3 & 1 \end{array}$$

we pass to

$$\begin{array}{cccccccccccc} \alpha & \beta & \gamma & \alpha & \beta & \alpha & \beta & \gamma & \alpha & \gamma & \beta & \gamma \\ 12 & 8 & 4 & 11 & 7 & 10 & 6 & 3 & 9 & 2 & 5 & 1, \end{array}$$

a permutation in which the descending orders indicated by

$$\begin{array}{cccc} 12 & 11 & 10 & 9 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{array}$$

are in evidence.

17. The first question is the enumeration of the partitions where the graph and the set of unequal numbers are given.

Let the graph contain a, b, c, ... nodes in the successive rows, and for a given set of $a+b+c+\ldots$ unequal numbers, let

$$(abc\ldots;)$$

denote the number of partitions.

Observe that above we found

$$(321;) = 16.$$

First we note that

$$(a;) = 1.$$

*Next take a graph of two rows. In any such graph

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* Compare "Problème des deux files de soldats," 'Théories des Nombres,' tome 1, p. 86, par EDOUARD LUCAS.

if ϵ be the smallest number involved, the arrangements are of two types, viz.,

$$\begin{array}{l}\bullet\;\bullet\;\bullet\;\bullet\;\bullet\;\bullet\;\bullet\;\epsilon\\ \bullet\;\bullet\;\bullet\end{array}$$

or

$$\begin{array}{l}\bullet\;\bullet\;\bullet\;\bullet\;\bullet\;\bullet\;\bullet\;\bullet\\ \bullet\;\bullet\;\epsilon\end{array}$$

except when the rows contain the same number of nodes; then there is the one type

$$\begin{array}{l}\bullet\;\bullet\;\bullet\;\bullet\;\bullet\;\bullet\\ \bullet\;\bullet\;\bullet\;\bullet\;\bullet\;\epsilon\end{array}$$

Hence, a moment's consideration establishes the relations

$$(ab\,;) = (a-1,\ b\,;)+(a,\ b-1\,;)$$

when $a > b$, and

$$(aa\,;) = (a,\ a-1\,;).$$

Treating these as difference equations it is easy to obtain the result

$$(ab\,;) = \frac{(a+b)\,!}{(a+1)\,!\ b\,!}(a-b+1) = \binom{a+b}{a}\frac{a-b+1}{a+1}\,;$$

$$(aa\,;) = \frac{(2a)\,!}{(a+1)\,!\ a\,!} = \binom{2a}{a}\frac{1}{a+1}.$$

18. This case of two rows is worth a special examination before proceeding to a greater number of rows. First consider the generating function of the numbers $(aa;)$:

$$u_x = \Sigma\,(aa;)\,x^a = 1+x+2x^2+5x^3+14x^4+\ldots$$

If we expand

$$(1-4x)^{\frac{1}{2}}$$

we find that the general term after the first is

$$-\frac{(2r)\,!}{(r+1)\,!\ r\,!}\,2x^{r+1},$$

and thence

$$u_x = \frac{1}{2x}\{1-(1-4x)^{\frac{1}{2}}\}$$

and

$$xu_x^{\,2}-u_x+1 = 0,$$

exhibiting a remarkable property of u_x.

Reverting to the difference equation

$$\begin{aligned}(aa;) &= (a,\ a-1\,;)\\ &= (a,\ a-2\,;)+(a-1,\ a-1\,;)\\ &= (a,\ a-3\,;)+2\,(a-1,\ a-2\,;)\\ &= (a,\ a-4\,;)+3\,(a-1,\ a-3\,;)+2\,(a-2,\ a-2),\end{aligned}$$

and observing that this last result may be written

$$(aa\,;) = (40\,;)(a,\ a-4\,;)+(31\,;)(a-1,\ a-3\,;)+(22\,;)(a-2,\ a-2\,;),$$

it is natural to suspect the law

$$(aa\,;) = \Sigma\,(st\,;)(a-t,\ a-s\,;),$$

where

$$s+t = \text{constant},$$

and it is easy to establish it.

For consider the graph

● ● ● ● ● ● ● ●x

● ● ● ● ● ●x ●x ●x

The four lowest numbers at the nodes are

(i) The last four of the second row,

(ii) The last of the first row and the last three of the second,

(iii) The last two of both rows.

