

IX. *The Approximate Arithmetical Solution by Finite Differences of Physical Problems involving Differential Equations, with an Application to the Stresses in a Masonry Dam.*

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§ 1. INTRODUCTION.—§ 1.0. The object of this paper is to develop methods whereby the differential equations of physics may be applied more freely than hitherto in the approximate form of difference equations to problems concerning irregular bodies.

Though very different in method, it is in purpose a continuation of a former paper by the author, on a “Freehand Graphic Way of Determining Stream Lines and Equipotentials” (‘Phil. Mag.’ February, 1908; also ‘Proc. Physical Soc.’ London, vol. xxi.). And all that was there said, as to the need for new methods, may be taken to apply here also. In brief, analytical methods are the foundation of the whole subject, and in practice they are the most accurate when they will work, but in the integration of partial equations, with reference to irregular-shaped boundaries, their field of application is very limited.

Both for engineering and for many of the less exact sciences, such as biology, there is a demand for rapid methods, easy to be understood and applicable to unusual equations and irregular bodies. If they can be accurate, so much the better; but 1 per cent. would suffice for many purposes. It is hoped that the methods put forward in this paper will help to supply this demand.

The equations considered in any detail are only a few of the commoner ones occurring in physical mathematics, namely:—LAPLACE’S equation $\nabla^2\phi = 0$; the oscillation equations $(\nabla^2 + k^2)\phi = 0$ and $(\nabla^4 - k^4)\phi = 0$; and the equation $\nabla^4\phi = 0$. But the methods employed are not limited to these equations.

The Number of Independent Variables.—In the examples treated in the paper this never exceeds two. The extension to three variables is, however, perfectly obvious. One has only to let the third variable be represented by the number of the page of a book of tracing paper. The operators are extended quite simply, and the same

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methods of successive approximation apply. But, of course, the labour would be greatly increased.

§ 1.1. *Finite differences* have, in themselves, but little importance to the student of matter and ether. They are here regarded simply as a makeshift for infinitesimals; and the understanding is always that we will eventually make the differences so small that the errors due to their finite size will be less than the errors of experiment or practical working, and may therefore be disregarded. That it is possible to make them small enough without much labour is illustrated by examples given hereafter.

In consequence of this point of view, the notation employed for finite differences is very similar to that for infinitesimal differences. Thus d and ∂ are differential operators, while δ and \mathfrak{S} are the corresponding finite difference operators. The oft-occurring symbol $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is represented in finite differences by \triangledown^2 .

The differences employed are "central differences," that is to say, they are considered as existing at the centre of the group of co-ordinate points from which they are derived. In this respect the notation differs from that used and defended by BOOLE ('Calculus of Finite Differences,' Art. 14), in which Δ_x^n is considered to exist at one end of the set of $n+1$ quantities which contribute to its value. The differencing operator δ and SHEPPARD'S* averager μ are defined by

$$\delta f(x) = f(x + \tfrac{1}{2}h) - f(x - \tfrac{1}{2}h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\mu f(x) = f(x + \tfrac{1}{2}h) + f(x - \tfrac{1}{2}h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where h is the co-ordinate difference. SHEPPARD shows that μ , δ , and d combine with one another according to the ordinary rules of algebra.

In this paper the differential coefficient $d^n f(x)/dx^n$ will be approximately represented when n is even by $h^{-n} \cdot \delta^n f(x)$ at the tabular points, and by $h^{-n} \cdot \mu \delta^n f(x)$ half-way between the tabular points. That is to say, these difference ratios are taken in place of the differential coefficients, and the error caused by so doing is left for consideration until after the difference equation has been solved. When n is odd, the symbolic expressions given above for $d^n f(x)/dx^n$ at tabular and at half-way points are simply interchanged. The representation is closer when the averager μ need not be introduced. Partial differential coefficients are represented by the difference ratios found by performing the above operations with μ and δ for each independent variable in turn. It will be convenient to have the representation of some of the commonest differential coefficients set forth explicitly.

Let ϕ be a function of x and y , and let lines be ruled on the plane xy parallel to the axes at equal distances h , of x and y , so as to divide the surface into a number of equal squares each of side h units. Let the arithmetical value of ϕ at the centre point of each square be written down in the square, forming a table of double entry.

* W. F. SHEPPARD, "Central-Difference Formulæ," 'Proc. Lond. Math. Soc.,' vol. xxxi., p. 460.

Let $(0\bar{1})$, $(2\bar{3})$, &c., for example, represent these numerical values at the co-ordinate points $x = 0$, $y = -h$; $x = 2h$, $y = -3h$, &c., reckoned from the centre of an arbitrarily chosen square as origin. Then at any tabular point 0, 0

$$\begin{aligned} \frac{\partial \phi}{\partial x} \text{ will be represented by } \mu \frac{\delta \phi}{\delta x} &= \frac{1}{2h} \{(10) - (\bar{1}0)\} \\ \frac{\partial^2 \phi}{\partial x^2} \quad \text{''} \quad \text{''} \quad \text{''} \quad \frac{\delta^2 \phi}{\delta x^2} &= \frac{1}{h^2} \{(10) - 2(00) + (\bar{1}0)\} \\ \frac{\partial^3 \phi}{\partial x^3} \quad \text{''} \quad \text{''} \quad \text{''} \quad \mu \frac{\delta^3 \phi}{\delta x^3} &= \frac{1}{2h^3} \{(20) - 2(10) + 2(\bar{1}0) - (\bar{2}0)\} \\ \frac{\partial^4 \phi}{\partial x^4} \quad \text{''} \quad \text{''} \quad \text{''} \quad \frac{\delta^4 \phi}{\delta x^4} &= \frac{1}{h^4} \{(20) - 4(10) + 6(00) - 4(\bar{1}0) + (\bar{2}0)\} \\ \frac{\partial^2 \phi}{\partial x \partial y} \quad \text{''} \quad \text{''} \quad \text{''} \quad \mu^2 \frac{\delta^2 \phi}{\delta x \delta y} &= \frac{1}{4h^2} \{(11) + (\bar{1}\bar{1}) - (\bar{1}1) - (1\bar{1})\} \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \quad \text{''} \quad \text{''} \quad \text{''} \quad \frac{\delta^2 \phi}{\delta x^2} + \frac{\delta^2 \phi}{\delta y^2} &= \nabla_1^2 \phi = \frac{1}{h^2} \{(10) + (01) + (10) + (0\bar{1}) - 4(00)\}. \end{aligned}$$

Thus $h^2 \nabla^2 f$ is the sum of the four nearest neighbours minus four times the value at the point considered.

$$\nabla_1^4 \phi = \left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \phi \text{ will be represented by } \left(\frac{\delta^4}{\delta x^4} + 2 \frac{\delta^4}{\delta x^2 \delta y^2} + \frac{\delta^4}{\delta y^4} \right) \phi = \nabla_1^4 \phi$$

$$\begin{aligned} \text{and } \nabla_1^4 \phi &= \frac{1}{h^4} [20(00) - 8\{(10) + (01) + (\bar{1}0) + (0\bar{1})\} + 2\{(11) + (\bar{1}\bar{1}) + (\bar{1}1) + (1\bar{1})\} \\ &\quad + \{(20) + (02) + (\bar{2}0) + (0\bar{2})\}]. \end{aligned}$$

Halfway between two tabular points, say at $\frac{1}{2}$, 0

$$\frac{\partial \phi}{\partial x} \text{ will be represented by } \frac{\delta \phi}{\delta x} = \frac{1}{h} \{(10) - (00)\}.$$

In the centre of four tabular values, *e.g.*, at $\frac{1}{2}$, $\frac{1}{2}$

$$\frac{\partial^2 \phi}{\partial x \partial y} \text{ will be represented by } \frac{\delta^2 \phi}{\delta x \delta y} = \frac{1}{h^2} \{(11) + (00) - (01) - (10)\}.$$

A point at which the difference equation obtaining throughout the body has to be satisfied will be called a *body-point*. There must be enough known values of the integral ϕ on the boundary side of any body-point to make the said difference equation completely determinate. Thus for ∇^2 there must be at least one layer of points with known values of ϕ on all sides of any body-point, for ∇^4 at least two layers. It will be seen in § 4 that at re-entrant angles a point may have sufficient known values outside it and yet not be a body-point, because the body-equation is not to be satisfied there. A point at which the body-equation is not satisfied, but at

which there is a value of ϕ which enters into the system of body-equations by way of the boundary-conditions will be called a *boundary-point*.

The values of any function of position at these two classes of points will be distinguished as *body values* and *boundary values*, or synonymously as *body-numbers* and *boundary-numbers*.

Problems are divided into two main classes according as the integral can or cannot be stepped out from a part of the boundary. They are discussed in § 2 and § 3 respectively.

§ 1.2. *Errors due to Finite Differences*.—Having solved an equation using the simple expressions of § 1.1 for the differential coefficients, it remains to enquire how much in error the integral may be. A rule of apparently universal application is to take smaller co-ordinate differences and repeat the integration; and, if necessary, extrapolate in the manner explained below.

It is known* that when central differences are used, the expansions of the differential coefficients of a function in terms of its differences contain only alternate powers of the co-ordinate difference h . The same is true for partial differential coefficients and for products of differential coefficients. Consequently the error of the representation of any differential expression by central differences is of the form $h^2F_2(x, y, z) + h^4F_4(x, y, z) +$ terms in higher powers of h^2 , where F_2, F_4 , &c., are independent of h .†

Next, as to the error of the finite-difference-integral ϕ . This is the infinitesimal integral of a differential equation having the error $h^2F_2(x, y, z) + h^4F_4(x, y, z) +$, &c. Let φ be the integral of the correct differential equation. Then, if we write

$$\phi(x, y, z) = \varphi(x, y, z) + m\psi_1(x, y, z) + m^2\psi_2(x, y, z) + \text{terms}$$

in higher powers of m , it follows that a differential expression of any order and degree for ϕ differs from the corresponding one for φ by

$$m \times (\text{a function of the differential coefficients of } \varphi \text{ and of } \psi_1) + \text{terms in } m^2, m^3, \text{ \&c.},$$

provided only that m is independent of the co-ordinates. Now, identifying m with h^2 , it follows that: *the errors of the integral and of any differential expressions derived from it, due to using the simple central differences of § 1.1 instead of differential coefficients, are of the form*

$$h^2f_2(x, y, z) + h^4f_4(x, y, z) + h^6f_6(x, y, z) +, \text{ \&c.}$$

Consequently, if the equation be integrated for several different values of h , extrapolation on the supposition that the error is of this form will give numbers very close to the infinitesimal integral. When h is small enough the error is simply

* W. F. SHEPPARD, "Central-Difference Formulæ," 'Proc. Lond. Math. Soc.,' vol. xxxi. (1899).

[† Note added January 21, 1910.—It is assumed that the co-ordinate axes in the tables which are compared are parallel, for the error at a fixed point and for a fixed value of h may depend on the direction of the axes.]

proportional to h^2 . Peculiarities present themselves on the boundary, but it is easy to see that errors will be of the form $h^2f_2+h^4f_4+$, &c., provided that in passing from one table to another each part is either kept infinitesimally correct, or else is worked by differences whose size in the one table bears a constant ratio to that in the other.

An extrapolation can only be made where the tabular points of the several tables coincide with one another. It is conceivable that in the future some method will be found of defining a continuous function in terms of the discrete body and boundary values, so that this continuous function shall have an error of the form $h^2f_2(x, y, z)+h^4f_4(x, y, z)+$, &c., everywhere. Extrapolation would then be possible everywhere.

An excellent illustration is afforded by Lord RAYLEIGH's account of the vibration of a stretched string of beads ('Sound,' vol. I., § 121). He gives the frequency of the fundamental for the same mass per unit length concentrated in various numbers of beads. This is reproduced below in the table. The co-ordinate difference h is inversely as one plus the number of beads, not counting beads at the fixed ends.

Number of free beads + one . .	2	3	4	5	10	20	40	∞
Ratio of frequency to that of continuous string }	·9003	·9549	·9745	·9836	·9959	·9990	·9997	unity
Error in representation of continuous string by string of beads }	·0997	·0451	·0255	·0164	·0041	·0010	·0003	·0000
Ratio of error to square of co-ordinate difference $\times a$ constant }	·3988	·4059	·4080	·4091	·4107	·4111	·4112	·4112

The degree of constancy of the last line shows that if we found the frequency for one bead and for three, then extrapolation, on the assumption that the error is proportional to h^2 , would give us the frequency for the continuous string to about one part in 1000; which is as near as we could get by twenty beads and no extrapolation. While extrapolation from the exact solutions for four beads and for nine would leave an error of only one in 50,000. Other examples of extrapolation will be found in § 3·1.

§ 2. *Procedure when the Conditions allow the Integral to be Marched out from a Part of the Boundary.*

§ 2·0. *Historical.*—Step-by-step arithmetical methods of solving ordinary difference equations have long been employed for the calculation of interest and annuities. Recently their application to differential equations has been very greatly improved by the introduction of rules allied to those for approximate quadrature. The papers referred to are:—

RUNGE, "Über die numerische Auflösung von Differentialgleichungen," 'Math. Ann.,' Bd. 46. Leipzig, 1895.

W. F. SHEPPARD, "A Method for Extending the Accuracy of Mathematical Tables," 'Proc. Lond. Math. Soc.,' XXXI.

KARL HEUN, "Neue methode zur approximativen Integration der Differentialgleichungen einer unabhängigen Veränderlichen," 'Zeitschrift Math. u. Phys.,' No. 45, 1900.

WILHELM KUTTA, "Beitrag zur näherungsweisen Integration totaler Differentialgleichungen," 'Zeitschrift Math. u. Phys.,' No. 46, 1901.

Further RICHARD GANZ, in a paper "Über die numerische Auflösung von partiellen Differentialgleichungen," 'Zeitschrift Math. u. Phys.,' No. 48, 1903, has extended the methods of RUNGE, HEUN, and KUTTA to partial equations of the type considered in this section. Those of the first order he turns into the form

$$\frac{\partial \phi}{\partial x} = F\left(x, y, \phi, \frac{\partial \phi}{\partial y}\right),$$

and starting from a boundary, where ϕ is a given function of y , he expands this function as a power series of y and integrates step by step in the x direction. The results he gives are of remarkable accuracy.

It is less accurate, but simpler, to work entirely by arithmetic in the manner illustrated in § 2.2 below.

§ 2.1.* *A simple Process and its Possibility.*—For ordinary equations the necessary and sufficient condition is that, for an n^{th} order equation, the integral and all its first $n-1$ differential coefficients should be given at the boundary. This is almost obvious at first sight. The complications in the following arise entirely from having to attend to the correct centering of the differences—an important thing in practice. Let the equation be

$$\frac{d^n \phi}{dx^n} = f\left(x, \phi, \frac{d\phi}{dx}, \frac{d^2\phi}{dx^2}, \frac{d^{n-1}\phi}{dx^{n-1}}\right).$$

Then if all the quantities of which f is a function are given at $x = x_0$, we can calculate $d^n \phi / dx^n$ at x_0 . Now, representing differentials by simple central differences, draw up a table in columns. The subscripts denote distance from x_0 .

