

IV. *The Diffraction of Electric Waves Round a Perfectly Reflecting Obstacle.*

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IN a previous communication\* it was verified that the effect at a point on a perfectly conducting sphere due to a Hertzian oscillator near to its surface was negligible in comparison with the effect that would have been produced at that point but for the presence of the sphere, when the point is at some distance from the oscillator and the radius of the sphere is large compared with the wave length of the oscillations. In what follows it is proposed to find the effect at all points produced by a Hertzian oscillator placed outside a conducting sphere whose radius is large compared with the wave length of the oscillations. For simplicity the axis of the oscillator will be assumed to pass through the centre of the sphere, but this assumption will not affect the generality of most of the results. An Appendix is added in which the more important mathematical relations required are established.

1. Let O be the centre of the conducting sphere of radius  $a$ , and let the oscillator be at a point  $O_1$ , the direction of the axis of the oscillator being  $OO_1$ , and the distance  $OO_1$  being  $r_1$ . In this case the lines of magnetic force are circles which have the line  $OO_1$  for common axis. If  $\gamma$  denotes the magnetic force at any point P,  $\rho$  the distance of P from  $OO_1$ , and  $z$  the distance of P from the plane through O perpendicular to  $OO_1$ ,  $\gamma\rho$  satisfies the differential equation

$$\frac{\partial^2}{\partial \rho^2}(\gamma\rho) - \frac{1}{\rho} \frac{\partial}{\partial \rho}(\gamma\rho) + \frac{\partial^2}{\partial z^2}(\gamma\rho) + \kappa^2 \gamma\rho = 0,$$

where  $2\pi/\kappa$  is the wave length of the oscillations. Transforming to polar co-ordinates  $(r, \theta)$ , where  $r$  is the distance OP and  $\theta$  the angle POO<sub>1</sub>,  $z = r \cos \theta$  and  $\rho = r \sin \theta$ ; hence, writing  $\cos \theta = \mu$ , the differential equation becomes

$$\frac{\partial^2}{\partial r^2}(\gamma\rho) + \frac{1-\mu^2}{r^2} \frac{\partial^2}{\partial \mu^2}(\gamma\rho) + \kappa^2 \gamma\rho = 0 \quad \dots \quad (1).$$

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The general solution of this equation is

$$\gamma\rho = r^{1/2} \sum_1^{\infty} \{A_n J_{n+1/2}(\kappa r) + B_n J_{-n-1/2}(\kappa r)\} (1-\mu^2) \frac{\partial P_n}{\partial \mu},$$

where  $J_{n+1/2}(\kappa r)$  is BESSEL'S function of order  $n+\frac{1}{2}$ , and  $P_n$  is the zonal harmonic of order  $n$ .

If  $\gamma_1$  is the magnetic force at the point P due to the oscillator,  $\gamma_1$  is the real part of  $C \frac{\partial}{\partial \rho} \frac{e^{-i\kappa(R-Vt)}}{R}$ , where V is the velocity of radiation and R is the distance  $O_1P$ ; this has to be expressed in the form given above for the solution of equation (1). Writing

$$\psi_1 = \rho \frac{\partial}{\partial \rho} \frac{e^{-i\kappa R}}{R},$$

where

$$R^2 = r^2 + r_1^2 - 2\mu r r_1,$$

it follows that

$$\psi_1 = \rho \frac{\partial}{\partial \rho} r^{-1/2} r_1^{-1/2} e^{1/4\pi i} \sum_0^{\infty} (2n+1) e^{1/2 n \pi i} K_{n+1/2}(\kappa r_1) J_{n+1/2}(\kappa r) P_n,$$

where  $r_1 > r$ , and

$$\psi_1 = \rho \frac{\partial}{\partial \rho} r^{-1/2} r_1^{-1/2} e^{1/4\pi i} \sum_0^{\infty} (2n+1) e^{1/2 n \pi i} K_{n+1/2}(\kappa r) J_{n+1/2}(\kappa r_1) P_n,$$

when  $r > r_1$ , where

$$K_{n+1/2}(iz) = \frac{\pi}{2 \cos n\pi} e^{-1/2(n+1/2)\pi i} \{J_{-n-1/2}(z) - e^{(n+1/2)\pi i} J_{n+1/2}(z)\}.$$

Now

$$\rho \frac{\partial}{\partial \rho} = (1-\mu^2) \left\{ r \frac{\partial}{\partial r} - \mu \frac{\partial}{\partial \mu} \right\},$$

$$(2n+1) \mu \frac{\partial P_n}{\partial \mu} = n \frac{\partial P_{n+1}}{\partial \mu} + (n+1) \frac{\partial P_{n-1}}{\partial \mu},$$

$$(2n+1) P_n = \frac{\partial P_{n+1}}{\partial \mu} - \frac{\partial P_{n-1}}{\partial \mu},$$

therefore,  $r_1 > r$ ,

$$\begin{aligned} \psi_1 = e^{1/4\pi i} r_1^{-1/2} \sum_1^{\infty} \left[ e^{1/2(n-1)\pi i} K_{n-1/2}(\kappa r_1) \left\{ r \frac{\partial}{\partial r} r^{-1/2} J_{n-1/2}(\kappa r) - (n-1) r^{-1/2} J_{n-1/2}(\kappa r) \right\} \right. \\ \left. - e^{1/2(n+1)\pi i} K_{n+3/2}(\kappa r_1) \left\{ r \frac{\partial}{\partial r} r^{-1/2} J_{n+3/2}(\kappa r) + (n+2) r^{-1/2} J_{n+3/2}(\kappa r) \right\} (1-\mu^2) \frac{\partial P_n}{\partial \mu} \right], \end{aligned}$$

that is

$$\psi_1 = -\kappa r_1^{-1/2} r^{1/2} e^{1/4\pi i} \sum_1^{\infty} \{e^{1/2(n-1)\pi i} K_{n-1/2}(\kappa r_1) + e^{1/2(n+1)\pi i} K_{n+3/2}(\kappa r_1)\} J_{n+1/2}(\kappa r) (1-\mu^2) \frac{\partial P_n}{\partial \mu},$$

or

$$\psi_1 = -r_1^{-3/2} r^{1/2} \sum_1^{\infty} (2n+1) e^{1/2(n+1/2)\pi i} K_{n+1/2}(\kappa r_1) J_{n+1/2}(\kappa r) (1-\mu^2) \frac{\partial P_n}{\partial \mu} \quad (2).$$

Similarly, when  $r > r_1$ ,

$$\psi_1 = -r_1^{-3/2} r^{1/2} \sum_1^{\infty} (2n+1) e^{1/2(n+1/2)\pi i} J_{n+1/2}(\kappa r_1) K_{n+1/2}(\iota \kappa r) (1-\mu^2) \frac{\partial P_n}{\partial \mu} \quad (3).$$

The solution of equation (1) that is required is that solution  $\psi$  which becomes infinite in the same way as  $\psi_1$  at the point  $(r_1, 0)$  and for which  $\partial\psi/\partial r$  vanishes when  $r = a$ ; the real part of  $C\psi e^{\iota \kappa V t}$  will be the required magnetic force at any point, for  $\partial(\gamma\rho)/\partial r$  will vanish when  $r = a$ , that is, the electric force tangential to the sphere will vanish. From the expression (2) for  $\psi_1$  it follows that the required solution is of the form

$$\psi = -r_1^{-3/2} r^{1/2} \sum_1^{\infty} (2n+1) e^{1/2(n+1/2)\pi i} K_{n+1/2}(\iota \kappa r_1) \{J_{n+1/2}(\kappa r) + B_n K_{n+1/2}(\iota \kappa r)\} (1-\mu^2) \frac{\partial P_n}{\partial \mu},$$

where  $r_1 > r > a$ , and the constants are determined so that  $\partial\psi/\partial r = 0$  when  $r = a$ , hence

$$\psi = -r_1^{-3/2} r^{1/2} \sum_1^{\infty} e^{1/2(n+1/2)\pi i} (2n+1) K_{n+1/2}(\iota \kappa r_1) \left\{ J_{n+1/2}(\kappa r) - \frac{\frac{\partial}{\partial a} \{a^{1/2} J_{n+1/2}(\kappa a)\}}{\frac{\partial}{\partial a} \{a^{1/2} K_{n+1/2}(\iota \kappa a)\}} K_{n+1/2}(\iota \kappa r) \right\} (1-\mu^2) \frac{\partial P_n}{\partial \mu},$$

when

$$r_1 > r > a \quad (4),$$

and

$$\psi = -r_1^{-3/2} r^{1/2} \sum_1^{\infty} (2n+1) e^{1/2(n+1/2)\pi i} \left\{ J_{n+1/2}(\kappa r_1) - \frac{\frac{\partial}{\partial a} \{a^{1/2} J_{n+1/2}(\kappa a)\}}{\frac{\partial}{\partial a} \{a^{1/2} K_{n+1/2}(\iota \kappa a)\}} K_{n+1/2}(\iota \kappa r_1) \right\} K_{n+1/2}(\iota \kappa r) (1-\mu^2) \frac{\partial P_n}{\partial \mu},$$

when

$$r > r_1 \quad (5).$$

Writing

$$\kappa r = z, \quad \kappa r_1 = z_1, \quad \kappa a = z_0, \quad z^{1/2} J_{n+1/2}(z) = 2^{1/2} \pi^{-1/2} u,$$

$$z^{1/2} J_{-n-1/2}(z) = (-1)^n 2^{1/2} \pi^{-1/2} v, \quad z_1^{1/2} J_{n+1/2}(z_1) = 2^{1/2} \pi^{-1/2} u_1, \text{ \&c.,}$$

$$u = R^{1/2} \sin \phi, \quad v = R^{1/2} \cos \phi, \quad u_1 = R_1^{1/2} \sin \phi_1, \quad v_1 = R_1^{1/2} \cos \phi_1,$$

$$u_0 = R_0^{1/2} \sin \phi_0, \quad v_0 = R_0^{1/2} \cos \phi_0,$$

it follows that

$$z^{1/2} K_{n+1/2}(\iota z) = 2^{-1/2} \pi^{1/2} e^{-1/2(n+1/2)\pi i} (v - \iota u)$$

that is

$$z^{1/2} K_{n+1/2}(\iota z) = 2^{-1/2} \pi^{1/2} R^{1/2} e^{-1/2(n+1/2)\pi i - \phi i}$$

Again

$$\frac{\partial}{\partial z} \{z^{1/2} K_{n+1/2}(\iota z)\} = 2^{-1/2} \pi^{1/2} e^{-1/2(n+1/2)\pi i - \phi i} \left\{ \frac{1}{2} R^{-1/2} \frac{\partial R}{\partial z} - \iota R^{1/2} \frac{\partial \phi}{\partial z} \right\};$$

now

$$v \frac{\partial u}{\partial z} - u \frac{\partial v}{\partial z} = 1, \quad \text{that is} \quad R \frac{\partial \phi}{\partial z} = 1,$$

therefore

$$\frac{\partial}{\partial z} \{z^{1/2} K_{n+1/2}(\iota z)\} = 2^{-1/2} \pi^{1/2} \iota e^{-1/2(n+1/2)\pi\iota - \phi\iota} \left\{ \frac{1}{2} \frac{\partial R}{\partial z} - \iota \right\} R^{-1/2},$$

and writing  $\frac{1}{2} \frac{\partial R}{\partial z} = -\tan \chi$ , this becomes

$$\frac{\partial}{\partial z} \{z^{1/2} K_{n+1/2}(\iota z)\} = -2^{-1/2} \pi^{1/2} e^{-1/2(n+1/2)\pi\iota - \phi\iota - \chi\iota} R^{-1/2} \sec \chi.$$

Similarly,

$$\frac{\partial}{\partial z} \{z^{1/2} J_{n+1/2}(z)\} = 2^{1/2} \pi^{-1/2} R^{-1/2} \cos(\phi + \chi) \sec \chi,$$

and therefore

$$\frac{\partial}{\partial \alpha} \{\alpha^{1/2} J_{n+1/2}(\kappa \alpha)\} \bigg/ \frac{\partial}{\partial \alpha} \{\alpha^{1/2} K_{n+1/2}(\iota \kappa \alpha)\} = \frac{2}{\pi} e^{1/2(n+1/2)\pi\iota + (\phi_0 + \chi_0)\iota} \cos(\phi_0 + \chi_0).$$

Hence the relations (4) and (5) become

$$\psi = -\frac{1}{\kappa r_1^2} \sum_{n=1}^{\infty} (2n+1) R^{1/2} R_1^{1/2} e^{-\phi_1\iota} \{\sin \phi - \iota e^{(\phi_0 + \chi_0 - \phi)\iota} \cos(\phi_0 + \chi_0)\} (1-\mu^2) \frac{\partial P_n}{\partial \mu},$$

when  $r_1 > r$ , that is

$$\psi = \frac{\iota}{2\kappa r_1^2} \sum_{n=1}^{\infty} (2n+1) R_1^{1/2} R^{1/2} \{e^{(\phi - \phi_1)\iota} + e^{(2\phi_0 + 2\chi_0 - \phi - \phi_1)\iota}\} (1-\mu^2) \frac{\partial P_n}{\partial \mu}. \quad (6),$$

when  $r_1 > r$ , and

$$\psi = \frac{\iota}{2\kappa r_1^2} \sum_{n=1}^{\infty} (2n+1) R_1^{1/2} R^{1/2} \{e^{(\phi_1 - \phi)\iota} + e^{(2\phi_0 + 2\chi_0 - \phi - \phi_1)\iota}\} (1-\mu^2) \frac{\partial P_n}{\partial \mu}. \quad (7),$$

when  $r > r_1$ .