Taking case (ii), the nodes marked x, the numbers may be

$$\begin{array}{ccccc} 2 & & 3 & & 4 \\ 431 & \text{or} & 421 & \text{or} & 321, \end{array}$$

and these arrangements are enumerated by (31;), we see, by subtracting each number from the number 5. Hence, in particular,

$$(88\,;) = (40\,;)(84\,;)+(31\,;)(75\,;)+(22\,;)(66\,;),$$

and, in general,

$$(aa\,;) = \Sigma\,(st\,;)(a-t,\ a-s\,;) \quad \text{where} \quad s+t = \text{constant}.$$

19. *Representation of $(aa\,;)$ as a Sum of Squares.*

Putting $s+t = a$, we find

$$(aa\,;) = (a0\,;)^2+(a-1,\ 1\,;)^2+(a-2,\ 2\,;)^2+\ldots,$$

the last term being

$$(\tfrac{1}{2}a,\ \tfrac{1}{2}a\,;)^2 \quad \text{or} \quad \{\tfrac{1}{2}(a+1),\ \tfrac{1}{2}(a-1)\}^2,$$

ascending as a is even or uneven.

Hence the identity

$$\frac{(2a)\,!}{(a+1)\,!\ a\,!} = 1^2+(a-1)^2+\{\tfrac{1}{2}a\,(a-3)\}^2+\left\{\frac{1}{3\,!}\,a\,(a-1)\,(a-5)\right\}^2+\ldots,$$

the last term of the series being the square of

$$\frac{a\,!}{\{\frac{1}{2}(a+2)\}\,!\,(\frac{1}{2}a)\,!} \quad \text{or of} \quad \frac{2\,.\,a\,!}{\{\frac{1}{2}(a+3)\}\,!\,\{\frac{1}{2}(a-1)\}\,!},$$

ascending as a is even or uneven.

The permutations enumerated by $(aa\,;)$ are those of $2a$ numbers

$$\begin{array}{c} m_1 m_2 \ldots m_a, \\ n_1 n_2 \ldots n_a, \end{array}$$

all different and subject to $a+2$ descending orders corresponding to the a columns and the 2 rows.

20. Consider now other permutations such that whilst the row numbers are in descending order, exactly s of the a column pairs are not in descending order.

Let $(aa\,;\,s)$ be the number of such permutations. I propose to show that

$$(aa\,;\,s) = (aa\,;\,0) = (aa\,;)$$

for all values of s, from $s = 0$ to $s = a$.

Ex. gr., the permutations enumerated by

$$\begin{array}{lccc} (22\,;\,0) & \text{are} & 43 & 42 \\ & & 21 & 31, \\ \\ (22\,;\,1) & \text{are} & 41 & 32 \\ & & 32 & 41, \\ \\ (22\,;\,2) & \text{are} & 31 & 21 \\ & & 42 & 43. \end{array}$$

To establish this theorem, since

$$u_x = 1+(11\,;)\,x+(22\,;)\,x^2+(33\,;)\,x^3+\ldots$$

and

$$xu_x^{\,2} = u_x - 1,$$

we find

$$(aa\,;) = (a-1,\ a-1\,;)+(a-2,\ a-2\,;)\,(11\,;)+\ldots+(11\,;)\,(a-2,\ a-2\,;)+(a-1,\ a-1\,;).$$

The right-hand side of this identity is equal to

$$(aa\,;\,1),$$

for it consists of a terms, of which the first enumerates the permutations in which the pair $m_1 n_1$ is out of order, the second those in which the pair $m_2 n_2$ is out of order, and so on. Hence

$$(aa\,;) = (aa\,;\,1).$$

Consider in general the arrangement

$$\alpha_1\alpha_2\beta_3\beta_4\beta_5\alpha_6\beta_7\ldots,$$

$$\alpha'_1\alpha'_2\beta'_3\beta'_4\beta'_5\alpha'_6\beta'_7\ldots,$$

where the α pairs are in order and the β pairs out of order.

For this particular arrangement the enumeration is given by

$$(22\,;)\,(33\,;)\,(11;)\,(11;)\,\ldots.$$

Put

$$u_x = 1+\mathrm{X},$$

$$u_y = 1+\mathrm{Y}.$$

The generating function to be considered is

$$1+\mathrm{X}+\mathrm{Y}+\mathrm{XY}+\mathrm{YX}+\mathrm{XYX}+\mathrm{YXY}+\mathrm{XYXY}+\mathrm{YXYX}+\ldots.$$

The X, Y in a product occurring *alternately* in all possible ways.