$x.$	$\phi.$	$\delta\phi.$	$\delta^2\phi.$..	$\delta^{n-1}\phi.$	$\delta^n\phi.$
x_0	ϕ_0	$(\delta\phi)_0$	$(\delta^2\phi)_0$..	$(\delta^{n-1}\phi)_0$	$(\delta^n\phi)_0$
$x_0 + \frac{1}{2}h$		$(\delta\phi)_{1/2h}$..	$(\delta^{n-1}\phi)_{1/2h}$	
$x_0 + h$	ϕ_h		$(\delta^2\phi)_h$..		
$x_0 + \frac{3}{2}h$		$(\delta\phi)_{3/2h}$				
$x_0 + 2h$	ϕ_{2h}					

Each difference is centred at values of x halfway between those for the difference of next lower order. Then, beginning with $\delta^n\phi$, each difference is added to the one of next lower order and the sum written down in the column of the lower order one, a step h after it. This process gives a table with a diagonal boundary. The next value of $\delta^n\phi$ is found from the difference equation and the process is repeated. The difference equation is satisfied at those values of x where the highest difference is tabulated. These processes involve moving certain differences through a step of $\frac{1}{2}h$, at the start and when satisfying the difference equation. Except for δ^n and δ^{n-1} this is done by $(\delta^r\phi)_{1/2h} = (\delta^r\phi)_0 + \frac{1}{2}h (\delta^{r+1}\phi)_0$. Note that this is a *central* formula with respect to the step $-\frac{1}{2}h$ to $+\frac{1}{2}h$. For δ^{n-1} and δ^n the motion of $\frac{1}{2}h$ is accomplished by

writing algebraic symbols for the unknown values of $\delta^n\phi$ and $\delta^{n-1}\phi$ and finding them from $\delta\delta^{n-1}\phi = \delta^n\phi$, and from the given difference equation simultaneously. On the other hand if any of the first $(n-1)$ differential coefficients are unknown at x_0 the summation from column to column across the table cannot be carried out, and so the step method is impossible. The conditions for stepwise integration of partial equations are not so simple and require investigation.

§ 2.2. As an *example of stepwise integration*, let us take the equation for diffusion in a tube, $\sigma\partial^2\phi/\partial x^2 = \partial\phi/\partial t$. In the first place, if $\phi = F(x, t)$ is a solution of $\partial^2\phi/\partial x^2 = \partial\phi/\partial t$, then $\phi = F(ax, a^2\sigma t)$ is a solution of the given equation. So we need only concern ourselves with $\partial^2\phi/\partial x^2 = \partial\phi/\partial t$, and the results will apply to bodies of any linear dimensions and any uniform diffusivity. Let us suppose the boundary conditions are: $\phi = 0$ when $x = \pm\frac{1}{2}$ for all values of t ; $\phi = 1$ for all values of x when $t = 0$; in fact the familiar case of a uniformly heated slab, the faces of which are suddenly cooled. The method of this example is so simple that it can hardly be novel. It is introduced to show how easy it sometimes is to obtain approximate integrals by arithmetic of equations usually treated by complex analysis. We draw up a table with a row for each 0.1 of x and a column at every 0.001 of t . (The reason for making the time step small will appear later, § 3.2.1.) The given boundary values are next inserted.

TABLE I.

	$t = 0.$	0.001.	0.002.	0.003.	0.004.	0.005.	0.005 correct by FOURIER'S method.	Errors.
$x = 0.5$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
0.4	1.0000	$a = 0.9090$	0.8356	0.7714	0.7209	0.6729	0.6828	-0.0099
0.3	1.0000	$b = 0.9959$	0.9834	0.9695	0.9492	0.9329	0.9545	-0.0216
0.2	1.0000	$c = 0.9998$	0.9993	0.9968	0.9945	0.9887	0.9980	-0.0093
0.1	1.0000	$d = 1.0000$	1.0000	0.9999	0.9994	0.9990	0.9996	-0.0006
0.0	1.0000	$e = 1.0000$	1.0000	1.0000	1.0000	0.9998	1.0000	-0.0002
-0.1	1.0000	$d = 1.0000$	1.0000	0.9999	0.9994	0.9990	0.9996	-0.0006

In satisfying the equation we must be careful to equate values of $\partial^2\phi/\partial x^2$ and $\partial\phi/\partial t$, which are centered at the same point. This causes a little difficulty at starting. When $t = 0.001$ let the values of ϕ be a, b, c, d, e , as indicated in Table I. Then if the difference equation be satisfied at $t = 0.0005$, it takes the form of 5 simultaneous equations involving a, b, c, d, e . Solving these equations, we find the numbers given in the column $x = 0.001$. Having got over this rather troublesome first step, we can find the rest much more simply by centering all differences on the columns $t = 0.001, 0.002, 0.003$, &c., and deducing each number from the two preceding columns. The errors resulting from the above process may be found by comparison with the Fourier solution $\phi = \sum_{m \text{ odd}} (-1)^{\frac{m-1}{2}} \frac{1}{m} e^{-m^2\pi^2 t} \cos(m\pi x)$. This series has been computed when $t = 0.005$, and the numbers so found are given in the table. It is seen that the step

method gives a fair general view of the temperature corresponding to a given time and position, and it can, of course, be employed with equal facility when the boundary temperatures vary in almost any assigned manner, including cases that would be difficult or impossible by the Fourier method.

§ 3. *It frequently happens that the integral can only be determined with reference to the boundary as a whole*, as, for example, in the calculation of the electrostatic potential at all points of a region when its value is given over the conducting boundaries. Here the differential equation is of the second order, and the first space rates of its integral are not given on the boundary, so that the step-method is inapplicable. The following § 3 contains an account of two methods for solving problems of the type indicated.

§ 3.0. *The Determinate Nature of the Problem.*—Let there be n body-points and s boundary-points in the region considered. Then the differential equation, to be satisfied in the body, is approximately represented at any body-point by an algebraic equation connecting the body value there with the surrounding values. This algebraic equation will be of the first, second, or higher degree, according as the differential equation is of the first, second, or higher degree in the function of position and its differentials. Forming this equation at every body-point, we have a system of n simultaneous integral equations between $s+n$ unknowns. To make the problem determinate, the boundary conditions must therefore supply s independent relations, involving the boundary values. The rules governing the arrangement of these s boundary equations, so as best to represent the given infinitesimal boundary conditions, have not yet been elaborated. In certain cases a choice of ways is open, as in the following example: Let the body equation be $(\partial^2/\partial x^2 + \partial^2/\partial y^2)f = \rho$, where ρ is a given function of position, and the boundary condition $\partial f/\partial n + \zeta f = 0$, where ζ is an arbitrary function of position on the boundary. Let the body values of f be denoted by ψ_1, ψ_2, \dots , and the boundary values by $\beta_1, \beta_2, \&c.$ Then in the annexed Table II. we are at liberty to choose between two alternative approximations. For we may take as values of f on the boundary

TABLE II.

	ψ_1	β_1
	ψ_3	β_2
	ψ_2	
	β_4	β_3

$$\frac{1}{2}(\psi_1 + \beta_1), \quad \frac{1}{2}(\psi_2 + \beta_2), \quad \frac{1}{2}(\psi_2 + \beta_3), \quad \frac{1}{2}(\psi_3 + \beta_4).$$

And as corresponding values of $\partial f/\partial n$,

$$\beta_1 - \psi_1, \quad \beta_2 - \psi_2, \quad \beta_3 - \psi_2, \quad \beta_4 - \psi_3.$$

So that we have one boundary condition for each β . Or else we may suppose the corner slightly bevelled, so that, while the above relations still hold for β_1 and β_4 , we now have at the corner $f = \frac{1}{2} \{ \psi_2 + \frac{1}{2}(\beta_2 + \beta_3) \}$, $\partial f/\partial n = \frac{1}{2}(\beta_2 + \beta_3) - \psi_2$. Here $\beta_2 + \beta_3$ enters as one variable. Also β_2 and β_3 only enter the body equations when combined in the form $\beta_2 + \beta_3$. We no longer seek to determine β_2 and β_3 separately, but merely their average, which is the value of f just outside the corner,

so that there is still one boundary equation for each boundary variable which we seek to determine separately. We have just considered a blunted angle of 90 degrees. By examining the remaining five angles possible with two co-ordinates, namely, 45, 135, 225, 270, 315 degrees, one may convince oneself that with a closed boundary it is always possible to arrange to have just as many boundary equations as boundary unknowns. It is in some cases necessary to suppose the corner slightly bevelled and to replace certain (β)'s by their averages. The representation of the boundary condition when both f and $\partial f/\partial n$ are given at each point will be considered in the theory of the dam.

When the infinitesimal boundary conditions are such as to make the problem determinate, it will be assumed that we can, and therefore do, represent them by a set of boundary equations equal in number to the boundary unknowns. If any case be discovered in which this is impossible, it will be an exception to the rest of § 3.

§ 3.1. The finite difference problem being thus made determinate the most direct way of finding the integral is to *solve the $n+s$ simultaneous algebraic equations for the n body and s boundary values of the integral*. To take an example:—At one pair of opposite edges of a square $\phi = 1$, at the other pair $\phi = 0$.

TABLE III.

Inside	0.5	1	1	1	1	← edge of square.
$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$						
everywhere. Find ϕ inside the square.	0	0.5	a	b	c	b
Now by symmetry the values of ϕ on the diagonals will be everywhere 0.5. In fact, we need only consider $\frac{1}{8}$ of the area of the square; all the rest follows from it. Taking finite differences, Table III. is drawn up with the given boundary and diagonal numbers in their proper places and	0		0.5	d	e	d
a, b, c, d, e, f for the	0			0.5	f	0.5
unknown body values.						
Now as the finite difference expression for $\nabla^2\phi$ given in § 1.1 has to vanish at all body-points, we have a relation between each of the letters a to f and its four nearest neighbours.	0				centre of 0.5 square.	
	↑ edge of square.					

The solution of these six simultaneous equations was accomplished in an hour and gave the following results:—

	<i>a.</i>	<i>b.</i>	<i>c.</i>	<i>d.</i>	<i>e.</i>	<i>f.</i>
By finite differences	0·693	0·772	0·794	0·601	0·6324	0·533
By infinitesimals	0·700	0·777	0·798	0·604	0·6354	0·534
Errors due to finite differences . .	0·007	0·005	0·004	0·003	0·003	0·001

The numbers for infinitesimal differences were obtained from

$$\phi = \frac{4}{\pi} \sum_{m \text{ odd}} (-1)^{\frac{m-1}{2}} \frac{1}{m} \operatorname{sech} \frac{m\pi}{2} \cos mx \cosh mz,$$

the separate terms of which satisfy $\nabla^2 \phi = 0$ at all points, and $\phi = 0$ when $x = \pm \frac{1}{2}\pi$, and by their addition make $\phi = 1$ when $z = \pm \frac{1}{2}\pi$. Adding up the series at these six points took 3 hours. It is seen that the greatest error is 1·4 per cent. of the range of potential between the side and diagonal.

Further, if we take co-ordinate differences of twice this size, leaving only one unknown in the same position in the square as *e* here occupies, its value is easily found to be 0·6250. Extrapolating as in § 1·2 we find for infinitesimal differences at this point $\phi = 0·6324 + \frac{1}{3}(0·6324 - 0·6250) = 0·6348$, and this is only $\frac{1}{10}$ th per cent. in error. To correct the other values we should have to halve the co-ordinate difference instead of doubling it, and this would require much more work.

As a second example of the use of simultaneous integral equations, let us take *the determination of the gravest period of vibration of a thin square plate with edges clamped in a plane*. It is known (LOVE's 'Elasticity,' ed. 1906, p. 469) that the displacement normal to the plate is of the form $W \cos (pt + \epsilon)$, and W satisfies the equation $\left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) W \equiv \nabla_1^4 W = c^4 W$, where c^4 is given as a function of p , the elastic constants, the thickness and the density. Now let us form a table such as (IV.) to represent W . In this table W is measured from the plane of the clamped boundary. The differential equation, when turned into finite differences, becomes a set of simultaneous equations, connecting each in turn of the unknown body values q, r, s, t with its twelve nearest neighbours. As the boundary numbers are all zero there are no constant terms in these equations, and they are only consistent when the determinant of the coefficients of q, r, s, t vanishes. There are a number of values of c^4 which cause the determinant to vanish, and of these the smallest is that

belonging to the gravest mode of vibration. Let $W_1 = f(x, y)$ be the appropriate integral for a square of unit side. Then $W_L = f(x/L, y/L)$ will bear a similar relation to a square of side L and to a new constant C_L^4 . Then $C_L^4 = \frac{\nabla_1^4 W_L}{W_L} = \frac{1}{L^4} \frac{\nabla_1^4 W_1}{W_1} = \frac{C_1^4}{L^4}$, so that $L^4 C_L^4$ is independent of the length of side of the square. It will be interesting to notice how this constant converges towards a limit as the number of co-ordinate differences in the side of the square is increased.

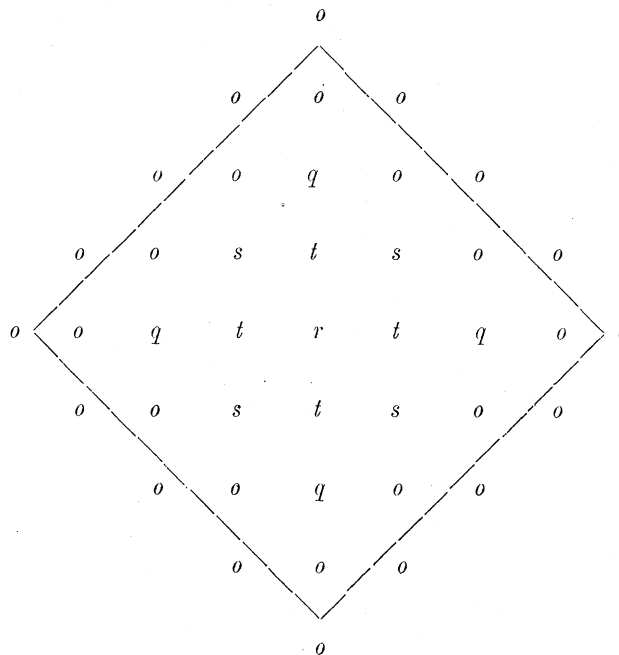
The configuration chosen was one in which the sides of the square are at 45 degrees to the rows and columns of the table. This gives a sharper boundary than the parallel arrangement. The symmetry of the gravest mode reduces the number of unknowns. As well as the arrangement in Table IV., two smaller ones were also considered, namely, those formed by cutting off in turn its first and second outer layers. C^4 was calculated in each case from the determinant, by approximation where necessary.

Collected results :—

Side of square = L .	C_L^4 for the gravest mode.	$L^4 C_L^4$.
$1\frac{1}{2}\sqrt{2}$	20.0000	405.0000*
$2\frac{1}{2}\sqrt{2}$	6.3058	985.28
$3\frac{1}{2}\sqrt{2}$	1.88843	1133.53

We may attempt a closer approximation by assuming that the error in C_L^4 is inversely as the square of the number of co-ordinate differences in the side of the square, in accordance with § 1.2. From the 1st and 2nd values extrapolation gives $L^4 C_L^4 = 1311.69$. And from the 2nd and 3rd $L^4 C_L^4 = 1287.96$. Or, if we assume that the error is of the form $e_2 h^2 + e_4 h^4$ and extrapolate from $L^4 C_L^4$ for the three values of h , we find $L^4 C_L^4 = 1282.62$. The way in which the values of $L^4 C_L^4$ converge

TABLE IV.



* Revised figures, March, 1910.

indicates that this last one can hardly be more than 1 in 1000 in error. The precise analytical theory of the vibrations of a square plate with clamped edges does not appear to be given anywhere, but we may obtain an upper limit to $L^4 C_L^4$ by the method in RAYLEIGH'S "Sound," § 89. Assuming $W = (a^2 - x^2)^2 (b^2 - y^2)^2$,* we have

$$C^4 \gg \int_{-b}^{+b} \int_{-a}^{+a} [\nabla^2 W]^2 dx dy / \int_{-b}^{+b} \int_{-a}^{+a} W^2 dx dy = (31.5000) (a^{-4} + b^{-4}) + (18.0000) (ab)^{-2},$$

when $a = b$ this becomes $1296.0/(2a)^4$, which is greater than $1283/(2a)^4$ obtained by finite differences.

§ 3.2. *Successive Approximation to the Integrals.*—Having illustrated the use of simultaneous integral equations, let us pass on to methods which have this property in common: that starting from a table of numbers, correct at the boundary, but otherwise merely as near as one can guess, one proceeds by definite methods to modify this table and thereby to cause it to approach without limit towards the true finite-difference integral.