2. The value of  $\psi$  at any point will now be compared with the value of  $\psi_1$  at the same point, and, the radius  $\alpha$  of the sphere being supposed to be great compared with the wave-length  $2\pi/\kappa$  of the oscillations, it will be sufficient to compare the principal parts of  $\psi$  and  $\psi_1$ . Now

$$\psi_1 = \rho \frac{\partial}{\partial \rho} \frac{e^{-\iota \kappa R}}{R},$$

that is

$$\psi_1 = (1-\mu^2) \left[ r \frac{\partial}{\partial r} - \mu \frac{\partial}{\partial \mu} \right] \frac{e^{-\iota \kappa R}}{R};$$

hence the principal part  $\bar{\psi}_1$  of  $\psi_1$  is given by

$$\bar{\psi}_1 = -\frac{\iota \kappa r^2}{R^2} (1-\mu^2) e^{-\iota \kappa R}.$$

In calculating the principal part of  $\psi$  it is convenient to consider first the contribution of the terms for which  $n + \frac{1}{2}$  is greater than the least of the two quantities

$z$  and  $z_1$ . In this case the approximations to be used for  $\phi_0$  and  $\chi_0$  are those given by relations (x.) of the Appendix ; from these

$$e^{2i(\phi_0+\chi_0)} = \frac{1+i \tan(\phi_0+\chi_0)}{1-i \tan(\phi_0+\chi_0)},$$

that is

$$e^{2i(\phi_0+\chi_0)} = -1 + i e^{2T_0} + \&c.,$$

where  $T_0$  is a large negative quantity. Hence

$$e^{i(\phi-\phi_1)} + e^{i(2\phi_0+2\chi_0-\phi-\phi_1)} = 2i \sin \phi e^{-i\phi_1} + i e^{2T_0} e^{-i(\phi+\phi_1)} + \&c.,$$

that is

$$e^{i(\phi-\phi_1)} + e^{i(2\phi_0+2\chi_0-\phi-\phi_1)} = i e^{2T} + i e^{2T_0} + \dots$$

and

$$(2n+1) R^{1/2} R_1^{1/2} \{e^{i(\phi-\phi_1)} + e^{i(2\phi_0+2\chi_0-\phi-\phi_1)}\} = \frac{2n+1}{\{\sinh \delta \sinh \delta_1\}^{1/2}} i \{e^{T-T_1} + e^{2T_0-T-T_1}\}.$$

Now

$$r_1 > r > a,$$

therefore

$$\delta_1 < \delta < \delta_0 \quad \text{and} \quad -T_1 < -T < -T_0;$$

hence the order of the corresponding term in  $\psi$ , compared with the principal part of  $\psi_1$ , is  $(\kappa r)^{-1/3}$ , and is multiplied by an exponential with negative exponent. Further, the portion of the series containing these terms is simply oscillatory on account of  $P_n$ , and therefore the sum of any number of these terms is not of higher order than  $(\kappa r)^{-1/3} \bar{\psi}_1$  by the appendix ; hence the part contributed by the terms of the series, for which  $n + \frac{1}{2} - z_1$  is of the same or higher order than  $z_1^{1/3}$ , is negligible compared with  $\bar{\psi}_1$ . Again, the terms for which  $n + \frac{1}{2} - z$  is of the same or higher order than  $z^{1/3}$ , but  $n + \frac{1}{2} - z_1$  is of lower order than  $z_1^{1/3}$ , are at most of the order  $(\kappa r)^{-1/3} \bar{\psi}_1$ , and therefore, as above, their sum is negligible in comparison with  $\bar{\psi}_1$ . Similar results hold when  $r > r_1$ . Hence the part of  $\psi$ , if any, which is of the same order as  $\psi_1$ , is contributed by terms for which  $n + \frac{1}{2}$  is less than  $z$  or exceeds  $z$  by a quantity of lower order than  $z^{1/3}$  when  $r > r_1$ , and by terms for which  $n + \frac{1}{2}$  is less than  $z_1$  or exceeds  $z_1$  by a quantity of lower order than  $z_1^{1/3}$  when  $r > r_1$ . Different treatment is necessary according to the form of approximate value of  $P_n(\mu)$  that is appropriate.

3. When  $\theta$  is small, the approximate value of  $P_n(\mu)$  is given by

$$P_n(\mu) = J_0\{(2n+1) \sin \frac{1}{2}\theta\},$$

whence to the same order

$$\frac{\partial P_n}{\partial \mu} = (n + \frac{1}{2}) \cos \frac{1}{2}\theta \operatorname{cosec} \theta J_1\{(2n+1) \sin \frac{1}{2}\theta\},$$

and the series for  $\psi$  is approximately

$$\psi = \frac{i}{\kappa r_1^2} \sin \theta \cos \frac{1}{2}\theta \sum_1 (n + \frac{1}{2})^2 R^{1/2} R_1^{1/2} \{e^{i(\phi-\phi_1)} + e^{i(2\phi_0+2\chi_0-\phi-\phi_1)}\} J_1\{(2n+1) \sin \frac{1}{2}\theta\},$$

when  $r < r_1$ .

Writing

$$S_1 = \frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2} \theta \sum_1 (n + \frac{1}{2})^2 R^{1/2} R_1^{1/2} e^{\iota(\phi - \phi_1)} J_1 \{ (2n + 1) \sin \frac{1}{2} \theta \},$$

the appropriate approximations for  $R$ ,  $R_1$ ,  $\phi$ , and  $\phi_1$  are

$$R = \sec \alpha, \quad \phi = z \cos \alpha - \frac{1}{2} n \pi + (n + \frac{1}{2}) \alpha, \quad \text{where } z \sin \alpha = n + \frac{1}{2}, *$$

$$R_1 = \sec \alpha_1, \quad \phi_1 = z_1 \cos \alpha_1 - \frac{1}{2} n \pi + (n + \frac{1}{2}) \alpha_1, \quad \text{where } z_1 \sin \alpha_1 = n + \frac{1}{2}; *$$

hence approximately

$$S_1 = \frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2} \theta \sum_1 (n + \frac{1}{2})^2 \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{\iota[z \cos \alpha - z_1 \cos \alpha_1 + (n + 1/2)(\alpha - \alpha_1)]} \cdot J_1 \{ (2n + 1) \sin \frac{1}{2} \theta \}.$$

Remembering that the oscillations of  $J_1 \{ (2n + 1) \sin \frac{1}{2} \theta \}$  depend on  $e^{\pm(2n+1)\iota \sin \frac{1}{2} \theta}$ , the principal part of  $S_1$ , arises from the terms in the neighbourhood of the term for which

$$\frac{d}{dn} \{ z \cos \alpha - z_1 \cos \alpha_1 + (n + \frac{1}{2})(\alpha - \alpha_1) \pm (2n + 1) \sin \frac{1}{2} \theta \}$$

vanishes; for this  $\alpha - \alpha_1$  is small since  $\theta$  is small, and therefore, unless  $r$  is nearly equal to  $r_1$ , which means that the point is close to the oscillator,  $n + \frac{1}{2}$  is small compared with  $z$  for the terms that contribute the principal part of  $S_1$ . Hence approximately

$$z \cos \alpha + (n + \frac{1}{2}) \alpha = z + \frac{1}{2z} (n + \frac{1}{2})^2,$$

$$z_1 \cos \alpha_1 + (n + \frac{1}{2}) \alpha_1 = z_1 + \frac{1}{2z_1} (n + \frac{1}{2})^2,$$

and the principal part of  $S_1$  is equal to the principal part of

$$\frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2} \theta \sum_1 (n + \frac{1}{2})^2 e^{\iota[z - z_1 + 1/2(n + 1/2)^2(\frac{1}{z} - \frac{1}{z_1})]} J_1 \{ (2n + 1) \sin \frac{1}{2} \theta \};$$

that is, the principal part of  $S_1$  is given by

$$\bar{S}_1 = \frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2} \theta e^{\iota(z - z_1)} \int_0^\infty e^{\frac{\iota}{2}(\frac{1}{z} - \frac{1}{z_1})\zeta^2} \zeta^2 J_1(2\zeta \sin \frac{1}{2} \theta) d\zeta;$$

therefore

$$\bar{S}_1 = -\frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2} \theta e^{\iota(z - z_1)} \frac{2 \sin \frac{1}{2} \theta}{(z^{-1} - z_1^{-1})^2} e^{-\iota \frac{2 \sin^2 \frac{1}{2} \theta}{(z^{-1} - z_1^{-1})}},$$

or

$$\bar{S}_1 = -\frac{\iota \kappa r^2 \sin^2 \theta}{(r_1 - r)^2} e^{-\iota \kappa (r_1 - r + \frac{2 r r_1 \sin^2 \frac{1}{2} \theta}{r_1 - r})}.$$

\* Appendix, Relations (ii).

Now the distance  $D$  from the oscillator to the point  $(r_1\theta)$  is given by

$$D^2 = r^2 + r_1^2 - 2rr_1 \cos \theta,$$

whence retaining only the most important terms

$$\bar{S}_1 = -\frac{\kappa r^2 \sin^2 \theta}{D^2} e^{-\kappa D},$$

and therefore  $\bar{S}_1 = \bar{\psi}_1$ .

The same result follows when  $r > r_1$ . Again writing

$$S_2 = \frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2}\theta \sum_1 (n + \frac{1}{2})^2 R^{1/2} R_1^{1/2} e^{\iota(2\phi_0 + 2\chi_0 - \phi - \phi_1)} J_1\{(2n+1) \sin \frac{1}{2}\theta\},$$

the principal part of  $S_2$  arises from the terms in the neighbourhood of the term for which

$$\frac{d}{dn} \{2\phi_0 + 2\chi_0 - \phi - \phi_1 \pm (2n+1) \sin \frac{1}{2}\theta\}$$

vanishes. When  $n + \frac{1}{2}$  is less than  $z_0$ , this requires that  $2\alpha_0 - \alpha - \alpha_1$  is small, since  $\theta$  is; when  $n + \frac{1}{2}$  is greater than  $z_0$ , it requires that  $\pi - \alpha - \alpha_1$  is small, that is  $\alpha$  and  $\alpha_1$  are each nearly a right angle, which means that the point is close to the oscillator, thus the case of  $2\alpha_0 - \alpha - \alpha_1$  small need only be considered. In this case the principal part of  $S_2$  is equal to the principal part of

$$\frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2}\theta \sum_1 (n + \frac{1}{2})^2 e^{\iota[2z_0 - z - z_1 + \frac{1}{2}(n + \frac{1}{2})^2(2z_0^{-1} - z^{-1} - z_1^{-1})]} J_1\{(2n+1) \sin \frac{1}{2}\theta\},$$

that is

$$\bar{S}_2 = \frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2}\theta e^{\iota(2z_0 - z - z_1)} \int_0^\infty e^{\frac{1}{2}\iota(2z_0^{-1} - z^{-1} - z_1^{-1})\zeta^2} \zeta^2 J_1(2\zeta \sin \frac{1}{2}\theta) d\zeta,$$

whence

$$\bar{S}_2 = -\frac{\iota}{\kappa r_1^2} \sin \theta \cos \frac{1}{2}\theta e^{\iota(2z_0 - z - z_1)} \frac{2 \sin \frac{1}{2}\theta}{(2z_0^{-1} - z^{-1} - z_1^{-1})^2} e^{-\frac{2\iota \sin^2 \frac{1}{2}\theta}{(2z_0^{-1} - z^{-1} - z_1^{-1})}},$$

or

$$\bar{S}_2 = -\frac{\iota\kappa}{r^2} \sin^2 \theta (2\alpha^{-1} - r^{-1} - r_1^{-1})^{-2} e^{-\iota\kappa[r + r_1 - 2\alpha + 2 \sin^2 \frac{1}{2}\theta (2\alpha^{-1} - r^{-1} - r_1^{-1})^{-1}]}.$$

This represents waves reflected from the neighbourhood of the points  $Q$  on the sphere, which are such that  $O_1Q$  makes an angle  $\alpha_0$  with the radius  $OQ$ , and if  $D_1$  is the distance  $O_1Q$ ,  $D_2$  the distance  $PQ$ , where  $P$  is the point  $(r, \theta)$ , the above becomes, after some reduction,

$$\bar{S}_2 = -\iota\kappa \sin^2 \alpha_1 e^{-\iota\kappa(D_1 + D_2)},$$

the result that would have been obtained by elementary methods. Hence the effect at a point  $P$  for which  $\theta$  is small is the sum of the direct waves and the waves reflected at the surface of the sphere.