This function is

$$\frac{(1+\mathrm{X})\,(1+\mathrm{Y})}{1-\mathrm{XY}}$$

or

$$\frac{u_x u_y}{u_x+u_y-u_x u_y}$$

or

$$\frac{xu_x-yu_y}{x-y},$$

since

$$xu_x^2-u_x+1 = yu_y^2-u_y+1 = 0.$$

Now

$$\frac{xu_x-yu_y}{x-y} = 1+(11;)\,(x+y)+(22\,;)\,(x^2+xy+y^2)+(33\,;)\,(x^3+x^2y+xy^2+y^3)+\ldots,$$

and the coefficient of $x^{a-s}y^s$ in the function is none other than

$$(aa\,;\,s).$$

Hence

$$(aa\,;\,s) = (aa\,;),$$

a remarkable theorem.

21. I will now obtain the generating function for the numbers $(ab\,;)$. Since

$$(ab\,;) = \binom{a+b}{a}\frac{a+1-b}{a+1},$$

$$\sum_b\sum_a (ab\,;)\,x^a y^b = \Sigma\Sigma\binom{a+b}{a}x^a y^b - \sum_0^\infty\sum_1^\infty\binom{a+b}{a}x^{a-1}y^{b+1}.$$

As yet we have assigned no meaning to $(ab\,;)$ when $a < b$, but retaining such terms and adding to the term

$$\Sigma\Sigma\binom{a+b}{a}x^{a-1}y^{b+1}$$

terms given by placing a equal to zero, we obtain the suggestive redundant generating function

$$\frac{1-\dfrac{y}{x}}{1-x-y},$$

in the expansion of which we require only those terms involving

$$x^a y^b$$

in which $a \geqq b$.

To eliminate the terms containing x^{-1} we have merely to add

$$\frac{\dfrac{y}{x}}{1-y},$$

and then we obtain

$$\frac{1-2y}{1-y,\ 1-x-y}.$$

We might also seek to remove those terms in $x^a y^b$ for which $a < b$, but for my present purpose the redundant form is quite as convenient and infinitely more suggestive.*

22. The result

$$(ab\,;) = \binom{a+b}{a}\frac{a-b+1}{a+1}$$

leads to the observation that the number obtained is precisely that obtained in Section 1 for the number of arrangements of the letters in

$$\alpha^a\beta^b$$

such that drawing a line between any two letters the number of α's to the left of the line $\geqq$ to the number of β's to the left of the line. Also that

$$(ab\,;) \div \binom{a+b}{a}$$

is the solution of the probability question for $a+b$ electors.

The one-to-one correspondence is easily established, for suppose

86531
742

* I have found that the reduced generating function is $\dfrac{1}{1-x-y} - \dfrac{1-\sqrt{(1-4xy)}}{2x(1-x-y)}$.

is an arrangement enumerated by (53 ;), I take the numbers 8, 7, 6, 5, 4, 3, 2, 1 in descending order and write down α when the number is in the first row and β when it is in the second, thus

$$\begin{array}{cccccccc} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \alpha & \beta & \alpha & \alpha & \beta & \alpha & \beta & \alpha, \end{array}$$

and we now have an arrangement of the letters in

$$\alpha^5\beta^3$$

such that, proceeding from left to right and stopping at any point, the number of α's met with is at least as large as the number of β's.

The result and correspondence are at once generalisable, for consider the arrangement

$$\begin{array}{l} 9765 \\ 832 \\ 4 \\ 1 \quad ; \end{array}$$

we are led to the permutation of $\alpha^4\beta^3\gamma\delta$, viz.,

$$\alpha\beta\alpha\alpha\gamma\beta\beta\delta,$$

which is such that, in passing from left to right, we have at any instant

(i) passed over at least as many α's as β's,
(ii) „ „ „ β's „ γ's,
(iii) „ „ „ γ's „ δ's.