Conditions.—The following methods of approximation have up to the present been applied only to a limited class of equations satisfying the conditions given below.

Let f be an arbitrary function of position having n body values $\psi_1, \psi_2, \dots, \psi_n$, and s boundary values $\beta_1, \beta_2, \dots, \beta_s$. Let the differential equation to be solved be $\mathfrak{D}\phi = 0$, where \mathfrak{D} is a differential operator, together with such boundary conditions as make the problem determinate. Let \mathfrak{D} be approximately represented by the finite difference operator \mathfrak{D}' , so that the body equations are

$$\mathfrak{D}'\psi_1 = 0, \quad \mathfrak{D}'\psi_2 = 0, \quad \dots, \quad \mathfrak{D}'\psi_n = 0 \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Then in order that the following justification of the approximation method may apply, it will be shown in the Appendix that \mathfrak{D}' and the boundary-equations must be linear; and the body- and boundary-equations must be the condition that a certain positive homogeneous quadratic function V of $\psi_1, \psi_2, \dots, \psi_n$ is a complete minimum. Also f , though otherwise arbitrary, is limited on the boundary to be the difference of two functions of position both of which satisfy the said boundary-conditions. Under these circumstances f can be expressed in the form $f = \sum_{k=1}^{k=n} A_k P_k$ where P_k is an integral of $(+\mathfrak{D}' - \lambda_k^2)\phi = 0$, which satisfies the same boundary-conditions as f , λ_k^2 being a positive constant, the sign before \mathfrak{D}' being the same as that of ψ_k in $\mathfrak{D}'\psi_k$. The proof of this fact and of various other properties of the (P)'s will be deferred to the Appendix. The (P)'s may be called the principal or normal modes of vibration of the system. V is analogous to potential energy. Some of the

* I am indebted to Prof. A. E. H. LOVE for pointing out this method and for giving me the numerical result for a square.

commoner possible forms of \mathfrak{D}' will be found in the table on p. 354. As well as the equation $\mathfrak{D}'\phi = 0$, the equation $\mathfrak{D}'\phi = (\text{a given function of the co-ordinates})$, and the equation $(\mathfrak{D}' - \lambda^2)\phi = 0$ may be treated by these successive-approximation-methods. They will be discussed in order.

§ 3·2·1. *The Equation $\mathfrak{D}'\phi = 0$.*—The approximation process proceeds as follows. Let ϕ_u be the correct finite-difference integral. Let ϕ_1 be a function (that is a table of numbers) satisfying the correct boundary-conditions, but arbitrary as to its body values. Next calculate the body-values of ϕ_2 by means of

$$\phi_2 = \phi_1 - \alpha_1^{-1} \mathfrak{D}'\phi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where α_1 is a number to be fixed ; and fill in such boundary-values of ϕ_2 as will satisfy the same boundary-conditions as ϕ_u . The succeeding steps are each of the form

$$\phi_{m+1} = \phi_m - \alpha_m^{-1} \mathfrak{D}'\phi_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for the body values, and by choosing the boundary values ϕ_{m+1} is made to satisfy the correct boundary condition. These are matters of simple arithmetic. It will be shown that by the judicious choice of $\alpha_1, \alpha_2, \dots, \alpha_t$ it is possible to make ϕ_{t+1} nearer to ϕ_u than ϕ_1 was. For since \mathfrak{D}' is linear and $\mathfrak{D}'\phi_u = 0$ we have from (2)

$$\phi_{m+1} - \phi_u = \phi_m - \phi_u - \alpha_m^{-1} \mathfrak{D}'(\phi_m - \phi_u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Now it is shown in the Appendix that $\phi_m - \phi_u$ may be expanded in a series of integrals of

$$(\mathfrak{D}' - \lambda_k^2) P_k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Put

$$\phi_1 - \phi_u = \Sigma A_k P_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Then by (4)

$$\mathfrak{D}'(\phi_1 - \phi_u) = + \Sigma A_k \lambda_k^2 P_k.$$

And therefore by (3)

$$\phi_2 - \phi_u = \Sigma A_k \left(1 - \frac{\lambda_k^2}{\alpha_1}\right) P_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Proceeding in the same manner after t operations we arrive at

$$\phi_{t+1} - \phi_u = \Sigma A_k \left(1 - \frac{\lambda_k^2}{\alpha_1}\right) \left(1 - \frac{\lambda_k^2}{\alpha_2}\right) \dots \left(1 - \frac{\lambda_k^2}{\alpha_t}\right) P_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

A measure of the deviation of two functions from one another which is used in the theory of Least Squares is the sum of the weighted squares of their differences. On the same principle let us measure the error of ϕ_{t+1} by

$$E_{t+1}^2 = S (\phi_{t+1} - \phi_u)^2 \times I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

where S stands for a summation over the body points and I is a certain one-signed

function of position. (See Appendix.) Squaring both sides of (7), multiplying by I and making the summation, we have by a property of the (P)'s (see Appendix, equations (31) and (32))

$$SI(\phi_{t+1} - \phi_u)^2 = \Sigma A_k^2 \left[\left(1 - \frac{\lambda_k^2}{\alpha_1}\right) \left(1 - \frac{\lambda_k^2}{\alpha_2}\right) \times \dots \times \left(1 - \frac{\lambda_k^2}{\alpha_t}\right) \right]^2 \quad \dots \quad (9).$$

Now it has been found that by a judicious choice of $\alpha_1, \alpha_2, \dots, \alpha_t$, the quantity $\left[\left(1 - \frac{\lambda_k^2}{\alpha_1}\right) \left(1 - \frac{\lambda_k^2}{\alpha_2}\right) \times \dots \times \left(1 - \frac{\lambda_k^2}{\alpha_t}\right) \right]^2$ may be made small for all possible values of λ_k^2 . (Thus fig. 1 shows this done for a set of seven (α)'s. This graph was arrived at by trial.) The error E_{t+1} of ϕ_{t+1} may therefore be made small in comparison with that of ϕ_1 . This is possible because the values of λ_k^2 lie in a finite range; corresponding to the fact that there are only a finite number of terms in the series $\Sigma A_k P_k$. A similar process will not work with the infinite series of sines, Bessel functions and other infinitesimal integrals of $(\mathfrak{D} - \lambda^2)P = 0$. In choosing $\alpha_1, \alpha_2, \dots, \alpha_t$, a diagram of the kind shown in figs. 1 and 2 is a great help. In this we take for the abscissa a variable λ^2 which takes in turn the values $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$, and as ordinate we consider

$$\omega = \left(1 - \frac{\lambda^2}{\alpha_1}\right) \left(1 - \frac{\lambda^2}{\alpha_2}\right) \times \dots \times \left(1 - \frac{\lambda^2}{\alpha_t}\right). \quad \dots \quad (10).$$

The value of ω at $\lambda^2 = \lambda_k^2$ is the ratio of the amplitude of the vibration P_k in the final approximation ϕ_{t+1} to its amplitude in the initial guess. The individual factors $(1 - \lambda^2/\alpha_r)$ in ω represent straight lines, all cutting the vertical axis at $\omega = 1$ and the horizontal axis at the points $\lambda^2 = \alpha_1, \alpha_2, \dots, \alpha_t$. By bringing any adjacent pair α_r and α_{r+1} closer together the values of ω corresponding to the range of λ^2 between α_r and α_{r+1} are diminished, provided that the other (α)'s remain fixed. It follows that by judiciously spacing the (α)'s along the horizontal axis and by taking a sufficient number of such points (that is of approximations) the successive maxima and minima of ω can be made all less, in absolute value, than any finite quantity ϵ however small. When this is done $S(\phi_{t+1} - \phi_u)^2 I$ being equal by (9) to $\Sigma A_k^2 \omega_k^2$ where ω_k is the value of ω at $\lambda^2 = \lambda_k^2$ must be less than $\epsilon^2 \Sigma A_r^2$. That is to say, the ratio of the error of the last approximation ϕ_{t+1} to that of the initial guess ϕ , being $\{SI(\phi_{t+1} - \phi_u)^2 / SI(\phi_1 - \phi_u)^2\}^{1/2}$, is less than ϵ , and ϵ can be made very small.

A knowledge of $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ is not necessary, but it is necessary to know the limits within which they range, or limits enclosing these. For the lower limit we require an estimate of λ_1^2 , which in the dynamical application is the square of the frequency of the gravest mode of vibration multiplied by a constant depending on density and elasticity. It is usually sufficiently close to take some boundary such as a rectangle or a sector of a circle, for which the frequency is known, and which fits in a rough way the irregular boundary under consideration.

Next, to find the upper limit, it is always possible to proceed as follows:—Make a guess at $A_n P_n$, that is to say, write down as sharply oscillatory a set of numbers as possible for the body values, such for example as Table V. when extended to fill the region, and add to it such boundary values as will satisfy the boundary equations. Call this guess χ_1 . Then $\chi_1 = A_n P_n + A_{n-1} P_{n-1} + \dots$, &c. Now operate on χ_1 with \mathfrak{D}' many times in succession, and after each operation readjust the boundary values so as to satisfy the boundary equations. By this process the coefficient of P_n is increased relatively to the coefficients of the other (P)'s in the expansion of the resulting table, because λ_n^2 is the greatest of the λ^2 . So that the table approaches a multiple of P_n . From P_n it is easy to find λ_n^2 . In any case a rough approximation to λ_n^2 suffices.

TABLE V.

+1	-1	+1
-1	+1	-1
+1	-1	+1

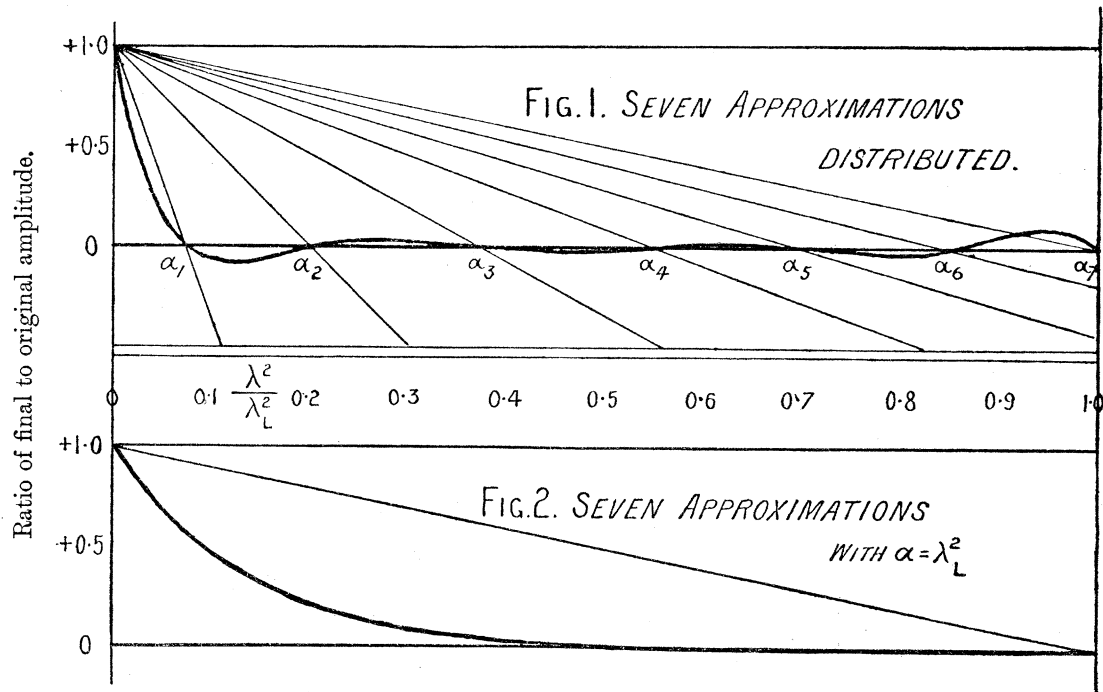
When the boundary values $\beta_1 \dots \beta_s$ vanish this labour is unnecessary, for then λ_n^2 cannot exceed (see Appendix) the greatest value of λ^2 pertaining to an integral of $(+\mathfrak{D}' - \lambda^2)\phi = 0$ with the given size of co-ordinate differences, and with ϕ vanishing at infinity; and this value of λ^2 depends only on the form of \mathfrak{D}' , and may be calculated once for all. It will be denoted by λ_L^2 . Thus when $\mathfrak{D}' = \mathfrak{D}^2/\mathfrak{D}x^2 + \mathfrak{D}^2/\mathfrak{D}y^2$ considerations of symmetry show that the most oscillatory integral is Table V. extended similarly in all directions, and from this we find $\lambda_L^2 = 4/(\mathfrak{D}x)^2 + 4/(\mathfrak{D}y)^2$.

Having thus found limits between which $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ must lie, it remains to choose the (α) 's so as to make ω small for all value of λ^2 in this range. In practice this has been done by drawing the graph of ω for arbitrary (α) 's and altering them or adding new ones until the maxima and minima of the curve were all sufficiently small. Figs. 1 and 2 are graphs of ω , representing two approximation processes requiring equal amounts of arithmetical labour. In fig. 1 the (α) 's are distributed over a wide range.* In fig. 2 all seven (α) 's are made equal to λ_L^2 . The curves show that these distributed (α) 's reduce the amplitudes corresponding to a wide range of λ^2 to less than one-tenth of their original value. On the other hand the (α) 's concentrated at λ_L^2 reduce the amplitudes in the neighbourhood of λ_L^2 much more perfectly, but leave the (P)'s of graver period less affected. The allowable type of curve for ω will depend on what is to be done with the integral of $\mathfrak{D}'\phi = 0$ when obtained. If its space-rates are required it is more important to abolish the modes of vibration having the largest values of λ^2 than it would be if volume integrals alone were needed.

When the ratio of λ_1^2/λ_L^2 is large, as in large tables, it is difficult to remove P_1 by the processes indicated by figs. 1 and 2. For example if $\mathfrak{D}' = \mathfrak{D}^2/\mathfrak{D}x^2 + \mathfrak{D}^2/\mathfrak{D}y^2$, and if the boundary is a square of ten co-ordinate differences side, on which ϕ vanishes, then the (P)'s having the lowest values of λ^2 will not differ greatly

* The advantage of distributing the (α) 's fairly uniformly was pointed out to me by Prof. A. E. H. LOVE.

from $\cos(\frac{1}{10}m\pi x) \cdot \cos(\frac{1}{10}n\pi y)$, and for these $\lambda^2 = (m^2 + n^2) \frac{1}{100}\pi^2$. So that the ratios of λ_1^2, λ_2^2 , &c., to λ_L^2 when set in order of size will run 0.0247, 0.0617, 0.0987, 0.1234, &c., up to nearly unity. Referring to fig. 1 it is seen that the seven approximations would reduce the amplitudes of P_1, P_2 , &c., in the ratios +0.48, +0.07, -0.065, -0.08, &c., the rest never exceeding $\frac{1}{10}$ and averaging about $\pm\frac{1}{40}$. Of course continued approximation would gradually reduce the amplitude of P_1 , but



Curves illustrating the process of approximation.

in cases like these it may be well to make a guess at the form of P_1 , to estimate λ_1^2 as in Appendix, equation (33), to find an approximation to the amplitude A_1 in $\phi_1 - \phi_u = \sum A_k P_k$ by the Fourier method (see Appendix, equations (29) and (22)), and so to remove the greater part of the first term of the series before the approximations are begun. This has been done in the problem of the dam, § 4.

We have so far supposed $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ unknown. If any one λ_k^2 of these be known, then making $\alpha = \lambda_k^2$ will entirely remove P_k from the series. This process may sometimes be useful for removing the gravest modes of vibration P_1, P_2 , &c.

Since the value of ω is independent of the order in which its factors are multiplied together, it follows that the result of ϕ_{t+1} of a series of operations of the type $\phi_{m+1} = \phi_m + \alpha_m^{-1} \mathfrak{D}'\phi_m$ depends on the initial guess ϕ_1 and on the values of the (α) 's, but not on the order in which the (α) 's are taken. The application of this result to practice is slightly limited because the number of significant figures retained is necessarily limited.