4. When  $\theta$  is nearly equal to  $\pi$  the appropriate approximation for  $P_n(\mu)$  is given by

$$P_n(\mu) = \cos n\pi J_0 \left\{ (2n+1) \sin \frac{1}{2}(\pi - \theta) \right\},$$

or, writing  $\pi - \theta = \theta'$ ,

$$P_n(\mu) = \cos n\pi J_0 \left\{ (2n+1) \sin \frac{1}{2}\theta' \right\},$$

and

$$\frac{\partial P_n}{\partial \mu} = - \left( n + \frac{1}{2} \right) \cos n\pi \cos \frac{1}{2}\theta' \operatorname{cosec} \theta' J_1 \left\{ (2n+1) \sin \frac{1}{2}\theta' \right\};$$

hence

$$\psi = - \frac{\iota}{\kappa r_1^2} \sin \theta' \cos \frac{1}{2}\theta' \sum_1 \left( n + \frac{1}{2} \right)^2 R^{1/2} R_1^{1/2} \cos n\pi \left\{ e^{\iota(\phi - \phi_1)} + e^{\iota(2\phi_0 + 2\chi_0 - \phi - \phi_1)} \right\} J_1 \left\{ (2n+1) \sin \frac{1}{2}\theta' \right\},$$

when  $r < r_1$ , where to the order required the terms for which  $n + \frac{1}{2}$  exceeds  $z$  by a quantity of the same or of higher order than  $z^{1/3}$  can be neglected. Writing

$$S_1 = - \frac{\iota}{\kappa r_1^2} \sin \theta' \cos \frac{1}{2}\theta' \sum_1 \left( n + \frac{1}{2} \right)^2 \cos n\pi R^{1/2} R_1^{1/2} e^{\iota(\phi - \phi_1)} J_1 \left\{ (2n+1) \sin \frac{1}{2}\theta' \right\},$$

the terms of this series, for which  $n + \frac{1}{2}$  is less than  $z$ , and  $z - n - \frac{1}{2}$  is of higher order than  $z^{1/3}$ , are at most of the order  $(\kappa r)^{1/3} (\kappa r_1)^{-2} \psi_1$ , and therefore their sum will be negligible in comparison with  $\psi_1$  unless  $\frac{d}{dn} [\phi - \phi_1 \pm n\pi \pm (2n+1) \sin \frac{1}{2}\theta']$  vanishes or is a multiple of  $2\pi$  for some value of  $n$  in the series. Using the appropriate approximations for  $\phi$  and  $\phi_1$ , this requires that  $\alpha - \alpha_1 \pm \pi$  is small or nearly equal to a multiple of  $2\pi$  since  $\theta'$  is small, and this is impossible, for  $\alpha$  and  $\alpha_1$  are each less than  $\frac{1}{2}\pi$ . The terms of the series  $S_1$ , for which  $|z - n - \frac{1}{2}|$  is of lower order than  $z^{1/3}$ , are at most of the order  $(\kappa r)^{1/3} (\kappa r_1)^{-2} \psi_1$ , and their sum will be negligible in comparison with  $\psi_1$  unless  $\frac{d}{dn} [\phi - \phi_1 \pm n\pi \pm (2n+1) \sin \frac{1}{2}\theta']$  is very small or nearly equal to a multiple of  $2\pi$  for a value of  $n$  in this series; substituting the appropriate approximations for  $\phi$  and  $\phi_1$ , this requires that  $\frac{1}{2}\pi - \alpha_1 - \pi$  is small, which is impossible, for  $\alpha_1$  is less than  $\frac{1}{2}\pi$ . Hence  $S_1$  is negligible in comparison with  $\psi_1$ . The same result follows when  $r > r_1$ . Again, writing

$$S_2 = - \frac{\iota}{\kappa r_1^2} \sin \theta' \cos \frac{1}{2}\theta' \sum_1 \left( n + \frac{1}{2} \right)^2 \cos n\pi R^{1/2} R_1^{1/2} e^{\iota(2\phi_0 + 2\chi_0 - \phi - \phi_1)} J_1 \left\{ (2n+1) \sin \frac{1}{2}\theta' \right\},$$

the terms of this series, for which  $n + \frac{1}{2}$  is less than  $z_0$ , and  $z_0 - n - \frac{1}{2}$  is of the same or of higher order than  $z_0^{1/3}$ , are at most of the order  $(\kappa \alpha)^{1/3} (\kappa r_1)^{-2} \psi_1$ , and therefore their sum will be negligible in comparison with  $\psi_1$  unless

$$\frac{d}{dn} [2\phi_0 + 2\chi_0 - \phi - \phi_1 \pm n\pi \pm (2n+1) \sin \frac{1}{2}\theta']$$

vanishes, or is nearly equal to a multiple of  $2\pi$  for some value of  $n$  in this series.



Substituting for  $\phi_0$ ,  $\chi_0$ , &c., their approximate values, this requires that  $2\alpha_0 - \alpha - \alpha_1 \pm \pi$  is small, since  $\theta'$  is small, and this is impossible for  $\frac{1}{2}\pi > \alpha_0 > \alpha$ ,  $\frac{1}{2}\pi > \alpha_0 > \alpha_1$ ; hence the sum of these terms is negligible in comparison with  $\psi_1$ . The terms of the series for which  $|z_0 - n - \frac{1}{2}|$  is of lower order than  $z_0^{1/3}$  are of the order  $(\kappa\alpha)^{3/2}(\kappa r_1)^{-2}\psi_1$ , and their sum is of highest order when  $\frac{d}{dn}(2\phi_0 + 2\chi_0 - \phi - \phi_1 \pm n\pi)$  is small, or nearly equal to a multiple of  $2\pi$ , for a value of  $n$  in this series. In this case the approximation for  $\phi_0 + \chi_0$  is  $\frac{1}{3}\pi + \frac{1}{4c}3^{-1/2}(3\mu)^2$  [Appendix (vi.)], and the above condition becomes that  $\pi - \alpha - \alpha_1 \pm \pi$  is small, or nearly a multiple of  $2\pi$ , which is satisfied if both  $\alpha$  and  $\alpha_1$  are small. The sum of these terms then is

$$-\frac{\iota}{\kappa r_1^2} \sin \theta' \cos \frac{1}{2}\theta' \sum (n + \frac{1}{2})^2 R^{1/2} R_1^{1/2} e^{\frac{2\pi}{3} - z - z_1 + \frac{1}{2}(n + \frac{1}{2})^2 (3^{1/6} 2^{2/3} c^{-1} z^{-2/3} - z^{-1} - z_1^{-1})} J_1 \{ (2n + 1) \sin \frac{1}{2}\theta' \},$$

retaining only the most important part, and the sum is therefore of the order  $(\kappa\alpha)^{3/2+1/3}(\kappa r_1)^{-2}\psi_1$ , which is of lower order than  $\psi_1$ , and therefore negligible in comparison with it. When  $n + \frac{1}{2}$  is greater than  $z_0$  and  $n + \frac{1}{2} - z_0$  is of the same or of higher order than  $z_0^{1/3}$ , but  $n + \frac{1}{2}$  is less than  $z$  and  $z - n - \frac{1}{2}$  is of the same or of higher order than  $z^{1/3}$ , when  $z < z_1$ , the terms are of the order  $(\kappa r)^{3/2}(\kappa r_1)^{-2}\psi_1$ , and their sum will be of the same order as  $\psi_1$ , if  $\frac{d}{dn}(2\phi_0 + 2\chi_0 - \phi - \phi_1 \pm n\pi)$  is small, or nearly a multiple of  $2\pi$ , for a value of  $n$  in this series. This condition is satisfied if there is a value of  $n$  in the series for which  $\pi - \alpha - \alpha_1 \pm \pi$  is small, or nearly a multiple of  $2\pi$ , which requires that both  $\alpha$  and  $\alpha_1$  should be small. For the other values of  $n$ , which have to be taken account of, the order of the terms is  $(\kappa r)^{5/3}(\kappa r_1)^{-2}\psi_1$ , and the condition that their sum should be of the same order as  $\psi_1$  is, that for some of these values of  $n$ ,  $\frac{1}{2}\pi - \alpha_1 \pm \pi$  is small, which is impossible. Hence the part of the series  $S_2$  which is of the order of  $\psi_1$  arises from the terms beginning with a term for which  $n$  is equal to  $z_0 + A z_0^{1/3}$ , where  $A$  is a positive quantity of the order of unity, and the principal part of  $S_2$  is therefore equal to the principal part of

$$-\frac{\iota}{2\kappa r_1^2} \sin \theta' \cos \frac{1}{2}\theta' \sum_{z_0 + A z_0^{1/3}} (n + \frac{1}{2})^2 R^{1/2} R_1^{1/2} e^{\iota(2\phi_0 + 2\chi_0 - \phi - \phi_1 - n\pi)} J_1 \{ (2n + 1) \sin \frac{1}{2}\theta' \} \\ - \frac{\iota}{2\kappa r_1^2} \sin \theta' \cos \frac{1}{2}\theta' \sum_{z_0 + A z_0^{1/3}} (n + \frac{1}{2})^2 R^{1/2} R_1^{1/2} e^{\iota(2\phi_0 + 2\chi_0 - \phi - \phi_1 + n\pi)} J_1 \{ (2n + 1) \sin \frac{1}{2}\theta' \}.$$

Now

$$e^{2\iota(\phi_0 + \chi_0)} = \frac{1 + \iota \tan(\phi_0 + \chi_0)}{1 - \iota \tan(\phi_0 + \chi_0)},$$

that is [Appendix (x.)]

$$e^{2\iota(\phi_0 + \chi_0)} = -1 + \iota e^{2T_0} \dots,$$

where  $T_0$  is a negative quantity and  $-T_0$  increases rapidly with  $n$ . Also, when  $\iota$

and  $r_1$  are so great compared with  $\alpha$  that  $\sin \alpha$  and  $\sin \alpha_1$  can be replaced by  $\alpha$  and  $\alpha_1$ , the principal part of  $S_2$  is equal to the principal part of

$$\frac{\iota}{\kappa r_1^2} \sin \theta' \cos \frac{1}{2} \theta' \sum_{z_0 + \Lambda z_0^{1/3}} (n + \frac{1}{2})^2 e^{-\iota [z + z_1 + 1/2 (n + 1/2)^2 (z^{-1} + z_1^{-1})]} J_1 \{ (2n + 1) \sin \frac{1}{2} \theta' \},$$

which is equal to

$$\frac{\iota}{\kappa r_1^2} \sin \theta' \cos \frac{1}{2} \theta' \int_{z_0}^{\infty} \zeta^2 e^{-\iota [z + z_1 + 1/2 \zeta^2 (z^{-1} + z_1^{-1})]} J_1 (2\zeta \sin \frac{1}{2} \theta') d\zeta.$$

When  $\theta'$  is very small the value of the above integral is approximately

$$-\frac{\iota \kappa r_1^2}{(r + r_1)^2} \sin^2 \theta' e^{-\iota \kappa (r + r_1)},$$

which is equal to the value of  $\psi_1$  at the point. These results correspond exactly with those obtained by the usual treatment of the effect of an opaque circular screen interposed between a source of light and the point of observation, when the source and the point of observation are both at distances from the screen great compared with the radius of the screen. It has been assumed in the above approximation that  $\frac{\alpha^2}{\lambda r} + \frac{\alpha^2}{\lambda r_1}$  is a small quantity whose square is negligible; when this condition is not satisfied the value of the integral involved depends on the quantity

$$\sqrt{(\kappa \alpha) (\alpha + \alpha_1) (\tan \alpha + \tan \alpha_1)^{-1/2}},$$

where  $\sin \alpha = \alpha/r$ ,  $\sin \alpha_1 = \alpha/r_1$ , and diminishes rapidly as this quantity increases.

5. When  $\theta$  is not small or nearly equal to  $\pi$ ,  $P_n(\mu)$  can be replaced by its approximate value

$$2^{1/2} \pi^{-1/2} \{ (n + \frac{1}{2}) \sin \theta \}^{-1/2} \cos \{ (n + \frac{1}{2}) \theta - \frac{1}{4} \pi \},$$

whence

$$\frac{dP_n}{d\mu} = (2n + 1)^{1/2} (\pi \sin^3 \theta)^{-1/2} \sin \{ (n + \frac{1}{2}) \theta - \frac{1}{4} \pi \},$$

and therefore, when  $r < r_1$ ,

$$\psi = \frac{\iota}{2\kappa r_1^2} \pi^{-1/2} (\sin \theta)^{1/2} \sum_1^{\infty} (2n + 1)^{3/2} R_1^{1/2} R_1^{-1/2} \{ e^{\iota(\phi - \phi_1)} + e^{\iota(2\phi_0 + 2\chi_0 - \phi - \phi_1)} \} \sin \{ (n + \frac{1}{2}) \theta - \frac{1}{4} \pi \},$$

approximately, that is

$$\psi = (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_1^{\infty} (n + \frac{1}{2})^{3/2} R_1^{1/2} R_1^{-1/2} \{ e^{\iota[\phi - \phi_1 + (n + 1/2) \theta - 1/4 \pi]} - e^{\iota[\phi - \phi_1 - (n + 1/2) \theta + 1/4 \pi]} + e^{\iota[2\phi_0 + 2\chi_0 - \phi - \phi_1 + (n + 1/2) \theta - 1/4 \pi]} - e^{\iota[2\phi_0 + 2\chi_0 - \phi - \phi_1 - (n + 1/2) \theta + 1/4 \pi]} \},$$

which may be written

$$\psi = S_1 - S_2 + S_3 - S_4,$$

where

$$\begin{aligned} S_1 &= (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_1^{\infty} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{i[\phi - \phi_1 + (n + 1/2)\theta - 1/4\pi]}, \\ S_2 &= (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_1^{\infty} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{i[\phi - \phi_1 - (n + 1/2)\theta + 1/4\pi]}, \\ S_3 &= (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_1^{\infty} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{i[2\phi_0 + 2\chi_0 - \phi - \phi_1 + (n + 1/2)\theta - 1/4\pi]}, \\ S_4 &= (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_1^{\infty} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{i(2\phi_0 + 2\chi_0 - \phi - \phi_1 - (n + 1/2)\theta + 1/4\pi)}. \end{aligned}$$

Since  $r < r_1$ , the formulæ (ii.) of the Appendix can be used for the terms of  $S_1$  and  $S_2$ , for which  $n + \frac{1}{2}$  is less than  $z$  and  $z - (n + \frac{1}{2})$  is of the same or higher order than  $z^{1/3}$ , and the corresponding parts of  $S_1$  and  $S_2$  are given by

$$\begin{aligned} S_{11} &= (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_1^{z - Az^{1/3}} (n + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{i[z \cos \alpha - z_1 \cos \alpha_1 + (n + 1/2)(\alpha - \alpha_1 + \theta) - 1/4\pi]}, \\ S_{21} &= (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_1^{z - Az^{1/3}} (n + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{i[z \cos \alpha - z_1 \cos \alpha_1 + (n + 1/2)(\alpha - \alpha_1 - \theta) + 1/4\pi]}, \end{aligned}$$

where  $z \sin \alpha = z_1 \sin \alpha_1 = n + \frac{1}{2}$  and  $A$  is of the order of unity.