Hence, if in an election four candidates have a, b, c, d (these numbers being in descending order of magnitude) supporters respectively, and at any instant they have respectively A, B, C, D votes, the probability that always

$$\mathrm{A} \geq \mathrm{B} \geq \mathrm{C} \geq \mathrm{D}$$

is

$$(abcd\,;) \div \frac{(a+b+c+d)!}{a!\,b!\,c!\,d!},$$

and, in general, if the votes polled at any instant show invariably the final order of the candidates, we have a state of affairs of which the probability is

$$(abcde\ldots\,;) \div \frac{(a+b+c+d+e+\ldots)!}{a!\,b!\,c!\,d!\,e!\ldots}.$$

23. From the result

$$(ab\,;) = \frac{(a+b)!}{(a+1)!\,b!}(a-b+1)$$

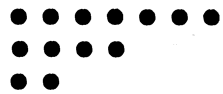
we find the analytical results

$$(a-1,\, a\,;) = 0,$$

$$(b-1,\, a+1\,;) = -\,(ab\,;),$$

and the latter of these is not so far interpretable.

Passing to three rows



containing a, b, and c nodes respectively, it is seen that the smallest of the $a+b+c$ different numbers must be situated at the right-hand nodes of some row, unless such row contains as many nodes as the row beneath.

Hence the difference equation

$$(abc\,;) = (a-1,\, b,\, c\,;) + (a,\, b-1,\, c\,;) + (a,\, b,\, c-1\,;),$$

provided that $(abc\,;) = 0$ when either

$$a-b+1 = 0 \quad \text{or} \quad b-c+1 = 0.$$

I find such a solution of the difference equation to be

$$(abc\,;) = \frac{(a+b+c)!}{(a+2)!\,(b+1)!\,c!}\,(a-b+1)\,(b-c+1)\,(a-c+2).$$

This is only interpretable when

$$a \geq b \geq c,$$

but *analytically*

$$+(abc\,;) = +(b-1,\, c-1,\, a+2\,;) = +(c-2,\, a+1,\, b+1\,;)$$

$$= -(a,\, c-1,\, b+1\,;) = -(b-1,\, a+1,\, c\,;) = -(c-2,\, b,\, a+2\,;),$$

relations which are useful for the manipulation of the functions.

The sum

$$\Sigma\Sigma\Sigma\,(abc;)\, x^a y^b z^c$$

with the inclusion of redundant terms I find to be

$$\frac{1-\dfrac{y}{x}\,.\,1-\dfrac{z}{y}\,.\,1-\dfrac{z}{x}}{1-x-y-z},$$

an expression which is remarkably suggestive.

To establish the expression it is merely necessary to verify that the coefficient of $x^a y^b z^c$, viz.:—

$$\frac{(a+b+c)!}{c!\,b!\,c!}-\frac{(a+b+c)!}{(a+1)!\,(b-1)!\,c!}-\frac{(a+b+c)!}{a!\,(b+1)!\,(c-1)!}$$
$$+\frac{(a+b+c)!}{(a+2)!\,(b-1)!\,(c-1)!}+\frac{(a+b+c)!}{(a+1)!\,(b+1)!\,(c-2)!}-\frac{(a+b+c)!}{(a+2)!\,b!\,(c-2)!}$$

reduces to the value of $(abc;)$ above given.

This is easy because it appears at once that $a-b+1$, $b-c+1$, and $a-c+2$ are factors.

It is easy now to conjecture the form of the general result.

The truth of the generating function for $(abc;)$ is, perhaps, best seen by writing

$$(abc;)=\binom{a+b+c}{a,\ b}\left(1-\frac{b}{a+1}\right)\left(1-\frac{c}{b+1}\right)\left(1-\frac{c}{a+2}\right),$$

where $\binom{a+b+c}{a,\ b}$ stands for $\dfrac{(a+b+c)!}{a!\,b!\,c!}$.

24. I will now establish the result

$$(a_1,\ a_2,\ a_3\ \ldots\ a_n;)$$
$$=\binom{a_1+a_2+a_3+\ldots+a_n}{a_1,\ a_2,\ \ldots\ a_{n-1}}$$
$$\times\left(1-\frac{a_2}{a_1+1}\right)\left(1-\frac{a_3}{a_2+1}\right)\left(1-\frac{a_4}{a_3+1}\right)\ldots\left(1-\frac{a_n}{a_{n-1}+1}\right)$$
$$\times\left(1-\frac{a_3}{a_1+2}\right)\left(1-\frac{a_4}{a_2+2}\right)\ldots\left(1-\frac{a_n}{a_{n-2}+2}\right)$$
$$\times\left(1-\frac{a_4}{a_1+3}\right)\ldots\left(1-\frac{a_n}{a_{n-3}+3}\right)$$
$$\cdots\cdots\cdots$$
$$\times\left(1-\frac{a_{n-1}}{a_1+n-2}\right)\left(1-\frac{a_n}{a_2+n-2}\right)$$
$$\times\left(1-\frac{a_n}{a_1+n-1}\right)$$