In carrying out an approximation-process with a set of (α) 's designed to make ω small, it is frequently only at the last stage that the predicted improvement

appears in the table. The intermediate stages may be wildly irregular if the values of α decided upon are used in ascending order of magnitude. If, on the contrary, the descending order is pursued the table tends to improve more regularly.

§ 3·2·2. *The Equation $\mathfrak{D}'\phi = \rho$, where ρ is a completely known Function of Position*—This can be treated in the same way as $\mathfrak{D}'\phi = 0$, except that the approximations must now be of the form

$$\phi_{m+1} = \phi_m - \alpha_m^{-1} \mathfrak{D}'\phi_m - \rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

for, since $\mathfrak{D}'\phi_u = \rho$, this may be written

$$\phi_{m+1} - \phi_u = \phi_m - \phi_u - \alpha^{-1} \mathfrak{D}'(\phi_m - \phi_u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which is the same as (3) of § 3·2·1.

§ 3·2·3. The equation

$$(\mathfrak{D}' - \lambda^2) P = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

together with s homogeneous boundary equations such as

$$\beta_j = f_{j1}\psi_1 + f_{j2}\psi_2 + \dots + f_{jn}\psi_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where $\beta_1 \dots \beta_s$ are the boundary values, $\psi_1 \dots \psi_n$ the body values and the (f) 's are given numbers. P is now written in place of ϕ , because by (4) of § 3·2·1 P is defined as satisfying equation (1) of this section. We will suppose that both P and λ^2 have to be determined. As equations (2) contain no terms independent of the (ψ) 's, it follows that when these expressions for the (β) 's are substituted in the body-equations the latter become homogeneous, and are only consistent for the particular values of λ^2 which we have already denoted by $\lambda_1^2, \lambda_2^2, \dots \lambda_n^2$. Further, on account of this homogeneity, any multiple of an integral satisfies the correct boundary conditions. Such a possibility does not arise with $\mathfrak{D}'\phi = 0$, for as \mathfrak{D}' contains no adjustable constant such as λ^2 it has no integral save $\phi = 0$, unless the boundary equations contain a term f_{j0} independent of the (ψ) 's. And if they do contain such a term, any multiple of an integral fails to satisfy them. Thus, in the theory of membranes, if we put $\mathfrak{D} = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and ϕ for the small displacement of the membrane from a fixed plane, then in the case of the membrane at rest $\mathfrak{D}'\phi = 0$, and ϕ is commonly given at the boundary, its values there being $f_{10}, f_{20}, \dots f_{s0}$.

On the other hand, for the vibrations of the membrane $(\mathfrak{D}' + \lambda^2)\phi = 0$ the boundary condition is commonly $\phi = 0$, corresponding to the vanishing of all the (f) 's.

To return to the general form of \mathfrak{D}' : let P_k be the integral desired; $(P_k)_1, (P_k)_2, \dots (P_k)_{t+1}$ the initial guess and successive approximations to P_k . The fact that any multiple of an integral is itself an integral allows us to put for the body-points of $(P_k)_{m+1}$

$$(P_k)_{m+1} = \gamma (P_k)_m - \alpha^{-1} \mathfrak{D}'(P_k)_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

Again, from the size and distribution of $\mathfrak{D}'\phi$ combined with a knowledge of various integrals of $\mathfrak{D}'\phi = f(x, y)$, a rough estimate of the errors in ϕ can frequently be made. Again for certain equations* the method of contour integration applied to a circle affords a check on the value of ϕ at its centre. This method is very rapid, and it is particularly advantageous when applied to a circle enclosing many body values, for then a repetition of the approximation process would be correspondingly tedious.

§ 3·2·5. *Routine of Approximation.—Time and Cost.*—To anyone setting out on a problem I offer the following experience as a guide in forming estimates:—It was found convenient to enter certain stages on a table with large squares, each divided into compartments. Thus for $\frac{\mathfrak{D}^4\phi}{\partial x^4} + 2\frac{\mathfrak{D}^4\phi}{\mathfrak{D}x^2\mathfrak{D}y^2} + \frac{\mathfrak{D}^4\phi}{\mathfrak{D}y^4} = 0$, one of the squares is shown in the annexed table. All the quantities in it refer to the central point of the square. The intermediate stages are done on rough paper and thrown away. So far I have paid piece rates for the operation $\mathfrak{D}_x^2 + \mathfrak{D}_y^2$ of about $\frac{n}{18}$

TABLE.

ϕ_1	$\nabla^2\phi_1$	$\nabla^4\phi_1$
ϕ_2	$\nabla^2\phi_2$	$\nabla^4\phi_2$
ϕ_3	&c.	

pence per co-ordinate point, n being the number of digits. The chief trouble to the computers has been the intermixture of plus and minus signs. As to the rate of working, one of the quickest boys averaged 2,000 operations $\mathfrak{D}_x^2 + \mathfrak{D}_y^2$ per week, for numbers of three digits, those done wrong being discounted.

§ 3·3. *Relative Merits of Simultaneous Equations and of Successive Approximation.*—The method of simultaneous equations may be applied to differential equations of any order and degree. It gives results which are exact for finite differences. It is necessary in discussions as to the existence and properties of the integrals of difference equations. But for actually calculating the integrals the labour becomes very great as the number of unknowns increases, and is of a sort which a clerk will not easily do. Large numbers of digits have to be dealt with, and a single mistake generally throws the result altogether out.

The successive approximation methods of § 3·2 have only been applied to a limited class of linear equations. The results are not exact even for finite differences. But the bulk of the work can be done by clerks who need not understand algebra or calculus. Small and infrequent mistakes, or taking only a small number of digits, do not prevent one arriving at a fairly correct result. Nevertheless, it has been found best to have everything worked in duplicate.

The method of successive approximation to the surface $z = \phi_u = f(x, y)$ reminds one of the manufacture of plane metallic surfaces. The initial form of the surface is arbitrary in both cases. The essential things in both cases are a method of testing the work at any stage, a tool with which to alter the surface and judgment in using it. Methods of testing the arithmetic have been described in § 3·2·4 above. Our

* These include $\nabla^2\phi = 0$, $\nabla_1^4\phi = 0$. See a paper by BOGGIO, 'Jahrb. Fortschritte Math.,' 1900, p. 740.

tools $\phi_{m+1} = \phi_m - \alpha_m^{-1} \mathfrak{D}'\phi_m$ and the Fourier method of removing principal modes of vibration may be compared to automatic grinding machines. The use of the hand-scraping tool corresponds to the adjustment of the numbers according to the judgment of the operator. This is always permissible.

§ 3.4. The treatment of *infinity* may be illustrated by the following hypothetical example:—Suppose we were to set out to determine the field due to a rotating mass of gravitating fluid of known shape, the gravitational potential would have to vanish at infinity. But to be determinable by the methods of § 3 the potential would have to be given over some boundary which could be included in the sheet of paper upon which its values were written. We might, for example, assume the potential at the edge of the paper equal to that due to the given mass of the liquid concentrated at its given centre of gravity, and find the figure of equilibrium on this hypothesis. (See treatment of base of dam in § 4.)

§ 4. The problem of *the determination of the stresses in a masonry dam* has been chosen as a final example for two reasons: (1) Its practical interest arising out of the great expense and productiveness of these structures and the destruction of life and property should they burst; (2) Its simplicity in that we have to determine a single quantity χ as a function of two co-ordinates only.

The methods developed are suitable for finding the stresses, not in dams only, but in a prism or cylinder of any section, acted upon by any distribution of surface stress which is normal to the axis of the prism, when the surface stress is given, provided that the shifts parallel to the axis are zero. § 4.1 is devoted to theorems relating to any shape of contour. In § 4.2 the special contour of the dam is introduced.

The discussion that follows is founded upon two papers in the Drapers' Company Research Memoirs (Dulau and Co.):

- (1) "On some Disregarded Points in the Stability of Masonry Dams," by L. W. ATCHERLEY and KARL PEARSON, F.R.S., 1904.
- (2) "An Experimental Study of the Stresses in Masonry Dams," by KARL PEARSON, F.R.S., and A. F. CAMPBELL POLLARD, assisted by C. W. WHEEN and L. F. RICHARDSON, 1907.

I also owe some ideas, *e.g.*, the use of equipollent loads in the base, to suggestions thrown out by Prof. PEARSON in the course of conversation. In these papers the dam is regarded as a prismatic-shaped body of indefinite length, so that the problem may be discussed in terms of two co-ordinates x and z lying in the vertical cross-section of the prism. The sluices are ignored—a serious omission.

§ 4.1.1. Prof. PEARSON and his collaborators lay much emphasis upon our ignorance as to the real *conditions at the base of the dam* and the consequent inapplicability of analysis based upon special assumptions at the base, such as linear, parabolic, or quartic distribution of shear. Admitting this ignorance, there seems to me one assumption more reasonable than any of the others, namely, that the dam may be

regarded as a ridge upon the surface of a very large slab of rock which, with the dam, forms one homogeneous isotropic elastic system; together with the further assumption that this substratum suffered from no internal stress before the dam was built, except, of course, that due to its own weight. The consideration of the results that follow from these assumptions will be continued in § 4.1.9 below.

§ 4.1.2. *Conventions as to Symbols.*—The Z axis vertically downwards.

The X axis horizontally directed from the water towards the tail.

θ , the angle which a line makes with the Z axis, to be reckoned positive when the rotation is from the Z axis to the X axis through 270° and so onwards.

The stresses \widehat{xx} and \widehat{zz} positive when they pull adjacent portions of the material together, and therefore the shear \widehat{xz} positive when the lower portion is pulling the upper towards the tail.

q and s distances along the outwardly directed normal to the masonry and rock and along the boundary drawn to the right of it.

N and T the normal and tangential stresses on the surface related to q and s in the same way that \widehat{zz} and \widehat{xz} are related to z and x .

l and n the cosines of the angles which the outwardly drawn normal makes with the X and Z axes respectively.

§ 4.1.3. *Position of the Origin.*—PEARSON and POLLARD (p. 37) take this at the join of the front and the top. It will be more convenient in what follows to take it vertically below this point at the level of the surface of our hypothetical slab of bed rock. This is done throughout. The water surface stands at $z = -p$, and the pressure due to it is accordingly $g\rho'(+p+z)$, where ρ' is the density of the water.

§ 4.1.4. *Specific Constants of the Masonry and Rock.*—Following PEARSON and POLLARD I take the density $\rho = 2.25$ times that of water (p. 29) and POISSON'S ratio η as $\frac{1}{4}$ (p. 33).

§ 4.1.5. *Units.*—Distances are reckoned in metres. Forces in metric tons, each equal to the weight of a cubic metre of water. Consequently $g\rho'$ the weight of unit volume of water is equal to unity. And $g\rho = 2.25$.

§ 4.1.6. *Equations to be Solved.*—The stresses may be expressed in terms of a single scalar χ which satisfies

$$\nabla_1^4 \chi \equiv \left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial z^4} \right) \chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

thus

$$\widehat{zz} = \frac{\partial^2 \chi}{\partial z^2} - g\rho z, \quad \widehat{xx} = \frac{\partial^2 \chi}{\partial x^2}, \quad \widehat{xz} = -\frac{\partial^2 \chi}{\partial x \partial z}. \quad . \quad . \quad (2), (3), (4).$$

The surface equations then become

$$\begin{aligned} -p-z &= \frac{\partial^2 \chi}{\partial z^2} - \frac{\partial^2 \chi}{\partial x \partial z} \epsilon \\ -\epsilon(p+z) &= -\frac{\partial^2 \chi}{\partial x \partial z} + \left(\frac{\partial^2 \chi}{\partial x^2} - g\rho z \right) \epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad (5) \end{aligned}$$

under the water. On the air surface the left-hand sides of these are zero. Here ϵ is the tangent of the angle which the surface, drawn to the right, makes with the z axis.

We shall require the surface stresses in terms of the space-rates of χ along the arc and normal. The theory of stress functions is fully discussed by Mr. J. H. MICHELL in 'Proc. London Math. Soc.,' Vol. XXI.; on p. 110 we find the required transformation, which expressed in our notation reads as follows:—

$$N_1 = \frac{\partial^2 \chi}{\partial s^2} - \frac{1}{R} \frac{\partial \chi}{\partial q}, \quad T_1 = -\frac{\partial^2 \chi}{\partial s \partial q} - \frac{1}{R} \frac{\partial \chi}{\partial s}.$$

These equations refer to a weightless solid, hence the suffix. To complete them we must add the stress $\widehat{zz} = -g\rho z$, $\widehat{xx} = 0 = \widehat{xz}$ after transforming it to components about the normal and arc. This gives

$$N = \frac{\partial^2 \chi}{\partial s^2} - \frac{1}{R} \frac{\partial \chi}{\partial q} - n^2 g\rho z, \quad T = -\frac{\partial^2 \chi}{\partial s \partial q} - \frac{1}{R} \frac{\partial \chi}{\partial s} + l g\rho z. \quad (6), (7).$$

§ 4.1.7. *The Size of the Dam.*—Suppose we have determined χ for a dam of a particular shape filled to a certain fraction of its height with water. Let us say $\chi_1 = f(x, z)$.

We wish to find the stresses in a dam b times as big every way and containing b times the height of water. Try $\chi_2 = f(x/b, z/b)$. Then $\partial^2 \chi_2 / \partial x^2$, $\partial^2 \chi_2 / \partial z^2$, and $\partial^2 \chi_2 / \partial x \partial z$ would all be $1/b^2$ times their former values at corresponding points of the surface, but the constant terms in the surface conditions, due to the water pressure and weight of masonry, are now b times their former values at corresponding points. Consequently $b^3 f(x/b, z/b)$ is the form of χ appropriate to a dam b times the size of the one for which $f(x, y)$ was determined, and the stresses in the former will be everywhere b times as great. For convenience in calculating, we will suppose that the co-ordinate difference, $\delta x = \delta z$, is equal to the unit of length, unless otherwise stated. The result can afterwards be applied to a dam of any size.

§ 4.1.8. *Simple Transformation Concerning the Density ρ of the Masonry.*—If the reservoir be empty, and χ_e is the integral for density ρ_1 , then $\frac{\rho}{\rho_1} \chi_e$ is the integral for density ρ , for it still satisfies the body equation $\nabla^4 \left(\frac{\rho}{\rho_1} \chi_e \right) = 0$ and also the surface equations (6) and (7), since both N and T vanish, so that the calculation of one integral suffices for all sizes and densities, if it be multiplied by the proper constants.

Now when the reservoir is full we may take account of any density by means of two independent integrals. Let us calculate χ_e for reservoir empty and density ρ_1 and χ_f for reservoir full and density ρ_2 , then the proper stress-function for reservoir

full and density ρ will be $\chi_f - \frac{\rho_2}{\rho_1} \chi_e + \rho \chi_e$, for, on account of linearity, the solutions of

$$\frac{\partial^2 \chi}{\partial s^2} - \frac{1}{R} \frac{\partial \chi}{\partial q} - n^2 g \rho z = N_1, \quad = N_2, \quad = N_3 \quad \text{in turn}$$

when added together satisfy $\frac{\partial^2 \chi}{\partial s^2} - \frac{1}{R} \frac{\partial \chi}{\partial q} - n^2 g \rho z = N_1 + N_2 + N_3$. Here N_1 is the normal stress due to the water pressure of a full reservoir, and N_2 and N_3 are the same when the reservoir is empty, that is zero. The above is the method of PEARSON and POLLARD (p. 28) translated into stress-function symbolism. It shows us how to find the stresses for a full reservoir sustained by masonry of any density by calculating in detail two cases only.

In this paper the case of full reservoir and density 2.25 is the only one treated.