Now

$$\frac{d}{dn} \{z \cos \alpha - z_1 \cos \alpha_1 + (n + \frac{1}{2})(\alpha - \alpha_1 + \theta)\} = \alpha - \alpha_1 + \theta,$$

and  $\alpha - \alpha_1 + \theta$  cannot be a small quantity for  $\alpha > \alpha_1$ , since  $r_1 > r$ ; hence  $S_{11}$  is at most of the order  $z_1^{-1/2} \psi_1$ , and therefore negligible in comparison with  $\psi_1$ .

Again

$$\frac{d}{dn} \{z \cos \alpha - z_1 \cos \alpha_1 + (n + \frac{1}{2})(\alpha - \alpha_1 - \theta)\} = \alpha - \alpha_1 - \theta,$$

and  $\alpha - \alpha_1 - \theta$  will vanish for a value of  $n$  between 1 and  $z - Az^{1/3}$ , provided the point  $P(r, \theta)$  lies inside the sphere described on  $OO_1$  as diameter, but not close to its boundary. If  $\alpha$  and  $\alpha_1$  now correspond to the value of  $n$  for which  $\theta = \alpha - \alpha_1$ ,  $\alpha$  is the angle  $OPT$  and  $\alpha_1$  the angle  $OO_1P$  in the figure,\* and the principal part of  $S_{21}$  is given by

$$\bar{S}_{21} = (2\pi)^{-1/2} \sin^{1/2} \theta (\kappa r_1^2)^{-1} (z_1 \sin \alpha_1)^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{i(z \cos \alpha - z_1 \cos \alpha_1 + 1/4\pi)} \int_{-\mu}^{\mu} e^{1/2 \zeta^2 \left( \frac{1}{z \cos \alpha} - \frac{1}{z_1 \cos \alpha_1} \right)} d\zeta,$$

where  $\mu$  is large, that is

$$\bar{S}_{21} = (2\pi)^{-1/2} \sin^{1/2} \theta (\kappa r_1^2)^{-1} (z_1 \sin \alpha_1)^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{i(z \cos \alpha - z_1 \cos \alpha_1 + 1/4\pi)} (2\pi)^{1/2} e^{1/4\pi i} \left\{ \frac{1}{z \cos \alpha} - \frac{1}{z_1 \cos \alpha_1} \right\}^{-1/2}.$$

\* The corresponding value of  $n$  is given by  $n + \frac{1}{2} = \kappa p$ , where  $p$  is the perpendicular from  $O$  on  $O_1P$ .

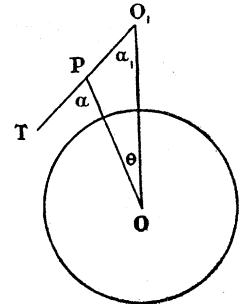


Fig. 1.

If  $O_1P = D$ , then

$$r_1 \cos \alpha_1 - r \cos \alpha = D, \quad r \sin \theta = D \sin \alpha_1,$$

and therefore

$$\bar{S}_{21} = \iota \kappa \sin^2 \theta r^2 D^{-2} e^{-\iota \kappa D},$$

that is for the points defined above  $\bar{S}_{21} = -\bar{\psi}_1$ .

The same formulæ (ii.) of the Appendix are applicable to the terms of the series  $S_3$  and  $S_4$ , for which  $n + \frac{1}{2}$  is less than  $z_0$  and  $z_0 - (n + \frac{1}{2})$  is of the same or higher order than  $z_0^{1/3}$ . The corresponding parts of the series  $S_3$  and  $S_4$  are given by

$$S_{31} = (2\pi)^{-1/2} \sin^{1/2} \theta (\kappa r_1^2)^{-1} \sum_1^{z_0 - A z_0^{1/3}} (n + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{\iota [2z_0 \cos \alpha_0 - z \cos \alpha - z_1 \cos \alpha + (n + 1/2)(2\alpha_0 - \alpha - \alpha_1 + \theta) - 1/4\pi]},$$

$$S_{41} = (2\pi)^{-1/2} \sin^{1/2} \theta (\kappa r_1^2)^{-1} \sum_1^{z_0 - A z_0^{1/3}} (n + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{\iota [2z_0 \cos \alpha_0 - z \cos \alpha - z_1 \cos \alpha_1 + (n + 1/2)(2\alpha_0 - \alpha - \alpha_1 - \theta) + 1/4\pi]},$$

where  $z_0 \sin \alpha_0 = z \sin \alpha = z_1 \sin \alpha_1 = n + \frac{1}{2}$ . Now

$$\frac{d}{dn} \{2z_0 \cos \alpha_0 - z \cos \alpha - z_1 \cos \alpha + (n + \frac{1}{2})(2\alpha_0 - \alpha - \alpha_1 + \theta)\} = 2\alpha_0 - \alpha - \alpha_1 + \theta,$$

and  $\alpha_0 > \alpha$  for  $r > a$ ,  $\alpha_0 > \alpha_1$  for  $r_1 > a$ , hence  $2\alpha_0 - \alpha - \alpha_1 + \theta$  is always finite, and therefore, as above, the sum of the terms  $S_{31}$  is of lower order than  $\psi_1$ . Also

$$\frac{d}{dn} \{2z_0 \cos \alpha_0 - z \cos \alpha - z_1 \cos \alpha_1 + (n + \frac{1}{2})(2\alpha_0 - \alpha - \alpha_1 - \theta)\} = 2\alpha_0 - \alpha - \alpha_1 - \theta,$$

and  $2\alpha_0 - \alpha - \alpha_1 - \theta$  will vanish for a value of  $n$  between 1 and  $z_0 - A z_0^{1/3}$ , provided the point  $P(r, \theta)$  satisfies conditions to be determined. For let  $n_0 + \frac{1}{2} = \kappa p$ , where  $p < a$ , and let  $O_1Q$  be a straight line through  $O_1$  at a distance  $p$  from  $O$ , meeting the sphere in  $Q$  (fig. 2), then the angle  $TQO_1$  is  $\alpha_0$ , where  $n_0 + \frac{1}{2} = z_0 \sin \alpha_0$ , and the angle  $OO_1Q$  is  $\alpha_1$ , where  $n_0 + \frac{1}{2} = z_1 \sin \alpha_1$ . Hence, drawing  $QP$  in the plane  $O_1OQ$  making the angle  $TQP$  equal to  $\alpha_0$ , and taking any point  $P$  on  $QP$ , the angle  $QPO$  is equal to  $\alpha$ , where  $n_0 + \frac{1}{2} = z \sin \alpha$ , and the angle  $POO_1$ , which is  $\theta$ , is equal to  $2\alpha_0 - \alpha - \alpha_1$ . For any position of  $P$  on the line  $QP$  the terms of the series  $S_{41}$  in the neighbourhood of the terms for which  $n = n_0$  contribute a sum which is of higher order than any of them, and the

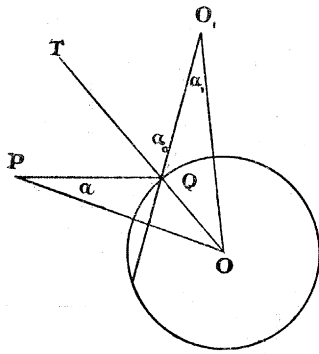


Fig. 2.

principal part of  $S_{41}$  is given by

$$\bar{S}_{41} = (2\pi)^{-1/2} \sin^{1/2} \theta (n_0 + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{\iota (2z_0 \cos \alpha_0 - z \cos \alpha - z_1 \cos \alpha_1 + 1/4\pi)}$$

$$\int_{-\mu}^{\mu} e^{1/2 \iota \zeta^2} \left( \frac{2}{z_0 \cos \alpha_0} - \frac{1}{z \cos \alpha} - \frac{1}{z_1 \cos \alpha_1} \right) d\zeta,$$

whence

$$\bar{S}_{41} = \iota (\kappa r_1^2)^{-1} \sin^{1/2} \theta (n_0 + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 \left( \frac{2}{z_0 \cos \alpha_0} - \frac{1}{z \cos \alpha} - \frac{1}{z_1 \cos \alpha_1} \right)^{-1/2} e^{\iota (2z_0 \cos \alpha_0 - z \cos \alpha - z_1 \cos \alpha_1)},$$

that is

$$\bar{S}_{41} = \iota \kappa \sin^2 \alpha_1 \sin^{1/2} \theta (z_1 \sin \alpha_1)^{-1/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 \left( \frac{2}{z_0 \cos \alpha_0} - \frac{1}{z \cos \alpha} - \frac{1}{z_1 \cos \alpha_1} \right)^{-1/2} e^{\iota (2z_0 \cos \alpha_0 - z \cos \alpha - z_1 \cos \alpha_1)},$$

which after some reduction can be put in the form

$$\bar{S}_{41} = \iota \kappa \sin^2 \alpha_1 \left( \frac{2 \cos \alpha_0}{a} + \frac{1}{QP} + \frac{1}{O_1 Q} \right)^{1/2} \left( \frac{2}{a \cos \alpha_0} + \frac{1}{QP} + \frac{1}{O_1 Q} \right)^{-1/2} e^{-\iota \kappa (O_1 Q + QP)},$$

and this represents the effect at P of the waves reflected from the sphere at Q.\*

From the diagram it appears that the points P for which the condition  $\theta = 2\alpha_0 - \alpha - \alpha_1$  can be satisfied are the points outside the tangent cone drawn from  $O_1$  to the sphere (that is the points outside the geometrical shadow), and as P approaches the surface of the cone the value of  $n_0 + \frac{1}{2}$  approaches  $\kappa \alpha$ ; hence the above result holds for all points outside the tangent cone and not close to its boundary.

Omitting for the present the discussion of the terms of the series  $S_3$  and  $S_4$  for

\* This result can be verified by elementary methods as follows: If M is the amplitude of the magnetic force at Q,  $M_1$  the amplitude of the magnetic force at P in the waves reflected from Q, the intensity at Q in a beam of rays from  $O_1$  of small cross section is

$$M^2 a^2 \sin (\alpha_0 - \alpha_1) \cos \alpha_0 d(\alpha_0 - \alpha_1) d\phi,$$

and the intensity in the same beam at P is

$$M_1^2 r^2 \sin \theta \cos \alpha d\theta d\phi,$$

and these intensities are equal, therefore

$$M^2 a^2 \sin (\alpha_0 - \alpha_1) \cos \alpha_0 d(\alpha_0 - \alpha_1) = M_1^2 r^2 \sin \theta \cos \alpha d\theta;$$

now

$$\theta = 2\alpha_0 - \alpha - \alpha_1,$$

$$z_0 \cos \alpha_0 d\alpha_0 = z \cos \alpha d\alpha = z_1 \cos \alpha_1 d\alpha_1,$$

$$r_1 \cos \alpha_1 - a \cos \alpha_0 = O_1 Q = a \sin (\alpha_0 - \alpha_1) / \sin \alpha_1,$$

whence

$$M_1^2 r^2 \sin^2 \theta = M a^2 \sin^2 (\alpha_0 - \alpha_1) \sin \theta \sec \alpha \sec \alpha_1 / z_1 \sin \alpha_1 \left( \frac{2}{z_0 \cos \alpha_0} - \frac{1}{z \cos \alpha} - \frac{1}{z_1 \cos \alpha_1} \right).$$

The amplitude of the  $\psi$  of the waves from  $O_1$  at Q is  $Ma \sin (\alpha_0 - \alpha_1)$ , and of the reflected waves at P is  $M_1 r \sin \theta$ , therefore the  $\psi$  of the reflected waves at P is

$$-\iota \kappa \sin^2 \alpha_1 \sin^{1/2} \theta \sec^{1/2} \alpha \sec^{1/2} \alpha_1 (z_1 \sin \alpha_1)^{-1/2} \left( \frac{2}{z_0 \cos \alpha_0} - \frac{1}{z \cos \alpha} - \frac{1}{z_1 \cos \alpha_1} \right)^{-1/2},$$

which is the result obtained from the series.