by showing that this expression satisfies the difference equation

$$(a_1,\ a_2,\ a_3,\ \ldots a_n;)=(a_1-1,\ a_2,\ a_3\ldots a_n;)+(a_1,\ a_2-1,\ a_3,\ \ldots a_n;)+\ldots+(a_1,\ a_2,\ a_3,\ \ldots a_n-1;).$$

Writing the co-factors of the multinomial coefficients in

$$(a_1,\ a_2,\ a_3,\ \ldots\ a_n;)\quad\text{and}\quad(a_1,\ a_2,\ \ldots\ a_s-1,\ \ldots\ a_n;)$$

as

$$\mathrm{C}\quad\text{and}\quad\mathrm{C}_s\ \text{respectively,}$$

we find

$$(a_1+a_2+a_3+\,.\,.+a_n)\,\mathrm{C}$$
$$=a_1\mathrm{C}_1+a_2\mathrm{C}_2+a_3\mathrm{C}_3+\ldots+a_n\mathrm{C}_n.$$

To prove this relation I will show that any factor

$$1-\frac{a_s}{a_t+s-t}(s>t)\text{ of C}$$

is also a factor of

$$a_1\mathrm{C}_1+a_2\mathrm{C}_2+\ldots+a_n\mathrm{C}_n.$$

First observe that $1-\dfrac{a_s}{a_t+s-t}$ is a factor of C_m unless m is equal to s or t. Therefore consider merely

$$a_t\mathrm{C}_t+a_s\mathrm{C}_s.$$

The factors of C which involve either a_t or a_s or both a_s and a_t are

$$\left(1-\frac{a_s}{a_t+s-t}\right)\prod_{m=1}^{m=t-1}\left(1-\frac{a_s}{a_m+s-m}\right)\prod_{m=t+1}^{m=s-1}\left(1-\frac{a_s}{a_m+s-m}\right)\prod_{p=s+1}^{p=n}\left(1-\frac{a_p}{a_s+p-s}\right)$$
$$\prod_{l=1}^{l=t-1}\left(1-\frac{a_t}{a_l+t-l}\right)\prod_{q=s+1}^{q=n}\left(1-\frac{a_q}{a_t+q-t}\right)\prod_{q=t+1}^{q=s-1}\left(1-\frac{a_q}{a_t+q-t}\right).$$

Hence, disregarding a common factor, C_t involves the factors

$$\frac{a_t-a_s+s-t-1}{a_t+s-t-1}\prod_{m=1}^{m=t-1}\frac{a_m-a_s+s-m}{a_m+s-m}\prod_{m=t+1}^{m=s-1}\frac{a_m-a_s+s-m}{a_m+s-m}\prod_{p=s+1}^{p=n}\frac{a_s-a_p-s+p}{a_s-s+p}$$
$$\prod_{l=1}^{l=t-1}\frac{a_l-a_t+t-l+1}{a_l+t-l}\prod_{q=s+1}^{q=n}\frac{a_t-a_q+q-t-1}{a_t+q-t-1}\prod_{q=t+1}^{q=s-1}\frac{a_t-a_q+q-t-1}{a_t+q-t-1},$$

and C_s involves the factors

$$\frac{a_t-a_s+s-t+1}{a_t+s-t}\prod_{m=1}^{m=t-1}\frac{a_m-a_s+s-m+1}{a_m+s-m}\prod_{m=t+1}^{m=s-1}\frac{a_m-a_s+s-m+1}{a_m+s-m}\prod_{p=s+1}^{p=n}\frac{a_s-a_p+p-s-1}{a_s+p-s-1}$$
$$\prod_{l=1}^{l=t-1}\frac{a_l-a_t+t-l}{a_l+t-l}\prod_{q=s+1}^{q=n}\frac{a_t-a_q-t+q}{a_t-t+q}\prod_{q=t+1}^{q=s-1}\frac{a_t-a_q-t+q}{a_t-t+q}.$$