§ 4.1.9. *Integration of Surface Equations.*—We see from equations (6) and (7) that if the shape of the boundary and the stresses upon it are given, then starting at a point s_0 , and assuming initial values of χ , $\partial \chi / \partial q$, and $\partial \chi / \partial s$, we can find a double row of values of χ all round the boundary by straightforward integration. The initial values of χ , $\partial \chi / \partial q$, and $\partial \chi / \partial s$ are not significant, for they depend only on the arbitrary function $Ax + Bz + C$, which may be added to any distribution of χ without affecting the stresses. The only outstanding uncertainty is at a sharp corner where $1/R$ becomes infinite. Prof. PEARSON has shown that at a sharp re-entrant corner the stress may become infinite ('History of Elasticity,' vol. ii., § 1711). In view of this it appeared possible that the boundary strip would be indeterminate at the corners, and the following investigation was made to settle the question. As it is a question of infinity, the finite forces $N ds$, $T ds$, and $-g \rho z ds dq$ in the immediate neighbourhood of the corner may be neglected, leaving simply

$$\frac{\partial^2 \chi}{\partial s^2} = \frac{1}{R} \frac{\partial \chi}{\partial q}, \quad \frac{\partial^2 \chi}{\partial s \partial q} = -\frac{1}{R} \frac{\partial \chi}{\partial s} \quad \dots \quad (8), (9).$$

Now let us suppose that the trace of the boundary on the y plane is a circle at the corner, with the intention of making the radius indefinitely small after integration, on integrating equations (8) and (9) with R constant it is found, after some work, that

$$\chi = C e^{-\frac{q}{R}} \sin\left(\frac{s}{R} + D\right) + H \quad \dots \quad (10)$$

is the complete integral, C , D , and H being arbitrary constants.

Now suppose that s/R , the angle turned through, increased from 0 to α . Then it can be shown from (10) that

$$\left(\frac{\partial \chi}{\partial s}\right)_\alpha = \left(\frac{\partial \chi}{\partial s}\right)_0 \cos \alpha + \left(\frac{\partial \chi}{\partial q}\right)_0 \sin \alpha, \quad \left(\frac{\partial \chi}{\partial q}\right)_\alpha = \left(\frac{\partial \chi}{\partial q}\right)_0 \cos \alpha - \left(\frac{\partial \chi}{\partial s}\right)_0 \sin \alpha \quad \dots \quad (11), (12).$$

It is remarkable that the changes in $\partial\chi/\partial s$ and $\partial\chi/\partial q$ depend on the angle turned through, but not upon the sharpness of the corner, as is evidenced by the absence of R from (11) and (12). These equations are the conditions that the tangent planes of the surface $\chi = f(x, y)$ just before and just after the corner shall be parallel to one another. Thus the boundary strip of χ can be integrated independently of the body equation $\nabla^4\chi = 0$.

§ 4.1.10. The following theorem about the total changes in $\partial\chi/\partial x$, $\partial\chi/\partial z$, and χ over

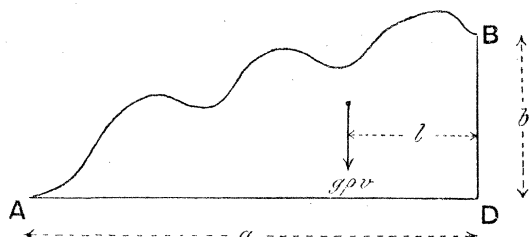


Fig. 4.

any length of boundary affords a useful check on the integration of equations (6) and (7). Let AB be any part of the boundary. Cut off in imagination a portion of the solid by horizontal and vertical lines through A and B meeting in D . It will be assumed at first that AB does not intersect AD , DB , except at A and B . Now apply

to AD a normal stress \widehat{zz}_0 and to BD a normal stress \widehat{xx}_0 , arranged in amount and distribution so as to balance the forces acting on the boundary AB , together with the weight of the portion ABD . Let the weight be $g\rho\nu$ acting in a line distant l from D , and let the stresses on the real surface AB be equivalent to a force X , Z acting through D , together with a couple G . The forces X , Z are to be reckoned positive when they are directed out of the solid. As the body ABD is in equilibrium χ , $\partial\chi/\partial x$, $\partial\chi/\partial z$ are single valued, and if there are no infinite stresses they will be continuous. The changes along AB can, therefore, be obtained by integrating along

AD and DB . To balance the horizontal and vertical forces $\int_D^A -\widehat{zz}_0 dx + Z + \nu g\rho\nu = 0$ and $\int_B^D -\widehat{xx}_0 dz + X = 0$, where ν is $+1$ if AB is above AD and -1 in the reverse

condition. Now, substituting $\widehat{zz} = \frac{\partial^2\chi}{\partial x^2} - g\rho z$ and $\widehat{xx} = \frac{\partial^2\chi}{\partial z^2}$, there results

$$\left| \frac{\partial\chi}{\partial x} \right|_D^A = g\rho z_0 \left| \frac{\partial\chi}{\partial x} \right|_D^A + Z + \nu g\rho\nu \quad \text{and} \quad \left| \frac{\partial\chi}{\partial z} \right|_B^D = X. \quad \text{Also, since the forces on } AD, DB \text{ are}$$

purely normal, $\partial\chi/\partial z$ is constant along DA and $\partial\chi/\partial x$ is constant along BD . Therefore, remembering that by (11) and (12) $\partial\chi/\partial x$, $\partial\chi/\partial z$ are continuous at the corners, we see that

$$\left| \frac{\partial\chi}{\partial x} \right|_A^B = -Z + g\rho(\nu z - z_0\alpha) \quad \text{and} \quad \left| \frac{\partial\chi}{\partial z} \right|_A^B = -X. \quad (13'), (14'),$$

where α is the length DA . These are the required total changes in $\partial\chi/\partial x$ and $\partial\chi/\partial z$. Lastly, to balance the moments about D , we must have

$$\int_D^A \widehat{zz}_0 (x - x_0) dx + \int_B^D \widehat{xx}_0 (z - z_0) dz + G + \mu l g\rho\nu = 0,$$

where x_0, z_0 are the co-ordinates of D, and where μ is +1 if AB is to the left of BD, and -1 if they are reversed. Integrate by parts, then

$$\left[(x-x_0) \int \widehat{zz_0} dx - \int \widehat{zz_0} (dx)^2 \right] + \left[(z-z_0) \int \widehat{xx_0} dz - \int \widehat{xx_0} (dz)^2 \right] + G + \mu l g \rho v = 0.$$

Substitute the values of the stresses and

$$\left|_A^B \chi = -G - a \left(\frac{\partial \chi}{\partial x} \right)_A - b \left(\frac{\partial \chi}{\partial z} \right)_B + g \rho \left(z_0 \frac{\alpha^2}{2} - \mu l v \right),$$

where b is the length BD. Or, since by (13') and (14') $\left(\frac{\partial \chi}{\partial z} \right)_B = \left(\frac{\partial \chi}{\partial z} \right)_A - X$,

$$\left|_A^B \chi = -G - a \left(\frac{\partial \chi}{\partial x} \right)_A - b \left(\frac{\partial \chi}{\partial z} \right)_A + bX + g \rho \left(z_0 \frac{\alpha^2}{2} - \mu l v \right). \quad \dots \quad (15'),$$

a useful formula.

When the boundary of the solid intersects AD, DB, as does the curly line in the fig. 5, then draw horizontal and vertical lines to form a figure enclosing the boundary. Apply the stresses to FH instead of AD and to LK instead of BD, leaving the rest of AFHJKLB unstressed. These stresses must balance the surface forces as before, and now the whole weight of the solid enclosed between AFJLB and the boundary. On account of the change in level from AD to KJ or BL the term $-g\rho z$ in \widehat{zz} is different, and this term being integrated along FJ and LB has the effect of subtracting the weight of AFJLBD from that of the solid part. Consequently we may integrate along AD, DB and keep everything else as before,

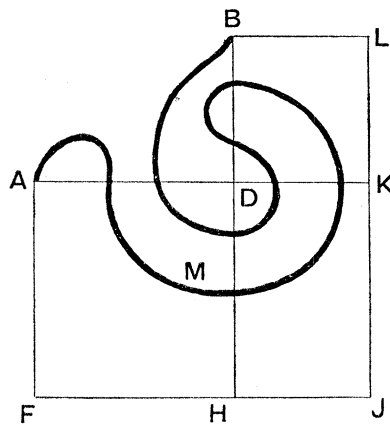


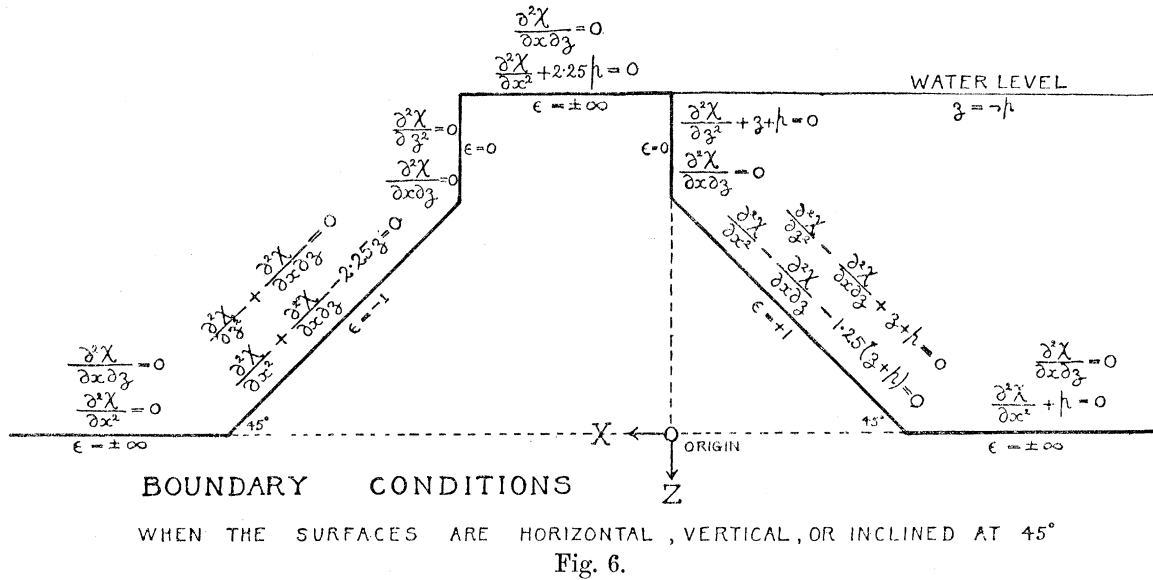
Fig. 5.

provided we replace $+vg\rho v$ by $\pm g\rho \int_A^B (z-z_0) dx$, where z follows the boundary curve.

Similarly in (15') we must replace $\mu l g \rho v$ by $\pm g \rho \int_A^B (z-z_0) (x-x_0) dx$.

§ 4.1.11. In order to simplify the arithmetic, the surface of the dam of the form chosen has been represented by straight lines, either horizontal, vertical, or sloping at 45 degrees. It is one of the peculiar advantages of successive approximation methods that a simple case like this comes in conveniently as the first stage of the solution for any rather different boundary.

Table of surface equations when the surfaces are horizontal, vertical, or at 45 degrees (fig. 6), deduced from equations above:—



The rather complicated conditions on the sloping surfaces require special reduction to finite differences. Suppose the flank runs through a set of values of χ as in Table VI., then at the point .5, 0

$$\partial^2 \chi / \partial x \partial z \text{ may be taken as } \frac{1}{2} \{ (0\bar{1}) + (11) - (1\bar{1}) - (01) \},$$

$$\partial^2 \chi / \partial z^2 \text{ as } \frac{1}{2} \{ (1\bar{1}) + (01) + (11) + (0\bar{1}) - 2(10) - 2(00) \}.$$

Therefore the surface condition

$$\frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^2 \chi}{\partial x \partial z} = 0 \quad \text{becomes} \quad 0 = (0\bar{1}) + (11) - (00) - (10) \quad . \quad . \quad (13).$$

TABLE VII.

TABLE VI.

Flank →	11̄	01̄	11̄
	10	00	10̄
	11	01	11̄

				Surface	
				- 2	- 1
				0	1
0	1	2	3		
Surface				- X	
2	3	4	b		
				Z	

In a similar way, at the point 0, .5 the second surface condition

$$0 = \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial x \partial z} - 2\frac{1}{4}z \quad \text{becomes} \quad 0 = (\bar{1}\bar{1}) + (10) - (0\bar{1}) - (00) - 2\frac{1}{4}z \quad . \quad (14).$$

The conditions on the front may be transformed in a similar way if necessary.

We proceed to show that the natural way of turning a corner in finite differences is consistent with (11) and (12). Take, for example, the corner in Table VII. representing the tail rising vertically from the base. On the horizontal surface $z = 0$, the equations $\frac{\partial^2 \chi}{\partial x^2} = 0$, $\frac{\partial^2 \chi}{\partial x \partial z} = 0$ are taken along the line $z = 0$ midway between the two boundary layers. When we come to the corner we may either continue this process and so determine $b = 5$, or else we may take $\frac{\delta^2 \chi}{\delta x^2} + \mu_x \mu_z \frac{\delta^2 \chi}{\delta x \delta z} = 0$ at $x = +\frac{1}{2}$; $z = 0$.

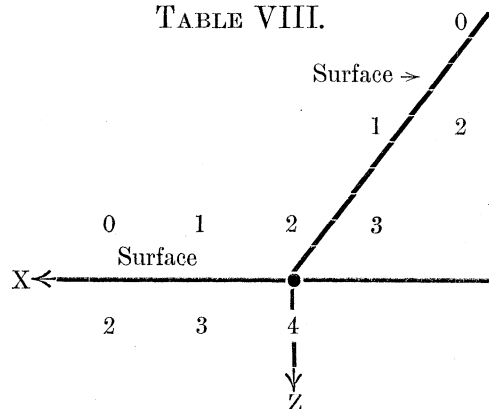
Either process leads to $\chi = 3$ at $x = -\frac{1}{2}$, $z = -\frac{1}{2}$, but one fixes b while the other leaves it arbitrary. Again we get $\chi = 1$ at $x = -\frac{1}{2}$, $z = -1\frac{1}{2}$, either by taking $\partial^2 \chi / \partial z^2 + \partial^2 \chi / \partial x \partial z = 0$ at $x = 0$, $z = -\frac{1}{2}$, or else by taking $b = 5$ and $\partial^2 \chi / \partial z^2 = 0$, and $\partial^2 \chi / \partial x \partial z = 0$, as holding at the middle of the two columns on $x = 0$. We may expect that when b is determined by the boundary equations a sharp corner is represented. When b is not so determined, a corner bevelled by the line $x - z = \frac{1}{2}$. In the absence of the foregoing analysis these ways of turning the corner would have seemed tempting but risky. But we shall see that they really do correspond to the analytical way. For, on the horizontal bed, we have $(\partial \chi / \partial s)_0 = 1$, $(\partial \chi / \partial q)_0 = -2$, and the angle turned through is $+\frac{1}{2}\pi$. Therefore, from (11) and (12) $\partial \chi / \partial s$ and $\partial \chi / \partial q$ on the tail above the corner should have the values -2 and -1 respectively. And these are exactly what we find in the finite difference table for the first differences.

Again, suppose the tail rises at 45 degrees to the base. Starting along the horizontal the value $\chi = 3$ at $x = -1$, $z = -\frac{1}{2}$ may be obtained by assuming (14) to hold at the origin, that is to say, at the corner.

The other numbers on the sloping surface were obtained by using (13) and (14) alternately, $g\rho z$ being left out. Now we have $(\frac{\partial \chi}{\partial s})_0 = 1$, $(\frac{\partial \chi}{\partial q})_0 = -2$, $\alpha = \frac{1}{4}\pi$; so that equations (11) and (12) give us -0.707 and -2.121 , respectively, for $\frac{\partial \chi}{\partial s}$ and $\frac{\partial \chi}{\partial q}$ on the slope just above the corner. And these are identical with the first differences of the table when δs and δq are given their proper values of $\sqrt{2}$ and $1/\sqrt{2}$, respectively.