which  $n + \frac{1}{2} - z_0$  is of lower order than  $z_0^{1/3}$ ,\* the terms for which  $z - n - \frac{1}{2}$  is of the same or of higher order than  $z^{1/3}$  will now be considered. Denoting these terms of the series  $S_3$  and  $S_4$  by  $S_{32}$  and  $S_{42}$ , and remembering that

$$e^{2i(\phi_0 + \chi_0)} = -1 + \iota e^{2T_0} \dots, \text{ \&c.,}$$

where  $T_0$  is a negative real quantity increasing rapidly with  $n$ ,  $S_{32}$  and  $S_{42}$  are to the required order of approximation given by

$$S_{32} = -(2\pi)^{-1/2} (\kappa r_1^2)^{-1} \sin^{1/2} \theta \sum^{z - \Lambda z^{1/3}} (n + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{i[n\pi - z \cos \alpha - z_1 \cos \alpha_1 - (n + 1/2)(\alpha + \alpha_1 - \theta) - 1/4\pi]},$$

$$S_{42} = -(2\pi)^{-1/2} (\kappa r_1^2)^{-1} \sin^{1/2} \theta \sum^{z - \Lambda z^{1/3}} (n + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{i[n\pi - z \cos \alpha - z_1 \cos \alpha_1 - (n + 1/2)(\alpha + \alpha_1 + \theta) + 1/4\pi]}.$$

The value of  $\frac{d}{dn} \{n\pi - z \cos \alpha - z_1 \cos \alpha_1 - (n + \frac{1}{2})(\alpha + \alpha_1 - \theta)\}$  is  $\pi - \alpha - \alpha_1 + \theta$ , which, since  $\alpha$  and  $\alpha_1$  are less than  $\frac{1}{2}\pi$  cannot be very small, and therefore the sum of the terms  $S_{32}$  is of lower order than  $\psi_1$ . Again the value of

$$\frac{d}{dn} \{n\pi - z \cos \alpha - z_1 \cos \alpha_1 - (n + \frac{1}{2})(\alpha + \alpha_1 + \theta)\}$$

is  $\pi - \alpha - \alpha_1 - \theta$ , and, if P is a point outside the geometrical shadow for which the angle  $OPO_1$  is less than a right angle, there is a value  $n_1$  of  $n$  in the series  $S_{42}$  for which  $\pi - \alpha - \alpha_1 - \theta$  vanishes, viz., that given by  $n_1 + \frac{1}{2} = \kappa p$  where  $p$  is the perpendicular from O on  $O_1P$ , and the principal part of  $S_{42}$  is given by

$$\bar{S}_{42} = -(2\pi)^{-1/2} (\kappa r_1^2)^{-1} (n_1 + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{-i(z \cos \alpha + z_1 \cos \alpha_1 + 1/4\pi)} \int_{-\mu}^{\mu} e^{-1/2 \zeta^2} \left( \frac{1}{z \cos \alpha} + \frac{1}{z_1 \cos \alpha_1} \right) d\zeta,$$

where

$$n_1 + \frac{1}{2} = z \sin \alpha = z_1 \sin \alpha_1,$$

that is by

$$\bar{S}_{42} = -(2\pi)^{-1/2} (\kappa r_1^2)^{-1} (n_1 + \frac{1}{2})^{3/2} \sec^{1/2} \alpha \sec^{1/2} \alpha_1 e^{-i(z \cos \alpha + z_1 \cos \alpha_1 + 1/2\pi)} (2\pi)^{1/2} \left( \frac{1}{z \cos \alpha} + \frac{1}{z_1 \cos \alpha_1} \right)^{-1/2},$$

or

$$\bar{S}_{42} = \iota \kappa \sin^2 \alpha_1 e^{-\iota \kappa D},$$

where D is the distance  $O_1P$ . Hence

$$\bar{S}_{42} = -\bar{\psi}_1.$$

The terms of the series  $S_1, S_2, S_3, S_4$  for which  $|n + \frac{1}{2} - z|$  is of lower order than  $z^{1/3}$  and  $z_1 - n - \frac{1}{2}$  is of the same or higher order than  $z_1^{1/3}$  are of the order  $z^{-1/3} \psi_1$ , and, that their

\* It will be proved below that these terms are important only in the neighbourhood of the geometrical shadow.

sum should be of the order of  $\psi_1$ ,  $\frac{1}{2}\pi - \alpha_1 - \theta$  must be small, that is, the point P must be on or close to the boundary of the sphere described on  $OO_1$  as diameter. In these terms of the series  $S_3$  and  $S_4$ ,  $e^{2i(\phi_0 + \chi_0)}$  can be replaced by  $-1$ , and then the sum of the terms of the series  $S_1 - S_2 + S_3 - S_4$  required is the same as the sum of the corresponding terms of the series

$$\frac{i}{2\kappa r_1^2} \sum_{n=1}^{\infty} (2n+1) R^{1/2} R_1^{1/2} \{e^{i(\phi - \phi_1)} - e^{-i(\phi + \phi_1)}\} (1 - \mu^2) \frac{dP_n}{d\mu}.$$

Now, for the values of  $\theta$  specified above,  $\frac{d}{dn} \{\pm \phi - \phi_1 \pm (n + \frac{1}{2})\theta\}$  can only be small when  $\alpha$  is a right angle or differs from a right angle by a small quantity, hence the principal part of the sum of the terms of the series  $S_1 - S_2 + S_3 - S_4$  is the same as the principal part of  $\psi_1$ , that is, it is equal to  $\bar{\psi}_1$ .

6. When  $r > r_1$ ,  $\phi$  and  $\phi_1$  are interchanged, the value of  $\psi$  may be written

$$\psi = S'_1 - S'_2 + S_3 - S_4,$$

and the discussion of the series is identical with that of the previous case. If a plane be drawn through  $O_1$  perpendicular to  $OO_1$ , then for points P on the side of this plane remote from the sphere and not close to the plane the principal part of  $S'_2$  is  $\bar{\psi}_1$ ; for points outside the geometrical shadow and not close to its boundary the terms of the series  $S_4$ , for which  $n$  is less than  $z_0$ , have a sum of the same order as  $\psi_1$ , which represents the effect of reflected waves; for points outside the geometrical shadow between the plane through  $O_1$  and the sphere centre O, radius  $OO_1$ , the terms of the series  $S_4$ , for which  $n$  is greater than  $z_0$ , have a sum whose principal part is  $-\bar{\psi}_1$ , and for points in the neighbourhood of the plane through  $O_1$  the terms of  $\psi$ , for which  $|n + \frac{1}{2} - z_1|$  is of lower order than  $z_1^{1/3}$ , have a sum whose principal part is  $\bar{\psi}_1$ . Thus for all points outside the geometrical shadow and not close to its boundary the principal part of the value of  $\psi$  obtained from the series is the same as that which would be given by applying the methods of geometrical optics.

7. The effect of the terms of the series  $S_3$  and  $S_4$ , beginning with those for which  $z_0 - n - \frac{1}{2}$  is of the same or of lower order than  $z_0^{1/3}$ , has now to be obtained. The corresponding part of  $S_4$  may be written

$$S_{43} = (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_{z_0 - \Lambda z_0^{1/3}} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{i[2\phi_0 + 2\chi_0 - \phi - \phi_1 - (n + 1/2)\theta + 1/4\pi]},$$

now for values of  $n$  greater than  $z_0$  it follows, from the corresponding expression for  $2\phi_0 + 2\chi_0$  [Appendix (vi.)], that the factor  $e^{2i(\phi_0 + \chi_0)}$  does not oscillate but approaches the value  $-1$  as  $n$  increases, hence for these values of  $n$  it may be written

$$e^{2i(\phi_0 + \chi_0)} = -1 - 2 \sum_{1}^{\infty} (-i)^k \cot^k (\phi_0 + \chi_0).$$

Writing

$$u_0 = w_0 e^{T_0}, \quad v_0 = w_0 e^{-T_0},$$

and remembering that

$$\cot(\phi_0 + \chi_0) = -u'_0/v'_0,$$

it follows that

$$\cot(\phi_0 + \chi_0) = \frac{1 + 2w_0 w'_0}{1 - 2w_0 w'_0} e^{2T_0},$$

where  $(1 + 2w_0 w'_0)/(1 - 2w_0 w'_0)$  differs from unity by a small quantity for all values of  $n$  greater than  $z_0$ , and  $T_0$  is a negative quantity increasing numerically with  $n$ ; whence

$$e^{2i(\phi_0 + \chi_0)} = -1 - 2 \sum_1^{\infty} (-i)^k \left( \frac{1 + 2w_0 w'_0}{1 - 2w_0 w'_0} \right)^k e^{2kT_0}$$

and

$$\begin{aligned} S_{43} = (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} & \left[ \sum_{z_0 - \Lambda z_0^{1/3}}^{z_0} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{i[2\phi_0 + 2\chi_0 - \phi - \phi_1 - (n + 1/2)\theta + 1/4\pi]} \right. \\ & - \sum_{z_0} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{-i[\phi + \phi_1 + (n + 1/2)\theta - 1/4\pi]} \\ & \left. - 2 \sum_{z_0} \sum_1^{\infty} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} \left( \frac{1 + 2w_0 w'_0}{1 - 2w_0 w'_0} \right)^k (-i)^k e^{2kT_0 - i[\phi + \phi_1 + (n + 1/2)\theta - 1/4\pi]} \right]. \end{aligned}$$

In the series

$$\sum_{z_0} \sum_1^{\infty} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} \left( \frac{1 + 2w_0 w'_0}{1 - 2w_0 w'_0} \right)^k (-i)^k e^{2kT_0 - i[\phi + \phi_1 + (n + 1/2)\theta - 1/4\pi]},$$

writing  $n + \frac{1}{2} = z_0 + \zeta$ , since  $2T_0 = \log \tan \phi_0$ , the coefficient of  $\zeta$  in  $2T_0$  is of the order  $z_0^{-1/3}$ ; hence, if the value of

$$\theta + \frac{\partial \phi}{\partial n} + \frac{\partial \phi_1}{\partial n}$$

where  $n + \frac{1}{2} = z_0$  is of higher order than  $z_0^{-1/3}$ , the sum of this double series is of lower order than its first term multiplied by  $z_0^{1/3}$ , and therefore to the order required negligible. Again, in the series

$$\sum_{z_0 - \Lambda z_0^{1/3}}^{z_0} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{i[2\phi_0 + 2\chi_0 - \phi - \phi_1 - (n + 1/2)\theta + 1/4\pi]}$$

the factor  $e^{2i(\phi_0 + \chi_0)}$  oscillates and, writing  $n + \frac{1}{2} = z_0 - \zeta$ ,

$$2\phi_0 + 2\chi_0 = \frac{1}{3}\pi + 3^{-1/2} (4c)^{-1} (3\mu)^2$$

[see Appendix (vi.)], where

$$3\mu = -6^{1/2} z_0^{-1/3} \zeta, \quad c = 2^{-1/3} \pi^{1/2} \Pi(-\frac{1}{6}),$$

and therefore the sum of this series is of lower order than its last term multiplied by  $z_0^{1/3}$  when the value of

$$\theta + \frac{\partial \phi}{\partial n} + \frac{\partial \phi_1}{\partial n}$$



for  $n + \frac{1}{2} = z_0$  is of higher order than  $z_0^{-1/3}$ . Hence the principal part of  $S_{43}$  is equal to the principal part of

$$-(2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_{z_0} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{-i[\phi + \phi_1 + (n + 1/2)\theta - 1/4\pi]}$$

when the value of

$$\theta + \frac{\partial \phi}{\partial n} + \frac{\partial \phi_1}{\partial n}$$

for  $n + \frac{1}{2} = z_0$  is of higher order than  $z_0^{-1/3}$ .\*

Now

$$\begin{aligned} & -(2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \sum_{z_0} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{-i[\phi + \phi_1 + (n + 1/2)\theta - 1/4\pi]} \\ &= (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} \left[ \sum_1^{z_0} (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{-i[\phi + \phi_1 + (n + 1/2)\theta - 1/4\pi]} \right. \\ & \quad \left. - \sum_1 (n + \frac{1}{2})^{3/2} R^{1/2} R_1^{1/2} e^{-i[\phi + \phi_1 + (n + 1/2)\theta - 1/4\pi]} \right]; \end{aligned}$$

the principal part of the second series has been found above, and arises from the terms on both sides of the term for which

$$\theta + \frac{\partial \phi}{\partial n} + \frac{\partial \phi_1}{\partial n} = 0.$$

If  $n_1$  is the corresponding value of  $n$ , and

$$n_1 + \frac{1}{2} = z \sin \alpha = z_1 \sin \alpha_1,$$

$n_1$  is given by

$$\theta = \pi - \alpha - \alpha_1,$$

hence, writing  $n = n_1 + \zeta$ , the principal part of the above series is

$$\begin{aligned} & (2\pi)^{-1/2} (\sin \theta)^{1/2} (\kappa r_1^2)^{-1} (n_1 + \frac{1}{2})^{3/2} \sec \alpha \sec^{1/2} \alpha_1 e^{-i(z \cos \alpha + z_1 \cos \alpha_1 + 1/4\pi)} \\ & \left[ \int_{-\infty}^{z_0 - n_1 - 1/2} e^{-1/2 i \zeta^2} \left( \frac{1}{z \cos \alpha} + \frac{1}{z_1 \cos \alpha_1} \right) d\zeta - \int_{-\infty}^{\infty} e^{-1/2 i \zeta^2} \left( \frac{1}{z \cos \alpha} + \frac{1}{z_1 \cos \alpha_1} \right) d\zeta \right]. \end{aligned}$$

Therefore, if  $D$  denotes the distance from  $O_1$  to the point  $P(r, \theta)$ ,

$$\sin \theta/D = \sin \alpha/r_1 = \sin \alpha/r,$$

and making the substitution

$$\zeta = \pi^{1/2} \left( \frac{1}{z \cos \alpha} + \frac{1}{z_1 \cos \alpha_1} \right)^{-1/2} \eta,$$

\* If  $\epsilon$  is an angle of higher order than  $z_0^{-1/3}$ ,  $\gamma$  the semivertical angle of the tangent cone to the sphere from  $O_1$ , and the tangent cones with vertices on  $OO_1$  whose semivertical angles are  $\gamma \pm \epsilon$  are constructed, the space excluded is that between these two tangent cones, which are both close to the boundary of the geometrical shadow.

the above expression becomes

$$-2^{-1/2} \kappa \sin^2 \alpha_1 e^{-\iota \kappa D - 1/4 \pi \iota} \int_{\eta_0}^{\infty} e^{-1/2 \pi \eta^2 \iota} d\eta,$$

where

$$\eta_0 = (a - r \sin \alpha) (2D)^{1/2} (\lambda r r_1 \cos \alpha \cos \alpha_1)^{-1/2},$$

hence

$$\bar{S}_{43} = -2^{-1/2} \bar{\psi}_1 e^{1/4 \pi \iota} \int_{\eta_0}^{\infty} e^{-1/2 \pi \eta^2 \iota} d\eta.$$

In the corresponding part of  $S_3$   $\theta$  has the opposite sign and therefore the sum is negligible in comparison with  $S_{43}$ .