Discarding common factors from these expressions we find that C_t involves the factors

$$\frac{a_t-a_s+s-t-1}{a_t+s-t-1}(a_{t-1}-a_s+s-t+1)(a_{s-1}-a_s+1)\frac{a_s-a_n-s+n}{a_s-s+n}$$
$$\times(a_1-a_t+t)\frac{a_t-a_{s+1}+s-t}{a_t+s-t}\cdot\frac{a_t-a_{t+1}}{a_t},$$

and that C_s involves the factors

$$\frac{a_t-a_s+s-t+1}{a_t+s-t}(a_1-a_s+s)(a_{t+1}-a_s+s-t)\frac{a_s-a_{s+1}}{a_s}$$
$$\times(a_{t-1}-a_t+1)\frac{a_t-a_n-t+n}{a_t-t+n}\cdot\frac{a_t-a_{s-1}+s-t-1}{a_t+s-t-1}.$$

Hence $a_tC_t+a_sC_s$ involves a factor which by elimination of a_t by means of the relation

$$1-\frac{a_s}{a_t+s-t}=0$$

may be written

$$\frac{(-1)(a_{t-1}-a_s+s-t+1)(a_{s-1}-a_s+1)(a_s-a_n-s+n)(a_1-a_s+s)(a_s-a_{s+1})(a_s-a_{t+1}-s+t)}{(a_s-1)\,a_s\,(a_s-s+n)}$$
$$+\frac{(+1)(a_1-a_s+s)(a_{t+1}-a_s+s-t)(a_s-a_{s+1})(a_{t-1}-a_s+s-t+1)(a_s-a_n-s+n)(a_s-a_{s-1}-1)}{a_s\,(a_s-s+\nu)\,(a_s-1)},$$

and this is at once seen to be zero. Hence

$$\sum_{m=1}^{m=n} a_mC_m$$

contains

$$1-\frac{a_s}{a_t+s-t}\quad(s>t)$$

as a factor. It therefore contains C as a factor.

It also contains another factor which is linear, and considerations of symmetry show it to be

$$a_1+a_2+a_3+\ldots a_n.$$

Hence the expression assumed for

$$(a_1,\ a_2,\ a_3,\ \ldots\ a_n\,;)$$

satisfies the difference equation, and is evidently the expression corresponding to the problem in hand.

25. We now observe that a redundant generating function, viz., an expression for the sum

$$\Sigma\,(a_1,\ a_2,\ a_3,\ \ldots\ a_n\,;)\,x_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots x_n^{a_n},$$

is

$$\frac{\prod\limits_{s=t+1}^{s=n}\ \prod\limits_{t=1}^{t=n-1}\left(1-\dfrac{x_s}{x_t}\right)}{1-(x_1+x_2+x_3+\ldots+x_n)}.$$

The question in probabilities, here, I believe, solved for the first time, may be stated as follows :—

If n candidates at an election have

$$a_1,\ a_2,\ a_3,\ \ldots\ a_n$$

voters in their favour respectively, where

$$a_1 \geqq a_2 \geqq a_3 \ldots a_{n-1} \geqq a_n,$$

and if any instant A_1, A_2, ... A_{n-1}, A_n voters have recorded their votes in favour of the several candidates respectively, the probability that *always*

$$A_1 \geqq A_2 \geqq A_3 \ldots A_{n-1} \geqq A_n$$

is

$$\prod_{s=t+1}^{s=n} \prod_{t=1}^{t=n-1} \left(1 - \frac{a_s}{a_t + s - t}\right).$$

The interesting point is the connection of the problem with the theory of the two-dimensional partitions of any set of Σa unequal numbers.

The graph solution proves at once the remarkable theorem that if $(b_1 b_2 b_3 \ldots)$ be the partition conjugate to $(a_1 a_2 a_3 \ldots)$,

$$(a_1 a_2 a_3 \ldots ;) = (b_1 b_2 b_3 \ldots ;),$$

a fact which it would be difficult to establish by pure algebra. In this case the corresponding probabilities are in the inverse ratio of

$$\frac{(a_1 + a_2 + a_3 + \ldots)\,!}{a_1!\ a_2!\ a_3! \ldots} \quad \text{to} \quad \frac{(b_1 + b_2 + b_3 + \ldots)\,!}{b_1!\ b_2!\ b_3! \ldots}.$$