§ 4.1.12. *A Possible Experimental Solution.*—Now that we have shown how to integrate the boundary conditions, the analogy with thin plates will help us. For it is known (LOVE'S 'Elasticity,' 1906, § 313) that for a thin weightless plate, originally

TABLE VIII.



plane and unacted upon by surface forces except at the edges, the displacement w normal to the plate satisfies the equation $0 = \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \equiv \nabla_1^4 w$, the edge conditions being such as to make the displacement purely normal to the undeformed plane and x and y being co-ordinates in this plane. So that if a sheet of thin steel "ferrotype" is taken and fixed at the edges in such a way as to make a double layer of values of w there equal to the integral of the boundary conditions of the dam, then the displacement w elsewhere would also be equal to the integral χ of the complete system of equations for the dam. The principal stresses in the dam are then equal to the principal curvatures of the plate. For a method of measuring them, see two letters to 'Engineering,' October 25 and November 1, 1907.

Whenever one has to solve the equation $\nabla_1^4 \chi = 0$, the form assumed by a piece of postcard bent in the fingers will be worth considering. As an accurate experimental method this would have the advantage over those of PEARSON and of WILSON and GORE, that displacements are applied instead of forces, and that the resulting displacement to be measured may be larger than theirs were.

§ 4.1.13. *To return to the Conditions at the Base of the Dam.*—Prof. PEARSON pointed out to me that the effect of the stresses in the base of the dam, at a distance where the base of the dam subtends but a small angle, will be the same as that of any other statically equivalent system over the base, in particular to the force-at-a-point found by compounding the pressure of the water acting through the centre of pressure with the weight of the dam acting through its centre of gravity. We may therefore, in imagination, remove the dam, leaving a horizontal plane with this force acting at a certain point, the pressure of the water in front and no pressure behind. Suppose we had the distribution of χ corresponding to this system of surface forces. Then the corresponding double set of values of χ at a considerable distance from the dam will differ exceedingly little from the true values, and if we keep them fixed and write in the upper surface values on the dam we may adjust the numbers inside by successive approximation to satisfy $\nabla^4 \chi = 0$, and the result will then be exceedingly close to the true integral in the neighbourhood of the dam. This is what has been done.

From the linearity of ∇^4 and of the stress equations, it follows that if the stresses corresponding to a number of solutions of $\nabla^4 \chi = 0$, when added, give the true stress, then the stresses derived from the sum of all these solutions will also be correct. With the aim of providing distributions of χ in the bedrock which shall enable engineers to solve "PEARSON'S Dam Problem" for any shape of dam boundary, I have considered the actual distribution of χ in the bedrock as made up of the three following parts, which must be added in the proper proportions as described below:—

(i.) The term $\psi_1 = -\frac{1}{18}g\rho z^3 = -\frac{1}{8}z^3$. This gives $3\widehat{xx} = -g\rho z$, and so combines with the stress due to the weight of the bedrock to make $s_x = 0$.

(ii.) In a weightless bedrock. The stress function ψ_2 , due to a point force in any direction at the origin.

(iii.) The stress function ψ_3 in a weightless medium, when on the surface $z = 0$, $\widehat{xz} = 0 = \widehat{zz}$ from $x = 0$ to $x = +\infty$, and $\widehat{zz} = -p$ from $x = 0$ to $x = -\infty$.

The stretch s_x is made to vanish at $x = +\infty$ by making $3\widehat{xx} = zz = -p$ in that region. At $x = \pm\infty$ the stresses due to (ii.) will vanish, and therefore so will also the corresponding s_x . Consequently, if we add together any multiples of the (χ) 's specified in (i.), (ii.), and (iii.), we have $s_x = 0$ at $x = +\infty$ when the rock has its proper density ρ . (iii.) is to be so placed that its origin is at the meet of the front and surface. (ii.) is to have the point of application of the forces placed where the resultant of the water pressure and weight of the dam cuts the plane $z = 0$.

It will be necessary to consider distributions ψ_2 and ψ_3 in some detail. We are indebted to Mr. J. H. MICHELL for both of them. (See 'Proc. Lond. Math. Soc.,' 1901.)

First, ψ_2 . For a point force of unit magnitude acting on the straight boundary at the origin of the polar co-ordinates r, θ' . The stress function χ is given by $\psi_2 = -\pi^{-1}r\theta' \sin \theta'$, and the force is in the positive direction of the line from which θ' is measured. The stresses are as follows :—

$$\widehat{rr} = \frac{1}{r^2} \frac{\partial^2 \psi_2}{\partial \theta'^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} = -\frac{2 \cos \theta'}{r\pi}, \quad \widehat{\theta\theta} = \frac{\partial^2 \psi_2}{\partial r^2} = 0, \quad \widehat{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi_2}{\partial \theta} \right) = 0 \quad (19), (20), (21).$$

Next, ψ_3 . *The Stress Function due to the Lake*.—MICHELL shows that if unit normal pressure be applied along a finite length of a straight boundary of an otherwise unlimited plate in which the sideways stretch vanishes, then the form of the stress function is

$$\psi_3' = (r^2\phi - r'^2\phi')/2\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

where r, ϕ and r', ϕ' are polar co-ordinates centred A and B, and AB is the initial line. He further proves that the axes of principal stress at any point P are the bisectors of the angle APB, and that if this angle is equal to α , the magnitudes of the principal stresses are

$$-(\alpha + \sin \alpha)/\pi \text{ along the internal bisector.} \quad (23),$$

$$-(\alpha - \sin \alpha)/\pi \text{ along the external bisector.} \quad . \quad . \quad . \quad . \quad . \quad (24).$$

We want the limit of ψ_3' in the neighbourhood of one end B when the other end A is removed to an infinite distance. Let the origin of our co-ordinates x, z coincide with B. Then let A be at a very great distance $x = -t, r'\phi' = +z, r' = t+x, \psi_3' = \frac{1}{2\pi} \{r^2\phi - (t+x)z\}$. Now remove the infinite but stressless tz and we have $\psi_3' = \{r^2\phi - xz\}/2\pi, \phi$ being equal to $\theta + \frac{1}{2}\pi$ in the notation of § 4.1.2. On the upper surface $\partial^2/\partial x^2 = \partial^2/\partial r^2$. Therefore the traction \widehat{zz} is equal to $+\phi/2\pi$.

$z = 0$, is 13·25 square metres. And this, at a density of 2·25, gives a mass of 29·81 metric tons above the rock surface per slice 1 metre thick. The centre of this mass was found by graphical construction to be at the point $x = +1·335$,* $z = -2·19$. The total horizontal thrust of the water on the same slice is 18·0 tons. In dividing the forces acting on the structure into lake pressure and a force-at-a-point, after the manner of § 4·1·13, I have considered the lake to extend right up to the vertical face. This leaves a force of 0·125 ton acting upwards near the corner, due to the bevel diminishing the depth of water just there to be included in the force at a point.

In finding the force at a point, the total horizontal thrust of the water of 18 tons acting through a point $\frac{2}{3}$ of the height of the dam from its top, is compounded with the weight of 29·82 tons acting through the centre of mass and with the 0·125 ton acting upwards near the corner. The resultant of 34·74 tons strikes the base at an angle, the tangent of which is 1·657 ($= 58^\circ 54'$), at a point the co-ordinates of which are $x = +2·55$, $z = 0$. If there had been a pronounced curve at the bottom of the front it would have been necessary to use a link polygon to find the load point of the horizontal $z = 0$.

§ 4·2·2. *Taking six co-ordinate differences in the height* the surface conditions were integrated by equations (13), (14), &c. A difficulty arose at top. For the dam there being only one co-ordinate difference thick, that is, χ being expressed by three columns of numbers only, the four boundary conditions cannot necessarily be simultaneously satisfied. It was avoided by making the total change in $\partial\chi/\partial x$ over the base equal to the total downward force, and by making the total change in $\partial\chi/\partial z$ over the base equal to the horizontal thrust of the water, and then integrating the boundary strips separately from in front and behind, and making χ one-valued where they met at the top.

On the lower boundary χ was made equal to the sum of the following four parts:—

(i.) For the lake pressure, the values of $6\psi_3$, found directly from the co-ordinates x and z . They are given in Table IX.

(ii.) For the force at a point $34·74\psi_2$, found from the factors r and $r \sin \theta$, measured on a large sheet of scale paper. $34·74\psi_2$ is also given in Table IX. They may be of use to anyone re-working the problem for a dam of a different shape.

(iii.) A linear function added to the above two in order to make the sum agree with the values of χ on the air surface behind the dam. It would have been simpler to have altered the surface values to make them agree with the base, but I did not think of that in time. This linear function is most conveniently specified by its first differences, which are $\partial\chi/\partial x = +9·735$, $\partial\chi/\partial z = -15·35$, and by the fact that it vanishes where the point-force cuts $z = 0$. These first differences were calculated from the first space-rates of the expression $-34·74\pi^{-1}r\theta \sin \theta$ along and normal to $\theta = -58^\circ 54'$, and were used to calculate the linear function.

(iv.) Finally, $-\frac{1}{8}z^3$ was calculated and added.

* By calculation more exactly 1·340.

The sum of these four functions of position is given on the lower boundary of Table X. Its lake and point-force components have been checked by plotting curves and recalculating any number which did not lie smoothly in the series. In this way an error of 1.0 could be detected, and therefore probably does not exist. The multiplications were made with a 20-inch slide rule. Altogether I expect the "standard deviation" of the errors in the lower boundary values is 0.15 or less.

The initial body values assumed may theoretically be arbitrary, but of course the approximation process need not be so long if they are near to the correct integral. In this case the initial body values were made equal to the sum of the four functions $6\psi_3$, $34.74\psi_2$, the linear function of (iii.) above, and $-\frac{1}{8}z^3$ in the region below $z = 0$. The numbers were read from graphs in a very rough way. Above $z = 0$ the initial body values were simply guesses.

The preliminary stages of approximation were full of experimental processes which did not completely succeed and are not worth describing,

Here $z = +1$.

-67.05 -143.6	-96.90 -163.7
-77.42 -143.8	-107.80 -164.3
-92.02 -143.5	-122.80 -164.0
-110.45 -143.1	-142.10 -163.4

Here $z = +1$.

-0.21 -46.8	-0.25 -37.4
-1.03 -35.2	-1.15 -26.5
-2.78 -25.5	-3.11 -18.0
-5.86 -18.1	-6.48 -11.7

TABLE IX.

The lower number in each square is the *stress function due to the point-force*.

It is hoped that this is correct to 0.1.

The upper number in each square is the *stress function due to the lake*, that is,

$$\frac{6\gamma^2}{4\pi} \{ -2(\theta + 90^\circ) + \sin 2(\theta + 90^\circ) \}.$$

It is hoped that the numbers are correct to 0.05 or less.

The side of each square is unity.

The position of this table is indicated by a dot on the right.

-105.15 -122.1	-105.70 -102.6	-129.20 -122.3	-132.45 -142.5
-11.58 -7.0	-20.62 -1.1	-23.05 -0.2	-12.89 -3.0
-52.93 -4.5	-59.28 -9.8	-67.24 -17.8	-76.39 -27.2
-87.13 -39.4	-100.65 -53.3	-116.9 -69.1	-136.3 -86.4
-66.02 -36.22	-49.13 -14.4	-56.70 -23.91	-77.57 -50.6
-91.75 -66.9	-109.45 -85.0	-130.8 -103.7	-105.70 -102.6
-129.20 -122.3	-132.45 -142.5	-142.5 -163.4	-163.4 -184.3

$z = -4.$
 $z = +7.$

The body region is enclosed by a black line.

[illegible]

especially as the arithmetic was somewhat faulty. Let us start with the table as left by them and call it ϕ_1 .

Next the processes recommended in § 3.2.1 were adhered to. The table being large it was thought best to remove P_1 and P_2 from $\phi_1 - \phi_u = \Sigma A_k P_k$ by making guesses at the form of P_1 and P_2 , calculating their coefficients A_1 and A_2 by the Fourier process and then subtracting $A_1 P_1 + A_2 P_2$ from $\phi_1 - \phi_u$. Of course P_k is defined so that $SIP_k^2 = 1$, where S denotes a summation over the body-points, whereas the guess is an approximation to $B_k P_k$, where B is an unknown constant. For $\mathfrak{D}' = \nabla^4$, and for zero boundary values of the (P)'s the general equations of the appendix give

$$I = 1, \quad \lambda_k^2 = V/T = S[\nabla^2(B_k P_k)]^2 / S[B_k P_k]^2.$$

Now $\nabla^4 \phi_1 = \Sigma \lambda_k^2 A_k P_k$. Therefore $S[P_k \cdot \nabla^4 \phi_1] = \lambda_k^2 A_k$. Therefore

$$A_k P_k = P_k \cdot S[P_k \cdot \nabla^4 \phi_1] \times S[B_k P_k]^2 / S[\nabla^2(B_k P_k)]^2 = (B_k P_k) \cdot S[B_k P_k \cdot \nabla^4 \phi_1] / S[\nabla^4(B_k P_k)]^2.$$

In the right-hand side P_k only appears in the combination $B_k P_k$, so when this is known $A_k P_k$ can be calculated at each point. A rough check on the accuracy of the guesses at $B_1 P_1$ and $B_2 P_2$ may be obtained by comparing the values of λ_1^2 and λ_2^2 obtained from them, namely, 0.18 and 2.3, with the corresponding quantities in infinitesimals for simpler geometrical figures of the same area. The area of this smaller dam table is 113 square units. And for a rectangle $\frac{3}{5}$ as broad as long the formula of § 3.1 gives $\lambda_1^2 \doteq 0.15$, while for a circular plate clamped at the edges the principal vibration with a single circular node has $\lambda^2 = 1.2$ (RAYLEIGH, 'Theory of Sound,' § 221A). It is possible that what has been called λ_2^2 above was really λ_3^2 .

Finally the table was given four approximations of the type $\chi_{n+1} = \chi_n - \alpha^{-1} \nabla^4 \chi_n$ where α was 10, 30, 50, and 64 in turn. These numbers were chosen because they gave a good "curve of ratio of final to initial amplitude." The result is shown in Table X. From it the stresses can easily be calculated. The values of $\nabla^4 \chi$ which should vanish are given there also.

§ 4.2.2.1. *Interaction of Body and Surface Equations. Bevelling of Re-entrant Angle at Front.*—It has been shown in § 4.1.9 above that the given stresses on

the upper surface determine a double layer of values of χ covering this surface. At almost all points of the double boundary layer it is impossible to evaluate $\nabla^4 \chi$ because one or more of the values of χ involved is lacking, so that χ over the surface must be determined by the surface condition only. The only exceptions are near a re-entrant angle. For example, at the points where $\chi = a$ or b in the

TABLE XI.

	0	0		
×	0	0		
×	a	0	0	0
×	c	b	0	0
×	×	×	×	

Table XI. sufficient surrounding numbers exist to determine $\nabla^4\chi$ completely. We might therefore determine a and b at these points by the body equation. But then we should be left with a boundary strip indicating a considerable amount of bevelling. Or we may go to the other extreme, and determine χ in the chequer marked c by assuming the surface equation $\partial^2\chi/\partial x\partial z$ to hold at the angle. I take it the boundary strip would then represent a perfectly sharp right angle. On these grounds I suppose that the case actually studied, in which a and b are determined by the surface equations and c by the body equation, represents an angle with the slight amount of bevelling caused by joining the points $x = 0, z = -\frac{1}{2}$, and $x = -\frac{1}{2}, z = 0$. This was not realized till about the 10th approximation, when Mr. BORCHARD pointed out some inconsistency. It was corrected along with other errors in the boundary conditions, and all final numbers refer to the bevelled angle as just stated.