If  $p$  is the perpendicular from  $O$  on  $O_1P$ ,  $p = r \sin \alpha$ , and when  $p$  is greater than  $a$  the point  $P$  is outside the boundary of the geometrical shadow,  $\eta_0$  is negative and increases rapidly as  $p$  increases, and the above value  $\bar{S}_{43}$  tends to the value found in the case where the point  $P$  was supposed not close to the boundary. When  $p$  is less than  $a$  the point  $P$  is inside the boundary of the geometrical shadow, and the above value of  $-\bar{S}_{43}$  is the principal part of  $\psi$ , that is

$$\bar{\psi} = 2^{-1/2} \bar{\psi}_1 e^{1/4 \pi \iota} \int_{\eta_0}^{\infty} e^{-1/2 \pi \eta^2 \iota} d\eta,$$

where  $\eta_0$  has the value given above.\* This result expresses the ratio of the magnetic force at a point inside the boundary of the geometrical shadow to the magnetic force due to the oscillator alone in terms of one quantity  $\eta_0$ . For values of  $\eta_0$  greater than 5 integration by parts will give a sufficient approximation, for values of  $\eta_0$  less than 5 it is convenient to write the above in the form

$$\bar{\psi} = \frac{1}{2} \bar{\psi}_1 [1 - (L + M) - \iota (L - M)],$$

where

$$L = \int_0^{\eta_0} \cos \frac{1}{2} \pi \eta^2 d\eta, \quad M = \int_0^{\eta_0} \sin \frac{1}{2} \pi \eta^2 d\eta,$$

and use GILBERT'S tables for these integrals. An important particular case is that for which  $O_1$  and  $P$  are both close to the surface of the sphere, as in the case of wireless telegraphy. The tables given below show how the amplitude of the oscillations diminishes as the distance along the earth's surface from the transmitter is increased. In calculating these tables the transmitter and receiver have been taken to be vertical antennæ, the fundamental wave-length is five times the height, and the results are given in two cases: for oscillations of wave-length one-fifth of a mile corresponding to antennæ of height 211 feet, and for oscillations of wave-length one-quarter of a mile corresponding to antennæ of height 264 feet. The first column gives the angular distance of the receiver from the transmitter, the second column

\* The investigation assumes that the series in  $\xi$  obtained by the substitution  $n = n_1 + \xi$  converge for values of  $n$  up to  $z_0$ , and this will be so if  $P$  is near the boundary of the shadow.

the distance  $d$  of the receiver from the transmitter in miles, the third column gives the corresponding value of  $\eta_0$  calculated from the formula

$$\eta_0 = 2 \left( \alpha - r \cos \frac{1}{2}\theta \right) (\lambda r \sin \frac{1}{2}\theta)^{-1/2},$$

the last column gives the ratio of the amplitude  $F$  of the oscillations at the receiver to the amplitude  $F_1$  of the oscillations due to the transmitter alone.

$\theta$ .	$d$ .	$\eta_0$ .	$F/F_1$ .	$\theta$ .	$d$ .	$\eta_0$ .	$F/F_1$ .
° ' ,				° ' ,			
1 0	70	·084	·46	1 0	70	·068	·47
1 20	93	·151	·43	1 20	93	·129	·44
1 40	116	·225	·40	1 40	116	·196	·41
2 0	140	·302	·37	2 0	140	·268	·38
2 20	163	·391	·34	2 20	163	·345	·36
2 40	186	·483	·32	2 40	186	·428	·33
3 0	209	·582	·29	3 0	209	·516	·30
3 20	233	·685	·26	3 20	233	·609	·28
3 40	256	·794	·24	3 40	256	·706	·25
4 0	279	·907	·22	4 0	279	·807	·23
4 20	302	1·025	·20	4 20	302	·914	·21
4 40	326	1·148	·18	4 40	326	1·024	·19
5 0	349	1·275	·16	5 0	349	1·137	·18
5 20	372	1·405	·15	5 20	372	1·255	·16
5 40	396	1·542	·14	5 40	396	1·377	·15
6 0	419	1·681	·13	6 0	419	1·501	·14
6 20	442	1·828	·12	6 20	442	1·630	·13
6 40	465	1·973	·11	6 40	465	1·762	·12
7 0	489	2·124	·10	7 0	489	1·897	·11
7 20	512	2·278	·09	7 20	512	2·035	·11
7 40	535	2·436	·09	7 40	535	2·177	·10
8 0	558	2·598	·08	8 0	558	2·321	·09
8 20	582	2·763	·07	8 20	582	2·469	·09
8 40	605	2·931	·07	8 40	605	2·619	·08
9 0	628	3·103	·07	9 0	628	2·773	·08
9 20	651	3·276	·06	9 20	651	2·929	·07
$\lambda = \cdot 2$ mile.				$\lambda = \cdot 25$ mile.			

## APPENDIX.—INVESTIGATION OF MATHEMATICAL RESULTS REQUIRED.

### 1. *Summation of Series.*

If  $u_n$  denotes the general term of a series, and  $S_m$  the sum of  $m$  terms, the first of which is  $u_n$ ,

$$S_m = u_n + u_{n+1} + \dots + u_{n+m-1},$$

which can be written symbolically

$$S_m = (1 + e^D + e^{2D} + \dots + e^{(m-1)D}) u_{n+t},$$

where  $D$  is  $d/dt$  and  $t$  is put equal to zero after the operations are performed. This is equivalent to

$$S_m = \frac{e^{mD} - 1}{e^D - 1} u_{n+t},$$

or

$$S_m = \frac{1}{D} \frac{D}{e^D - 1} (e^{mD} - 1) u_{n+t},$$

which can be expressed by

$$S_m = \int_0^m \frac{D}{e^D - 1} u_{n+t} dt. *$$

The present object is to replace this integral by expressions suitable for calculation in different cases. The first case is that in which  $u_{n+t}$  contains an exponential factor  $e^{at}$ , where  $\alpha - 2k\pi i$  ( $k$  an integer) does not vanish or is not very small. Writing

$$u_{n+t} = e^{at} w_{n+t},$$

where  $w_{n+t}$  contains no other exponential factor of the form  $e^{at}$ ,

$$\frac{D}{e^D - 1} u_{n+t} = \frac{D}{e^D - 1} e^{at} w_{n+t} = e^{at} \frac{D + \alpha}{e^{D+\alpha} - 1} w_{n+t},$$

that is

$$\frac{D}{e^D - 1} u_{n+t} = e^{at} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{d\alpha^k} \left( \frac{\alpha}{e^\alpha - 1} \right) D^k w_{n+t},$$

or

$$\frac{D}{e^D - 1} u_{n+t} = e^{at} \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \alpha \frac{d^k}{d\alpha^k} \left( \frac{1}{e^\alpha - 1} \right) + k \frac{d^{k-1}}{d\alpha^{k-1}} \left( \frac{1}{e^\alpha - 1} \right) \right] D^k w_{n+t},$$

hence

$$\frac{D}{e^D - 1} u_{n+t} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{d\alpha^k} \left( \frac{1}{e^\alpha - 1} \right) [\alpha e^{at} D^k w_{n+t} + e^{at} D^{k+1} w_{n+t}],$$

that is

$$\frac{D}{e^D - 1} u_{n+t} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{d\alpha^k} \left( \frac{1}{e^\alpha - 1} \right) \frac{d}{dt} (e^{at} D^k w_{n+t}).$$

Therefore

$$S_m = \int_0^m \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{d\alpha^k} \left( \frac{1}{e^\alpha - 1} \right) \frac{d}{dt} (e^{at} D^k w_{n+t}) dt,$$

whence

$$S_m = \left\{ \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{d\alpha^k} \left( \frac{1}{1 - e^\alpha} \right) D^k w_{n+t} \right\}_{t=0} - \left\{ \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{d\alpha^k} \left( \frac{1}{1 - e^\alpha} \right) e^{at} D^k w_{n+t} \right\}_{t=m}$$

The sum  $S_m$  is thus expressed as the difference of two series which, in general, are diverging series, but they can be used for approximate calculation if the last term taken is within that part of the series which converges. When the series  $S_m$

\* Cf. BOOLE, 'Finite Differences,' ch. v.

converges to a limit when  $m$  increases without limit, the limit  $S$  to which it converges will be given approximately\* by

$$S = \left\{ \sum_0 \frac{1}{k!} \frac{d^k}{d\alpha^k} \left( \frac{1}{1-e^\alpha} \right) D^k w_{n+t} \right\}_{t=0} \quad \dots \quad (i.),$$

it being understood that only the terms of this series where it converges are to be taken.

The above clearly fails if  $\alpha = 2k\pi i$ , for in that case  $1-e^\alpha$  vanishes; it is also inapplicable if  $\alpha - 2k\pi i$  is very small, for then  $1-e^\alpha$  is very small, and the terms of (i.) diverge at once. The result shows that the sum  $S_m$ , however great  $m$  is, is not of an order higher than that of the terms that compose it when  $\alpha - 2k\pi i$  does not vanish or is not very small; this may be compared with the known case of DIRICHLET'S integral which vanishes unless the range of integration incloses the origin.

The second case to be considered is that in which  $\alpha = 0$  (to which the case  $\alpha = 2k\pi i$  is always reducible) and  $w_{n+t}$  contains an exponential factor  $e^{\beta t^2}$ ; writing

$$w_{n+t} = e^{\beta t^2} w_{n+t},$$

and remembering that

$$\frac{D}{e^D - 1} = 1 - \frac{1}{2}D + \sum_1 (-)^{k-1} \frac{1}{2k!} B_{2k-1} D^{2k},$$

where  $B_{2k-1}$  are BERNOULLI'S numbers, it follows that

$$S_m = \int_0^m e^{\beta t^2} w_{n+t} dt + \left\{ \left( -\frac{1}{2} + \sum_1 (-)^{k-1} \frac{1}{2k!} B_{2k-1} D^{2k-1} \right) e^{\beta t^2} w_{n+t} \right\}_{t=m} \\ - \left\{ \left( -\frac{1}{2} + \sum_1 (-)^{k-1} \frac{1}{2k!} B_{2k-1} D^{2k-1} \right) e^{\beta t^2} w_{n+t} \right\}_{t=0}.$$

When the real part of  $\beta$  is not greater than zero the important part of the integral  $\int_0^m e^{\beta t^2} w_{n+t} dt$  is that contributed by small values of  $t$ , and writing

$$w_{n+t} = w_n + t w'_n + \frac{1}{2} t^2 w''_n + \&c.,$$

it follows that

$$\int_0^m e^{\beta t^2} w_{n+t} dt = w_n \int_0^m e^{\beta t^2} dt + w'_n \int_0^m e^{\beta t^2} t dt + \&c.^\dagger$$

Now

$$\int_0^m e^{\beta t^2} dt = \int_0^\infty e^{\beta t^2} dt - \int_m^\infty e^{\beta t^2} dt,$$

that is

$$\int_0^m e^{\beta t^2} dt = \frac{1}{2} \pi^{1/2} (-\beta)^{-1/2} + e^{\beta m^2} / 2\beta m - \&c.,$$

\* When  $w_{n+t}$  only involves  $t$  as a polynomial the series has only a finite number of terms which represent the exact sum.

† For the determination of the important part of the integral it is sufficient that  $w_{n+t}$  should be expressible in powers of  $t$  for small values of  $t$ .

and therefore,  $m$  not being small, the important part of  $\int_0^m e^{\beta t^2} dt$  is  $\frac{1}{2}\pi^{1/2}(-\beta)^{-1/2}$ ; similarly, the important part of  $\int_0^m e^{\beta t^2} t dt$  is  $-\frac{1}{2}\beta^{-1}$ ; hence, the important part of  $\int_0^m e^{\beta t^2} w_{n+t} dt$  is  $\frac{1}{2}w_n\pi^{1/2}(-\beta)^{-1/2}$ , provided  $w'_n$  is of lower order than  $w_n\beta^{1/2}$  with corresponding conditions for  $w''_n$ , &c. Therefore, when  $\beta^{-1/2}$  is of an order higher than unity, the most important part of  $S_m$  is  $\frac{1}{2}w_n\pi^{1/2}(-\beta)^{-1/2}$ , in this case the sum of an order higher than that of the terms that compose it.