§ 4.2.2.2. *Error due to a Point-Force having been substituted for the Actual Distribution.*—We see from Table X. that the actual distribution of stress differs from the assumed point-force in being spread out well over the base. We may form some estimate of the order of the error involved by comparing the stresses over the lower boundary due to a vertical point-force with the same due to a statically equivalent pressure spread uniformly over the base. MICHELL'S stress function $\psi_3 = (r^2\phi - r'^2\phi')/2\pi$ enables us to do this. (See equations (22), (23), (24).) For when the point P on the lower boundary is vertically below the centre of the stressed surface, the total force is proportional to the stressed area, which is equal to $2r \sin \frac{1}{2}\alpha$. So that if the total force is to be constantly unity as the area alters, the stress function must be equal to $\frac{1}{2r \sin \frac{1}{2}\alpha} \cdot \frac{r^2\phi - r'^2\phi'}{2\pi}$. By (23) and (24) the principal stresses at P are

$$-\frac{2}{r\pi} \left\{ 1 - \frac{\alpha^2}{4 \cdot 3!} + \text{terms in } \alpha^4, \&c. \right\}, \quad -\frac{1}{r\pi} \left\{ \frac{\alpha^2}{3!} + \text{terms in } \alpha^4, \&c. \right\}.$$

In the limit when the force is at a point we have $\alpha = 0$, and these reduce to $-2/r\pi$ and zero. In the case of a point on the lower boundary, eight units below the level of the rock surface, the base of the dam subtends an angle of about $4\frac{1}{2}/8$ radius. Then $\alpha^2/4 \cdot 3! = 0.013$: that is to say, the errors in the stresses at the lower boundary, due to substituting a point-force for this statically equivalent pressure, spread uniformly over the whole width of the base, are about 2 per cent. of the greater of the two principal stresses at the lower boundary. This uniform spreading is not, of course, exactly what has happened in the approximation process, but it is sufficiently similar for the question at issue (see Table X.). The correct lake stresses and the correct upper boundary will tend to swamp this 2 per cent., which is therefore quite negligible.

§ 4.2.2.3. *Errors due to Incomplete Approximation.*—The values of $\nabla^4\chi$ given in Table X. for the last approximation look as though they consisted chiefly of a principal mode of vibration which had a single nodal line sloping parallel to the flank of the dam across the middle of the table. For this mode of vibration λ^2 will be considerably

greater than for the gravest mode which has $\lambda_1^2 = 0.15$; it will probably be as great as unity. Then the errors in χ must be roughly equal to $\nabla^4 \chi$. If we imagine the tabulated values of $\nabla^4 \chi$ smoothed to eliminate all vibrations, except the one with the single nodal line already referred to, and if we then take second differences of the smoothed $\nabla^4 \chi$, these second differences will be the errors in the stresses. It is easy to see that they will be less than unity.

§ 4.2.3. *Halving the Differences.*—Some notion of the error due to having so few as $4\frac{1}{2}$ co-ordinate differences in the base of the dam may be formed from the error in k^4 , found from the equation $(\nabla^4 - k^4)\phi = 0$, for the gravest mode of a square of seven differences to the diagonal. This is shown in § 3.1 to be 13 per cent. It was therefore thought desirable to halve the differences and reapproximate. Body values half-way between those of the smaller table were filled in by interpolation.

Since the boundary formerly lay half-way between two sets of numbers, a set of interpolated values now lies directly upon it, and consequently the surface conditions resolve themselves into relations between the three outermost layers of numbers instead of the two outermost as before.

§ 4.2.3.1. *The Method of Approximation.*—After some preliminary experiments, the numbers just inside the boundary were corrected by taking $\chi_{m+1} = \chi_m - \alpha^{-1} \nabla^4 \chi_m$, and after each approximation the numbers just outside the boundary were corrected so as to keep $\delta\chi/\delta q$ equal to $\partial\chi/\partial q$, calculated analytically. The following values of α were taken in turn: 10, 30, 50, 64, and again 2, 5, 20, 40, 40, 50, 60. These numbers were chosen, as all such have been, because they gave a suitable curve of reduction of amplitudes.

§ 4.2.3.2. *An analytic integration of the boundary conditions* was carried out, using equations (6), (7) together with (11), (12) at the corners, and starting from $\chi = 0$, $\partial\chi/\partial q = 0$, behind the tail of the dam, as in the small table. The expressions deduced for χ and $\partial\chi/\partial q$ are given in a schedule. Values of χ calculated from them were set down in their places on the boundary, and the first difference across the boundary was made equal to $\partial\chi/\partial q$. These numerical values may be seen in Table XII. They have been carefully checked at a number of points by the theorem about total changes given in § 4.1.9, and have been found to be correct to 0.01 or less. The exact values of χ obtained from the boundary have been compared on $z = 0$ with those found from the point-force and lake pressure in the bedrock—

$x.$	χ , calculated strictly along boundary.	χ , calculated from point-force assumption.
-4.5	270.30	270.20
-5.5	329.987	329.90

a quite satisfactory agreement.

TABLE XII.

TABLE XII.

The stress function $-\chi$ for a dam 6 metres high. Co-ordinate difference one-half. The boundary of the solid passes through the series of dots. The black line encloses the body region. The small numbers are $-\frac{1}{16} \nabla^4 \chi$.

17.20 26.91 39.98

10.86 17.20 26.89 39.98 56.43
-0.38

10.97 17.20 27.00 40.15 56.54
-0.06

10.82 17.20 27.30 40.54 56.84
+0.34

6.06 10.40 17.60 28.00 41.31 57.54
-0.02 +0.05

3.57 5.86 10.60 18.40 29.19 42.58 58.73
+0.42 -0.16 +0.33

3.99 5.86 11.02 19.59 30.87 44.48 60.41
-1.81 +0.49 -0.16

4.20 5.86 12.06 21.46 33.34 47.13 62.88
-0.01 -0.44 +1.02

2.25 3.00 7.05 14.25 24.37 36.67 50.65 66.21
+1.23 -1.25 +0.77 +0.51

1.23 1.27 3.97 9.49 17.63 28.23 40.95 55.17 70.49
+0.80 +1.14 +0.75 +0.26 +0.09

0.90 0.37 2.11 6.25 12.88 21.86 33.00 46.28 60.81 75.82
+0.49 -0.01 -0.37 +0.53 -1.77 +0.25

0.65 0.85 0.93 0.05 1.22 4.42 9.84 17.50 27.21 39.02 52.77 67.71 82.90 99.84
+0.50 -0.20 -0.37 +0.45 -0.83 -0.43 +0.03

0.00 0.00 0.00 0.00 0.00 0.97 3.67 8.22 14.80 23.37 33.83 46.17 60.28 75.59 91.55 108.64 127.24 147.33
-0.26 +0.17 -0.56 +0.39 +0.12 +0.14 -0.97 -0.09 +0.70

0.47 0.57 0.65 0.85 1.67 3.84 7.78 13.52 21.03 30.44 41.57 54.43 68.75 84.25 100.53 117.84 136.57 156.69
+0.10 -0.67 +0.09 -0.34 -0.08 +0.91 -0.69 -0.12 -0.58 +0.64 -0.19 +0.59 -0.52 -0.32

2.25 2.57 2.96 3.65 5.28 8.50 13.46 20.13 28.56 38.71 50.38 63.55 78.07 93.70 110.23 127.57 146.13 166.73
+0.46 -0.52 -1.17 -0.82 +0.18 -0.44 +0.08 +0.69 +0.19 -0.25 +0.37 +0.08 +1.28 +0.15

5.19 5.98 6.91 8.35 10.85 14.98 20.68 28.11 37.22 47.93 60.07 73.53 88.16 103.91 120.62 138.28 157.26 177.59
+0.13 -0.12 -1.21 +0.57 -0.41 +0.11 -0.01 -0.04 -0.15 +0.32 -0.36 +0.27 +0.62 +0.13

9.58 10.86 12.47 14.77 18.13 22.95 29.29 37.31 46.96 58.14 70.67 84.35 99.12 114.93 131.73 149.56 168.49 189.28
-0.13 -0.87 +0.53 +0.15 +0.02 -0.27 0.00 +0.14 +0.70 +0.24 +0.23 -0.23 -0.33 -0.09

17.24 19.62 22.83 26.71 32.19 39.08 47.66 57.71 69.26 82.08 96.01 110.98 126.99 143.95 162.11 181.22
-0.09 -0.14 -0.13 -0.22 -0.04 +0.22 -0.20 -0.48 -0.22 -0.37 -0.43 +0.10

28.28 32.33 36.43 42.65 49.95 59.09 69.40 81.37 94.50 108.68 123.84 140.14 157.13 175.71

42.35 47.35 54.12 61.80 71.52 82.06 94.34 107.60 122.26 137.32 154.10 171.28

tesimally correct upper boundary of the fine-difference table did not at first fit the lower part derived from the coarse-difference table. A correction to the coarse-difference table was therefore calculated, which, if added to it, would have made its

Position (see fig. 7).	X.	$\frac{\partial X}{\partial q}$.
Up to S_0	Zero	Zero
S_0 to S_1	$-gp(S-S_0)^3/12\sqrt{2}$	$+gp(S-S_0)^2/4\sqrt{2}$
S_1 to S_2	$-gp125/48$	$+gp25/8$
S_2 to S_3	$-gp\left\{\frac{(S-S_2)^3}{12\sqrt{2}} + \frac{3\cdot5(S-S_2)^2}{4} + \frac{25(S-S_2)}{8\sqrt{2}} + \frac{125}{48}\right\}$	$+gp\left\{\frac{(S-S_2)^2}{4\sqrt{2}} + \frac{3\cdot5(S-S_2)}{2} + \frac{25}{8\sqrt{2}}\right\}$
S_3 to S_4	$-gp(7\cdot6467)$	$+gp(7\cdot125)$
S_4 to S_5	$-gp\{3(S-S_4)^2 + 7\cdot125(S-S_4) + 7\cdot6467\}$	Zero
S_5 to S_6	$-gp(17\cdot771)$ $-g\sigma\frac{(S-S_5)^3}{6}$	$-gp(13\cdot125)$
S_6 to S_7	$-gp\left\{-\frac{(S-S_6)^3}{12\sqrt{2}} + \frac{(S-S_6)^2}{8} + \frac{13\cdot125(S-S_6)}{\sqrt{2}} + 17\cdot771\right\}$ $-g\sigma\left\{\frac{(S-S_6)^3}{6\sqrt{2}} + \frac{(S-S_6)^2(5\cdot5)}{2} + \frac{15\cdot125(S-S_6)}{\sqrt{2}} + 27\cdot73\right\}$	$+gp\left\{\frac{(S-S_6)^2}{4\sqrt{2}} - \frac{(S-S_6)}{4} - \frac{13\cdot125}{\sqrt{2}}\right\}$ $+g\sigma\frac{15\cdot125}{\sqrt{2}}$
S_7 onwards	$-gp\{13\cdot250(S-S_7) + 24\cdot375\}$ $-g\sigma\{3(S-S_7)^2 + 2\cdot875(S-S_7) + 36\cdot706\}$	$+g\sigma(18\cdot000)$

upper boundary infinitesimally correct as regards χ , when the value of χ was found by interpolation* ($\partial\chi/\partial q$ appeared to be near enough already). And this correction was used to bring the lower boundary of the fine-difference table into harmony with its upper boundary. Its greatest value in the region of the fine-difference table was 1.5. *Secondly*, a principal mode of vibration, having maximum value 0.5 and λ^2 , roughly 1.5, was left in the top of the dam not completely removed by the approximation with $\alpha = 2$. It was removed by guesswork satisfactorily. *Thirdly*, a number of slips in the arithmetic had to be corrected. The result of these processes is shown in Table XII.

§ 4.2.3.4. *Errors Due to Incomplete Approximation.*—As a justification of the body values of Table XII., I propose to consider the final distribution of $\nabla_1^4\chi$ belonging to them, and not the process by which they were obtained, for owing to errors and experiments this process was long and complicated. Let us imagine the distribution of $\nabla_1^4\chi$ given in Table XII., to be expanded in the series $\frac{1}{16}\nabla_1^4\chi = A_1\lambda_1^2P_1 + A_2\lambda_2^2P_2 + \dots + \dots + A_n\lambda_n^2P_n$, where P_k is an integral of $(\frac{1}{16}\nabla_1^4 - \lambda_k^2)P_k = 0$, and has $P_k = 0$ and $\partial P_k/\partial n(\text{normal}) = 0$ on the boundary. Then the error in χ is $AP_1 + A_2P_2 + \dots + A_nP_n$. An inspection of $\nabla_1^4\chi$ shows that it changes sign many times in a small area. Hence it obviously contains the principal modes of vibration for which λ^2 is large. The greatest value, namely, λ_n^2 , is about 64, so for this the error in χ will be $\frac{1}{64}$ of $\frac{1}{16}\nabla_1^4\chi$. The greatest value of $\frac{1}{16}\nabla_1^4\chi$ is seen to be 1.81, corresponding to an error of 0.03 in χ . If this had been distributed all over with alternate + and - signs the corresponding error in the stresses would be $(4 \times 0.03)/h^2 = 16 \times 0.03 = 0.48$. This is appreciable, but not serious. The mean value of $\frac{1}{16}\nabla_1^4\chi$, formed by squaring, adding, and taking the square root, would be much smaller than 1.81, and the error in the stresses due to the presence of the higher modes of vibration P_n, P_{n-1}, P_{n-2} , &c., will be correspondingly less than 0.48.

Next, as to the gravest mode P_1 : it is possible that P_1 may be prominent in the error in χ , and yet $\lambda_1^2P_1$ obscured in $\nabla_1^4\chi$ by the presence of $\lambda_n^2P_n$, since λ_1^2 is a small fraction of λ_n^2 . We can estimate it very roughly by means of the theorem that at the centre of a circle an arbitrary function f differs from an integral χ of $\nabla^4\chi = 0$, which coincides with it, as to value and normal space rate, everywhere on the circumference, by the integral over the interior of the circle of the product

$$\nabla^4 f \times \frac{\alpha^2}{16\pi} \left[\left(\frac{r}{\alpha} \right)^2 \log \left(\frac{r}{\alpha} \right)^2 + 1 - \left(\frac{r}{\alpha} \right)^2 \right] = \nabla^4 f \times B, \text{ say,}$$

where α is the radius of the circle. The integration has been effected with sufficient accuracy by drawing contours of B on tracing paper and laying it over Table XII., and then adding up the values of $\nabla_1^4\chi$ situated between each pair of contours.

* The internal distribution of this correction was calculated by contour integration applied to freehand graphs in a way which the author hopes to publish shortly.

The following results have been found:—

Co-ordinates of centre of circle.		Radius.	P.*	Q.†
$x.$	$z.$			
+2	+1	2·25	+0·24	+0·09
+2	+1	1·5	+0·03	+0·03
+4·5	+1·5	1·5	−0·43	−0·38
0	+1·5	1·5	+0·21	−0·19
+1·5	−1·0	1·5	−0·05	−0·04
+0·5	+0·5	1·0	+0·027	+0·05
+4	+1	1·0	−0·110	−0·22
+3	0	1·0	−0·015	−0·03
+1·5	0	1·0	−0·054	−0·09

* P is the excess at the centre of the tabulated number $-\chi$ over an integral of $\nabla_1^4\phi = 0$, which has the same value and radial space rate on the circumference.

† Q is twice the previous column divided by square of radius.

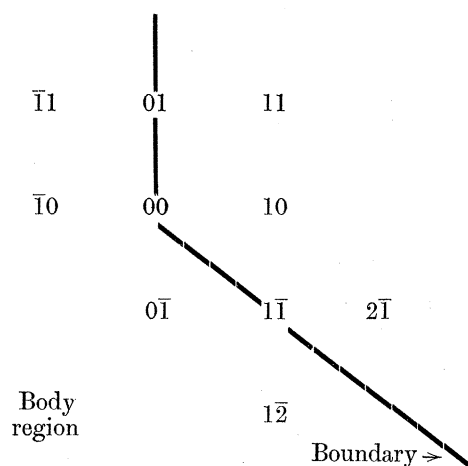
The corresponding errors in the stresses cannot be determined easily. As a very rough guide to them, take the second differences coefficient at the centre of the circle measured along a diameter with co-ordinate difference equal to the radius, that is $-(\text{twice error at centre})/(\text{radius})^2$. These differences are also tabulated and are seen to be small.