The third case is when  $\alpha$  is small and  $u_{n+t}$  contains an exponential factor of the form  $e^{\alpha t + \beta t^2}$ ; as in the immediately preceding case, the result depends on the value of the integral  $\int_0^m e^{\alpha t + \beta t^2} w_{n+t} dt$ .

If the real part of  $\beta$  is negative, or if  $\alpha$  and  $\beta$  are both pure imaginaries, the important part of this integral is  $w_n \int_0^\infty e^{\alpha t + \beta t^2} dt$ , with the same restrictions as in the previous case.

The fourth case is that when  $\beta=0$ ,  $\alpha$  is small, and  $u_{n+t}$  contains an exponential factor  $e^{\alpha t + \gamma t^3}$ ; as above, the result depends on the value of the integral  $\int_0^m e^{\alpha t + \gamma t^3} w_{n+t} dt$ , and, if the real part of  $\gamma$  is negative, or if both  $\alpha$  and  $\gamma$  are pure imaginaries, the important part of this integral is  $w_n \int_0^\infty e^{\alpha t + \gamma t^3} dt$ , with restrictions similar to those of the two preceding cases. The integrals in the third and fourth cases have been fully investigated and tabulated.

## 2. *Approximate Expressions for the Bessel Functions.*

Most of the expressions to be investigated in what follows have been given by L. LORENZ,\* who obtained them from expressions for the sum of the squares of the two solutions and the product of the two solutions.

The investigation that follows derives them directly from the fundamental expression for a solution of BESSEL's equation and the passage from the periodic form of the expressions to the form involving real exponentials, which is insufficiently treated by LORENZ, is traced. Writing

$$z^{1/2}J_{n+1/2}(z) = 2^{1/2}\pi^{-1/2}u_n, \quad (-)^n z^{1/2}J_{-n-1/2}(z) = 2^{1/2}\pi^{-1/2}v_n,$$

and using SCHLÄFLI's† formula for  $J_n$ , it follows that

$$v_n - u_n = 2^{-1/2}\pi^{-1/2}z^{1/2} \left[ \int_0^\pi [(-)^n \cos \{(n+\frac{1}{2})\theta + z \sin \theta\} - i \cos \{(n+\frac{1}{2})\theta - z \sin \theta\}] d\theta \right. \\ \left. + (-)^n \sin(n+\frac{1}{2})\pi \int_0^\infty e^{-z \sinh \psi + (n+1/2)\psi} d\psi - i \sin(n+\frac{1}{2})\pi \int_0^\infty e^{-z \sinh \psi - (n+1/2)\psi} d\psi \right]. \ddagger$$

\* 'Œuvres Scientifiques,' tome I., p. 405.

† Math. Ann. Band III, p. 143.

‡ In what follows  $n$  will be taken to be a positive integer, and  $u$  and  $v$  will be written for  $u_n$  and  $v_n$ .

Hence

$$v - u = 2^{-1/2} \pi^{-1/2} z^{1/2} \left[ -i \int_0^\pi e^{-iz \sin \theta + (n+1/2)\theta} d\theta + \int_0^\infty e^{-z \sinh \psi + (n+1/2)\psi} d\psi \right. \\ \left. - i \sin(n + \frac{1}{2}) \pi \int_0^\infty e^{-z \sinh \psi - (n+1/2)\psi} d\psi \right].$$

It is required to find approximations for the integrals above when  $z$  is large. Taking first the case where  $n + \frac{1}{2}$  is less than  $z$ , the important part of  $v - u$  arises from the first integral on the right-hand side, and the most important part of that integral is contributed by the values of  $\theta$  for which the exponent is nearly stationary; writing

$$w = -z \sin \theta + (n + \frac{1}{2}) \theta,$$

$w$  is stationary when  $z \cos \theta = n + \frac{1}{2}$ , and putting  $n + \frac{1}{2} = z \sin \alpha$ , the corresponding value of  $\theta$  is  $\frac{1}{2}\pi - \alpha$ . Hence substituting  $\theta = \frac{1}{2}\pi - \alpha + \vartheta$ ,

$$w = -z \cos \alpha + (n + \frac{1}{2}) (\frac{1}{2}\pi - \alpha) + \frac{1}{2} z \cos \alpha \vartheta^2 + \frac{1}{3!} z \sin \alpha \vartheta^3 + \dots,$$

and

$$\int_0^\pi e^{-iz \sin \theta + (n+1/2)\theta} d\theta = \int_{\alpha-1/2\pi}^{\alpha+1/2\pi} e^{-iz \cos \alpha + (n+1/2)(1/2\pi - \alpha) + 1/2 z \cos \alpha \vartheta^2 + 1/6 z \sin \alpha \vartheta^3} d\vartheta;$$

that is

$$\int_0^\pi e^{-iz \sin \theta + (n+1/2)\theta} d\theta = e^{-iz \cos \alpha + (n+1/2)(1/2\pi - \alpha)} \int_{\alpha-1/2\pi}^{\alpha+1/2\pi} e^{1/2 z \cos \alpha \vartheta^2 + 1/6 z \sin \alpha \vartheta^3 + \dots} d\vartheta.$$

Now unless  $\frac{1}{2}\pi - \alpha$  is small, the important part of the integral on the right-hand side is

$$\int_{\alpha-1/2\pi}^{\alpha+1/2\pi} e^{1/2 z \cos \alpha \vartheta^2} d\vartheta = \int_{-\infty}^{\infty} e^{1/2 z \cos \alpha \vartheta^2} d\vartheta - \int_{\alpha+1/2\pi}^{\infty} e^{1/2 z \cos \alpha \vartheta^2} d\vartheta - \int_{-\infty}^{\alpha-1/2\pi} e^{1/2 z \cos \alpha \vartheta^2} d\vartheta,$$

and unless  $\{(\frac{1}{2}\pi - \alpha) z \cos \alpha\}^{-1}$  is of the same or of higher order than  $(z \cos \alpha)^{-1/2}$ , the second and third integrals on the right are negligible in comparison with the first, that is, if  $z - n - \frac{1}{2}$  is of an order higher than  $z^{1/3}$ ,

$$\int_{\alpha-1/2\pi}^{\alpha+1/2\pi} e^{1/2 z \cos \alpha \vartheta^2} d\vartheta = 2^{1/2} \pi^{1/2} (-z \cos \alpha)^{-1/2} = 2^{1/2} \pi^{1/2} (z \cos \alpha)^{-1/2} e^{1/4 \pi i},$$

and therefore, with the same restriction as to the magnitude of  $n$ ,

$$\int_0^\pi e^{-iz \sin \theta + (n+1/2)\theta} d\theta = 2^{1/2} \pi^{1/2} (z \cos \alpha)^{-1/2} e^{i[-z \cos \alpha + (n+1/2)\pi - (n+1/2)\alpha]},$$

the term of highest order only being retained. Hence

$$v - u = (\cos \alpha)^{-1/2} e^{i[-z \cos \alpha + 1/2 n \pi - (n+1/2)\alpha]} *$$

\* The parts contributed by the second and third integrals in the first expression for  $v - u$  are of order  $z^{-1}$  and therefore negligible.





therefore

$$R = 2^{1/3} 3^{-1/3} \pi^{-1} z^{1/3} [\{\Pi(-\frac{2}{3})\}^2 + 6\mu \Pi(-\frac{1}{3}) \Pi(-\frac{2}{3}) \cos \frac{1}{3}\pi + 9\mu^2 \{\Pi(-\frac{1}{3})\}^2] \cos^2 \frac{1}{6}\pi.$$

Now,

$$\Pi(-\frac{2}{3}) = 2^{-2/3} \pi^{-1/2} \Pi(-\frac{1}{3}) \Pi(-\frac{5}{6}),$$

whence

$$\{\Pi(-\frac{2}{3})\}^2 = 2^{1/3} 3^{-1/2} \pi^{1/2} \Pi(-\frac{5}{6}),$$

similarly,

$$\{\Pi(-\frac{1}{3})\}^2 = 2^{2/3} 3^{-1/2} \pi^{1/2} \Pi(-\frac{1}{6});$$

hence

$$R = 3^{-5/6} \pi^{-1/2} z^{1/3} [\Pi(-\frac{5}{6}) + 2^{2/3} \Pi(-\frac{1}{2}) \cdot 3\mu + 2^{1/3} \Pi(-\frac{1}{6}) (3\mu)^2] \quad \text{. . . (iv).}$$

Again,

$$\tan \phi = \frac{\Pi(-\frac{2}{3}) - 3\mu \Pi(-\frac{1}{3})}{\Pi(-\frac{2}{3}) + 3\mu \Pi(-\frac{1}{3})} \tan \frac{1}{6}\pi,$$

therefore  $\phi = \frac{1}{6}\pi - \beta$ , where  $\beta$  is given by

$$\tan \beta = \frac{3\mu \sin \frac{1}{3}\pi \Pi(-\frac{1}{3})}{\Pi(-\frac{2}{3}) + \frac{3}{2}\mu \Pi(-\frac{1}{3})},$$

whence

$$\beta = 3\mu c (1 - \frac{3}{2}\mu c) \sin \frac{1}{3}\pi,$$

where

$$c = \Pi(-\frac{1}{3})/\Pi(-\frac{2}{3}) = 2^{-1/3} \pi^{-1/2} \Pi(-\frac{1}{6}),$$

and

$$\phi = \frac{1}{6}\pi - 3\mu c (1 - \frac{3}{2}\mu c) \sin \frac{1}{3}\pi \quad \text{. . . . . (v).}^*$$

The value of  $\chi$ , where

$$\tan \chi = -\frac{1}{2} \frac{dR}{dz}$$

is also required; it is, however, more convenient to calculate  $\phi + \chi$ . From the relations

$$-\frac{1}{2} \frac{dR}{dz} = u \frac{du}{dz} + v \frac{dv}{dz}, \quad v \frac{du}{dz} - u \frac{dv}{dz} = 1,$$

it follows that

$$\tan \chi = -\frac{uu' + vv'}{u'v - uv'}.$$

Now

$$\tan \phi = \frac{u}{v},$$

therefore

$$\frac{v'}{u'} = \frac{\tan \phi + \tan \chi}{\tan \phi \tan \chi - 1}, \quad \text{that is } \tan(\phi + \chi) = -\frac{v'}{u'};$$

to obtain  $\phi + \chi$  to the second power of  $\mu$ , the third power of  $\mu$  must be retained in  $u$  and  $v$ , hence

$$\tan(\phi + \chi) = \frac{\{\Pi(-\frac{1}{3}) + \frac{1}{2}\Pi(\frac{1}{3})(3\mu)^2\} \cos^2 \frac{1}{6}\pi}{\Pi(-\frac{1}{3}) \sin \frac{1}{6}\pi \cos \frac{1}{6}\pi + \frac{1}{2}\Pi(\frac{1}{3})(3\mu)^2 \sin \frac{5}{6}\pi \cos \frac{5}{6}\pi},$$

\* These expressions for  $R$  and  $\phi$  are to the order given, the same as those given by LORENZ.

that is

$$\tan(\phi + \chi) = \frac{\Pi(-\frac{1}{3}) + \frac{1}{2}\Pi(\frac{1}{3})(3\mu)^2}{\Pi(-\frac{1}{3}) - \frac{1}{2}\Pi(\frac{1}{3})(3\mu)^2} \tan \frac{1}{3}\pi,$$

therefore  $\phi + \chi = \frac{1}{3}\pi + \gamma$  where

$$\tan \gamma = \frac{\Pi(\frac{1}{3})(3\mu)^2 \tan \frac{1}{3}\pi}{\Pi(-\frac{1}{3}) \sec^2 \frac{1}{3}\pi}, \quad \text{that is } \gamma = \frac{1}{4c} 3^{-1/2} (3\mu)^2,$$

whence

$$\phi + \chi = \frac{1}{3}\pi + \frac{1}{4c} 3^{-1/2} (3\mu)^2 \quad \dots \dots \dots \text{(vi.)}$$

When  $z - n - \frac{1}{2}$  is of the same order as  $z^{1/3}$ , the series for  $v - u$  in ascending powers of  $\mu$  is not suitable for obtaining approximate values, and must therefore be replaced by a series involving inverse powers of  $\mu$ . To effect this it is necessary to obtain the principal part of the integrals in (iii.); as in this case  $\mu$  is negative, the principal part is contributed by the second integral. Writing

$$w = -3\mu e^{-1/3\pi i} \zeta - \zeta^3,$$

the important part of the integral arises from values of  $\zeta$  in the neighbourhood of the value of  $\zeta$  that makes  $w$  stationary; this value is given by  $\xi = (-\mu)^{1/2} e^{-1/6\pi i}$ , and substituting

$$\zeta = (-\mu)^{1/2} e^{-1/6\pi i} + \zeta,$$

it follows that

$$e^{-1/3\pi i} \int_0^\infty e^{-3\mu e^{-1/3\pi i} \zeta - \zeta^3} d\zeta = e^{-2i(-\mu)^{3/2-1/3\pi i}} \int_{-(-\mu)^{1/2} e^{-1/6\pi i}}^\infty e^{-3(-\mu)^{1/2} e^{-1/6\pi i} \zeta^3 - \zeta^3} d\zeta;$$

when  $\mu$  is not small the principal part of the integral on the right-hand side is the same as that of the integral whose lower limit is  $-\infty e^{-1/6\pi i}$ , and therefore the principal part only being retained,

$$e^{-1/3\pi i} \int_0^\infty e^{-3\mu e^{-1/3\pi i} \zeta - \zeta^3} d\zeta = 3^{-1/2} \pi^{1/2} (-\mu)^{-1/4} e^{-2i(-\mu)^{3/2-1/4\pi i}}.$$

Hence, the principal part only being retained,

$$v - u = 6^{-1/6} z^{1/6} (-\mu)^{-1/4} e^{-2i(-\mu)^{3/2-1/4\pi i}};$$

therefore

$$R = 6^{-1/3} z^{1/3} (-\mu)^{-1/2}, \quad \phi = 2(-\mu)^{3/2} + \frac{1}{4}\pi,$$

and substituting for  $\mu$  its value, this becomes

$$R = z^{1/2} \{2(z - n - \frac{1}{2})\}^{-1/2}, \quad \phi = \frac{1}{3} 2^{3/2} (z - n - \frac{1}{2})^{3/2} z^{-1/2} + \frac{1}{4}\pi. \quad \dots \text{(vii.)}$$

These are the forms that the expressions for  $R$  and  $\phi$ , when  $z - n - \frac{1}{2}$  is of higher order than  $z^{1/3}$ , take when  $\alpha$  is near to  $\frac{1}{2}\pi$ ; for, writing  $\frac{1}{2}\pi - \alpha = \epsilon$ , it follows that

$$\phi = z \cos \alpha - \frac{1}{2}n\pi + (n + \frac{1}{2})\alpha$$

becomes

$$\phi = z \sin \epsilon - (n + \frac{1}{2}) \epsilon + \frac{1}{4} \pi ;$$

now

$$z - n - \frac{1}{2} = z (1 - \cos \epsilon) = \frac{1}{2} z \epsilon^2,$$

therefore

$$\phi = \frac{1}{3} z \epsilon^3 + \frac{1}{4} \pi = \frac{1}{3} 2^{3/2} (z - n - \frac{1}{2})^{3/2} z^{-1/2} + \frac{1}{4} \pi$$

and

$$R = z \{z^2 - (n + \frac{1}{2})^2\}^{-1/2} = z \{z - n - \frac{1}{2}\}^{-1/2} \{2z - (z - n - \frac{1}{2})\}^{-1/2};$$

that is, retaining the principal part only,

$$R = z^{1/2} \{2(z - n - \frac{1}{2})\}^{-1/2}.$$

The leading term in  $(v - u) z^{-1/6}$  being known, a further approximation can be obtained from the differential equation

$$\frac{d^2 z}{d\mu^2} - 9\mu y = 0,$$

the result is

$$v - u = 6^{-1/6} z^{-1/6} (-\mu)^{-1/4} e^{-2i(-\mu)^{3/2 - 1/4\pi i}} \left\{ 1 + \frac{1 \cdot 5}{144} \cdot \frac{i}{(-\mu)^{3/2}} - \frac{1 \cdot 5 \cdot 7 \cdot 11}{2 \cdot 144^2} \cdot \frac{1}{(-\mu)^3}, \&c. \right\}^*.$$

The same remark applies to the approximation for  $v - u$  when  $z - n - \frac{1}{2}$  is of higher order than  $z^{1/3}$ , the ordinary differential equation for BESSEL'S functions being made use of.

When  $n + \frac{1}{2} - z$  is of the same order as  $z^{1/3}$  the corresponding series when  $\mu$  is positive is required. The principal part of  $v - u$  now arises from the integral  $\int_0^\infty e^{3\mu\zeta - \zeta^3} d\zeta$ , the principal part of which is contributed by values in the neighbourhood of the value of  $\zeta$  that makes  $3\mu\zeta - \zeta^3$  stationary; that is, in the neighbourhood of  $\zeta = \mu^{1/2}$ , and writing

$$\zeta = \mu^{1/2} + \zeta_1,$$

it follows that

$$\int_0^\infty e^{3\mu\zeta - \zeta^3} d\zeta = e^{2\mu^{3/2}} \int_{-\mu^{1/2}}^\infty e^{-3\mu^{1/2}\zeta_1^2 - \zeta_1^3} d\zeta_1,$$

therefore the principal part is

$$e^{2\mu^{3/2}} \int_{-\infty}^\infty e^{-3\mu^{1/2}\zeta_1^2} d\zeta_1 = 3^{-1/2} \pi^{1/2} \mu^{-1/4} e^{2\mu^{3/2}}.$$

This result gives the leading term in the value of  $v$ , to obtain the leading term in the value of  $u$  it is necessary to calculate the principal part of the imaginary terms, that is, the principal part of

$$\frac{1}{2} e^{-1/3\pi i} \int_0^\infty e^{-3\mu e^{-1/3\pi i} \zeta - \zeta^3} d\zeta - \frac{1}{2} e^{1/3\pi i} \int_0^\infty e^{-3\mu e^{1/3\pi i} \zeta - \zeta^3} d\zeta.$$

\* Cf. STOKES, 'Camb. Phil. Trans.,' vol. x., p. 105.



The value of  $\chi$  is given by the relations

$$\tan(\phi + \chi) = 2e^{4\mu^{3/2}}, \quad \text{or} \quad \tan \chi = e^{4\mu^{3/2}} \quad \dots \quad (\text{ix.}),$$

the leading term only being retained.

When  $n + \frac{1}{2} - z$  is great compared with  $z^{1/3}$ , the necessary approximations are obtained from the original formula for  $v - u$ , viz. :—

$$v - u = 2^{-1/2} \pi^{-1/2} z^{1/2} \left[ -\iota \int_0^\pi e^{-\iota z \sin \theta + (n+1/2) \theta \iota} d\theta + \int_0^\infty e^{-z \sinh \psi + (n+1/2) \psi} d\psi \right. \\ \left. - \iota \sin(n + \frac{1}{2}) \pi \int_0^\infty e^{-z \sinh \psi - (n+1/2) \psi} d\psi \right].$$

The principal part of this expression arises from the second integral, and is contributed by the values of  $\psi$  near to the value that makes  $z \sinh \psi - (n + \frac{1}{2}) \psi$  stationary; if this value of  $\psi$  is  $\delta$ , then

$$z \cosh \delta = n + \frac{1}{2},$$

and writing  $\psi = \delta + \psi_1$ , it follows that

$$\int_0^\infty e^{-z \sinh \psi + (n+1/2) \psi} d\psi = \int_{-\delta}^\infty e^{(n+1/2) \delta - z \sinh \delta - 2z \sinh \delta \sinh^2 1/2 \psi_1 - (n+1/2) (\sinh \psi_1 - \psi_1)} d\psi_1,$$

the principal part of which is equal to

$$e^{(n+1/2) \delta - z \sinh \delta} \int_{-\infty}^\infty e^{-1/2 z \sinh \delta \psi_1^2} d\psi_1 = 2^{1/2} \pi^{1/2} z^{-1/2} (\sinh \delta)^{-1/2} e^{(n+1/2) \delta - z \sinh \delta},$$

the part which depends on the integral  $\int_{-\infty}^{-\delta}$  being of lower order when  $n + \frac{1}{2} - z$  is of higher order than  $z^{1/3}$ ; hence

$$v = (\sinh \delta)^{-1/2} e^{(n+1/2) \delta - z \sinh \delta},$$

where  $\delta$  is given by  $z \cosh \delta = n + \frac{1}{2}$ .

To obtain  $u$  it is necessary to calculate the leading terms of the imaginary part of the expression for  $v - u$ , and this arises from the first integral; the part required is the principal part of

$$-\frac{1}{2} \iota \int_0^\pi e^{-\iota z \sin \theta + (n+1/2) \theta \iota} d\theta - \frac{1}{2} \iota \int_0^\pi e^{\iota z \sin \theta - (n+1/2) \theta \iota} d\theta.$$

The exponent in the first of these is stationary when  $\theta = \iota \delta$ , and the exponent in the second when  $\theta = -\iota \delta$ , writing in the first  $\theta = \iota \delta - \theta_1$  and in the second  $\theta = \iota \delta + \theta_1$ , the expression becomes

$$-\frac{1}{2} \iota e^{-(n+1/2) \delta + z \sinh \delta} \left\{ \int_{-\iota \delta}^{\pi - \iota \delta} e^{-2z \sinh \delta \sin^2 1/2 \theta_1 + \iota z \cosh \delta (\theta_1 - \sin \theta_1)} d\theta_1 + \int_{\iota \delta}^{\pi + \iota \delta} e^{-2z \sinh \delta \sin^2 1/2 \theta_1 - \iota z \cosh \delta (\theta_1 - \sin \theta_1)} d\theta_1 \right\},$$



further by using the differential equation satisfied by  $z^{-1/2}(v-w)$ , which is BESSEL'S equation.\*

The various approximations are collected together below for convenience of reference :—

When  $z-n-\frac{1}{2}$  is great compared with  $z^{1/3}$ ,

$$R = \sec \alpha, \quad \phi = z \cos \alpha - \frac{1}{2}n\pi + (n + \frac{1}{2})\alpha \quad \dots \quad \text{(ii.)}$$

where  $z \sin \alpha = n + \frac{1}{2}$ .

When  $z-n-\frac{1}{2}$  is of the same order as  $z^{1/3}$ ,

$$R = z^{1/2}\{2(z-n-\frac{1}{2})\}^{-1/2}, \quad \phi = \frac{1}{3}2^{3/2}z^{-1/2}(z-n-\frac{1}{2})^{3/2} + \frac{1}{4}\pi \quad \dots \quad \text{(vii.)}$$

When  $|z-n-\frac{1}{2}|$  is of lower order than  $z^{1/3}$ ,

$$R = 3^{-5/6}\pi^{-1/2}z^{1/3}[\Pi(-\frac{5}{6}) + 2^{2/3}\Pi(-\frac{1}{2}) \cdot 3\mu + 2^{1/3}\Pi(-\frac{1}{6})(3\mu)^2\mu] \quad \dots \quad \text{(iv.)}$$

$$\phi = \frac{1}{6}\pi - 3\mu c (1 - \frac{3}{2}\mu c) \sin \frac{\pi}{3} \quad \dots \quad \text{(v.)}$$

$$\phi + \chi = \frac{1}{3}\pi + \frac{1}{4c} 3^{-1/2}(3\mu)^2 \quad \dots \quad \text{(vi.)}$$

where

$$3\mu = 6^{1/3}z^{-1/3}(n + \frac{1}{2} - z), \quad c = 2^{-1/3}\pi^{-1/2}\Pi(-\frac{1}{6}).$$

When  $n + \frac{1}{2} - z$  is of the same order as  $z^{1/3}$ ,

$$R = 6^{-1/3}z^{1/3}\mu^{-1/4}[e^{4\mu^{3/2}} + \frac{1}{4}e^{-4\mu^{3/2}}], \quad \tan \phi = \frac{1}{2}e^{-4\mu^{3/2}} \quad \dots \quad \text{(viii.)}$$

$$\tan \chi = e^{4\mu^{3/2}} \quad \dots \quad \text{(ix.)}$$

When  $n + \frac{1}{2} - z$  is great compared with  $z^{1/3}$

$$R = (\sinh \delta)^{-1}[e^{-2\tau} + \frac{1}{4}e^{2\tau}], \quad \tan \phi = \frac{1}{2}e^{2\tau} \quad \dots \quad \text{(x.)}$$

where

$$\tau = z \sinh \delta - (n + \frac{1}{2})\delta, \quad z \cosh \delta = n + \frac{1}{2}.$$

When  $n$  is not an integer the corresponding results can be obtained by writing

\* Another method of approximating to the value of  $R$  is to make use of the relation

$$R = \frac{4z}{\pi} \int_0^\infty K_0(2n \sinh \zeta) \cosh(2n+1)\zeta d\zeta,$$

which is not difficult to establish, and then deduce  $\phi$  from the result; the method given is more direct, and avoids the difficulties that arise in determining the constants for the different forms of  $\phi$  in this other method.

$e^{-n\pi\iota}$  for  $(-)^n$  in the original expression, but, with a view to the special case where  $n+\frac{1}{2}$  is an integer, it is preferable to use the solutions  $K_{n+1/2}(\iota z)$  and  $K_{n+1/2}(-\iota z)$ ; the expressions to be approximated to are  $2^{1/2}\pi^{-1/2}z^{1/2}K_{n+1/2}(\iota z)$  and  $2^{1/2}\pi^{-1/2}z^{1/2}K_{n+1/2}(-\iota z)$ . With the same notation as when  $n$  is an integer

$$\begin{aligned} 2^{1/2}\pi^{-1/2}z^{1/2}K_{n+1/2}(\iota z) &= R^{1/2}e^{-\iota\phi-1/2(n+1/2)\pi\iota}, \\ 2^{1/2}\pi^{-1/2}z^{1/2}K_{n+1/2}(-\iota z) &= R^{1/2}e^{\iota\phi+1/2(n+1/2)\pi\iota}, \end{aligned}$$

where  $R$  and  $\phi$  have the same values as when  $n$  is an integer.

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