§ 4·2·4. *Peculiarities at the angles of 135 degrees on the Upper Boundary.*—There are three alternative ways of treating the relation between the body and boundary equations at the re-entrant angles. (See Table XIII.)

(a) The number ($\bar{1}0$) is determined so as to satisfy $\nabla_1^4\chi = 0$, and does not enter into the boundary equations at all. (10) is derived from ($0\bar{1}$) by means of the known value of $\partial\chi/\partial q$, half-way between them. Also (11) is derived from ($\bar{1}1$) in a similar way. This is the procedure which has been adopted in calculating Table XII.

(b) The number ($\bar{1}0$) is determined so as to satisfy $\nabla_1^4\chi = 0$, but it also enters

TABLE XIII.



into the boundary equations, in that the average of (11) and (10) is derived from the average of ($\bar{1}0$) and ($\bar{1}1$) by means of the value of $\partial\chi/\partial q$ at the centre of the four points where these values are situated.

(c) (11) is derived from ($\bar{1}1$) by means of the value of $\partial\chi/\partial q$ half-way between them and ($2\bar{1}$) is similarly derived from ($1\bar{2}$). But ($0\bar{1}$) is derived from (10) by $\partial\chi/\partial q$ at $\frac{1}{2}$, $\frac{\bar{1}}{2}$ and ($\bar{1}0$) is also derived from (10) by $\partial\chi/\partial q$ at 00, where in the latter q is normal to the line joining 01 and 00. Thus ($\bar{1}0$) and ($0\bar{1}$) are both connected to (10), and are no longer independent. It is therefore impossible to satisfy the body equations completely. Large values of $\nabla_1^4\chi$ may appear at $\bar{1}0$ and $0\bar{1}$. In some work by this method, now rejected, the approximation was so arranged that these values of $\nabla_1^4\chi$ tended to become equal and opposite. The reason for choosing (a) was that the system (c) gave large oscillations in the stresses near the corners, oscillations for which it was difficult to find any physical reason; (b) gave extraordinary oscillations in the shears on the boundary, but (a) gave smooth curves of stress.

TABLE XIV.

			0
0	0		0
0	0	a	

To confirm this choice the co-ordinate differences have been altered so that the boundary lies half-way between two tabular numbers. The ambiguities now disappear. Thus, at the hind toe it is found on stepping out the integral of boundary equations that we have zero values, as in Tables X. and XIV., whatever be the size of the co-ordinate difference.

Then at the corner $\frac{\partial^2\chi}{\partial x^2} = \frac{a}{2}$, $\frac{\partial^2\chi}{\partial x \partial z} = -\frac{a}{2}$. Therefore $-\frac{\partial^2\chi}{\partial x^2} / \frac{\partial^2\chi}{\partial x \partial z} = 1$.

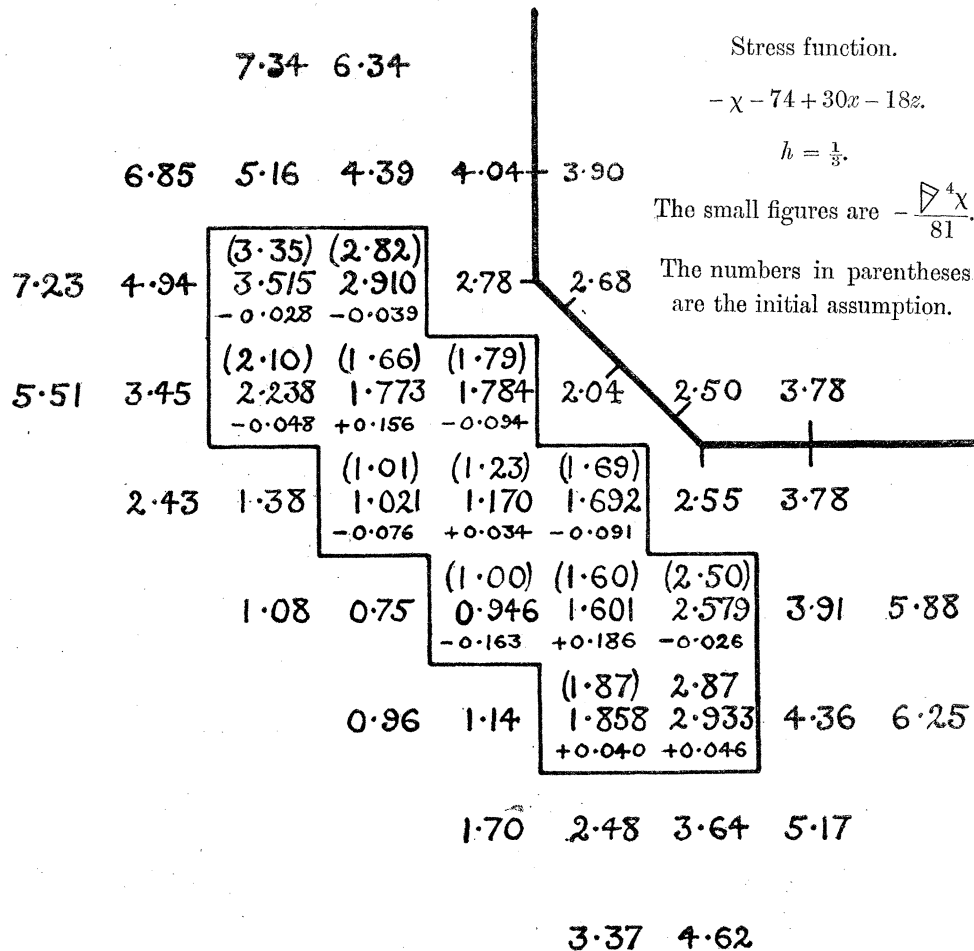
Now test Table XII. to see whether it gives the same result.

At the hind toe $\frac{\partial^2\chi}{\partial x^2} = +3.88$, $\mu_x\mu_z \frac{\partial^2\chi}{\partial x \partial z} = -3.79$, and $\frac{-3.88}{-3.79} = 1.02$, which is satisfactory.

At the front toe a more elaborate check has been made by means of a new table with smaller co-ordinate differences, $h = \frac{1}{3}$. This is Table XV. It was prepared as follows:—First, the number of digits in the values of χ , near the corner, were reduced by taking instead $-\chi' = -\chi - 74 + 30x - 18z$, so that χ and χ' have identical second-differences. Next, curves were plotted showing the variation of χ' along lines parallel to the co-ordinate axes. Values of χ' were read from the curves at the new points required. These values provided the boundary numbers of Table XV., on the side of it which is in the middle of the dam. They also provided the initial body values, which are given in parentheses in Table XV. The boundary numbers on the masonry-water surface were, however, not obtained by interpolation from Table XII., but instead from the infinitesimal integral of the boundary conditions. Each of them

was derived from the value of χ' at the *nearest* point on the masonry-water surface by adding or subtracting the value of $\frac{\partial \chi'}{\partial q}$ at that point, multiplied by the distance to that point. The final body values of χ' were arrived at by the process which has been compared to hand-scraping in § 3.3 above. They are justified by the very small

TABLE XV.



values of $\nabla_1^4 \chi'$ belonging to them. They are given to three places of decimals merely in order to get smooth curves of stress. The interpolated boundary values may well be ± 0.02 in error, so that the third decimal has no real significance. The stresses deduced from this table are shown in two cases in figs. 8 and 9. It is seen that the sharp oscillations revealed in the stress by the fine differences are passed over by the coarse differences. The relation between the stresses near the front toe is, however, not consistent with the proportionality of the error to h^2 . This indicates that the assumptions which have been made as to boundary conditions have not been quite consistent with one another.* Nevertheless, the scheme adopted with $h = \frac{1}{2}$ is to be

* Or that the term in h^4 in the error cannot be neglected.

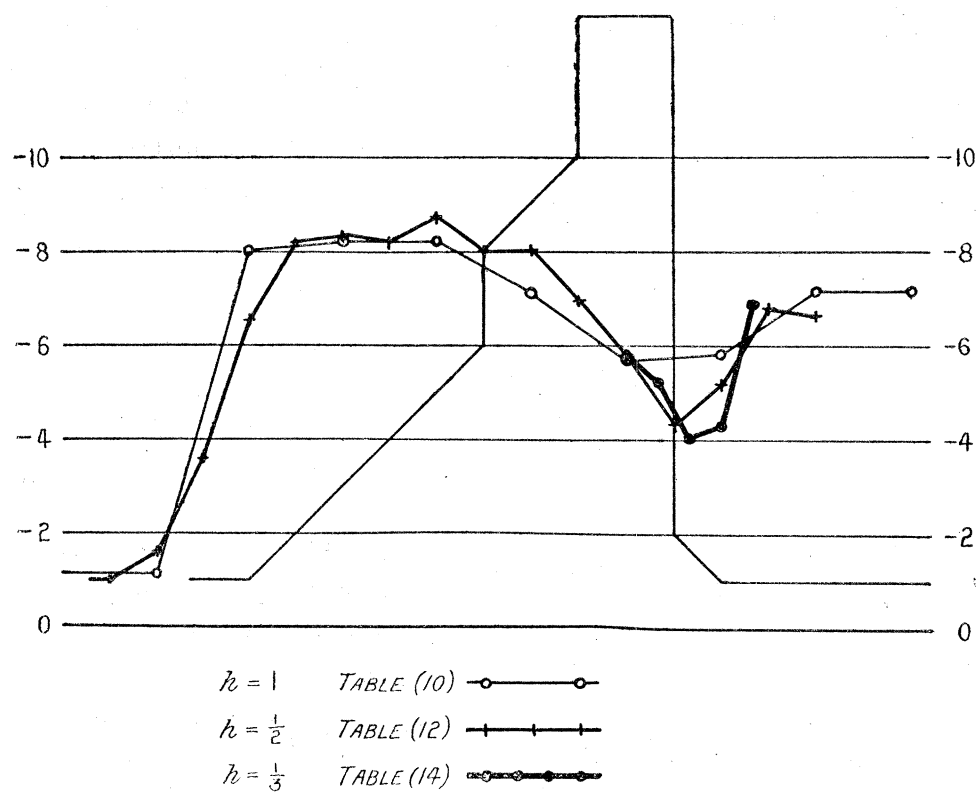


Fig. 8. Vertical stress \widehat{zz} on $z = +\frac{1}{2}$ for a dam 6 units high. [Table (14) should read (15).]

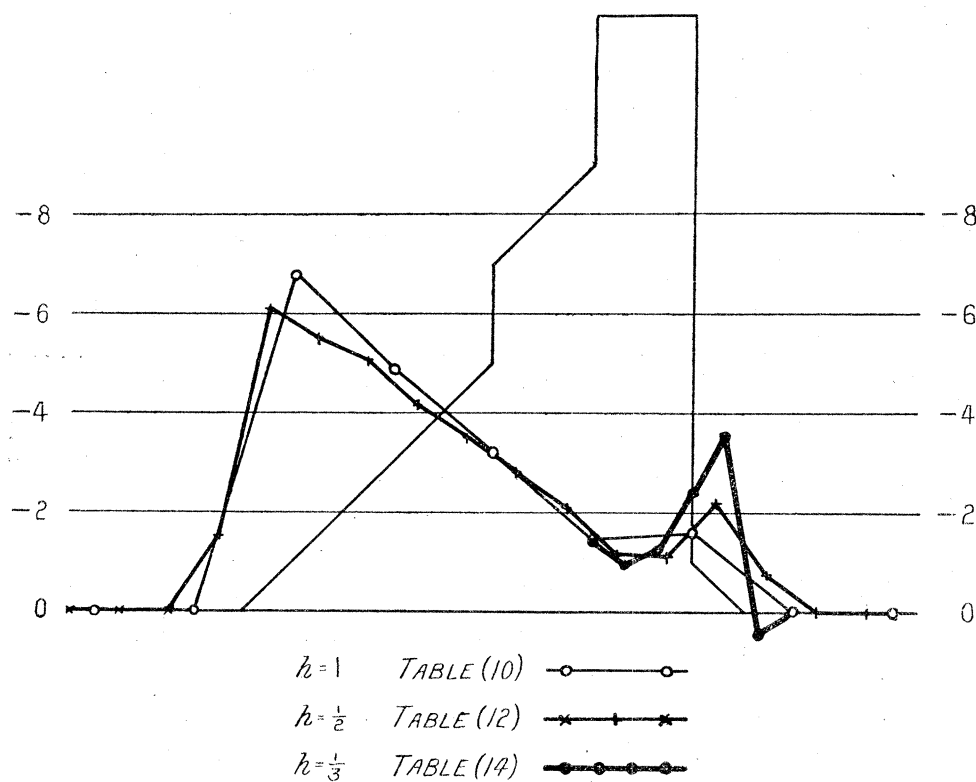


Fig. 9. Shear \widehat{xz} on $z = 0$ for a dam 6 units high. [Table (14) should read (15).]

preferred to the alternatives b and c , for with them the inconsistency was found to be greater. It would be interesting to rework Table XV., taking, on the masonry-water surface, numbers stepped out by finite differences exactly as was done for the table with $h = 1$. It is not possible to adjust the boundary values of Table XV. so as to give by interpolation the infinitesimally correct values of χ' and $\frac{\partial \chi'}{\partial q}$ on the masonry-water boundary, for this would necessitate four independent adjustments in the three values nearest to either corner.

At the protuberant angles of 135 degrees the method (a) adopted for the re-entrant angles will not apply. For according to this method number (10) is not determined by the boundary conditions (see Table XIII.). Neither is it determined by the body equation, for it is right outside the boundary. But it cannot be omitted, for it enters into the body equation at 1, 0.

It has been derived from (10), as in the method (c). Fortunately a protuberant angle of 135 degrees only occurs at one point, and even if the method adopted is not quite correct, the stresses will only be affected in the immediate neighbourhood of the angle. This point is more than half-way up the flank.

§ 4.3. *Conclusion.*—An attempt has been made to provide tables of the stress-function, giving by their second differences stresses sufficiently accurate to be of use to the practical designers of dams. The evidence that this has been done is:—
(i.) The discussion of the boundary conditions in § 4.1.11, § 4.2.3.1, and § 4.2.4;
(ii.) The discussion as to the completeness of the approximation in § 4.2.2.3 and § 4.2.3.4; (iii.) The general agreement of the stresses derived from the three sizes of co-ordinate difference in figs. 8 and 9. In this statement about the accuracy it is assumed that the stress is taken from the table which has the smallest co-ordinate difference. That is, from Table XV. having $h = \frac{1}{3}$ near the front toe; from Table XII. having $h = \frac{1}{2}$ elsewhere in the dam; from Table X. having $h = 1$ deep down in the bedrock.

An additional confirmation is the strong resemblance of the curves in figs. 8 and 9 to those found by Messrs. WILSON and GORE* by stressing an india-rubber model with a rounded angle at the front toe.

The arithmetic has been carried out by a number of people, principally Messrs. TILLEY, H. BORCHARD, W. SHEPPARD, C. H. MASTERS, and G. ROBINSON, and I am grateful to them for the care they have bestowed upon it.

§ 5. This work on the solution of physical problems by finite differences has been carried on at intervals during three years. My thanks are due to Mr. G. A. SCHOTT for convincing me of the desirability of discovering new methods for solving physical problems; to Mr. A. BERRY, Mr. G. F. C. SEARLE, and Prof. KARL PEARSON for their encouragement and advice in the early stages; to Mr. A. CAMPBELL, Mr. J. C. M. GARNETT, and Mr. H. H. JEFFCOTT for references and the loan of books;

* 'Institution of Civil Engineers,' February, 1908.

to Prof. A. E. H. LOVE for several vital improvements which were introduced on re-writing the paper, especially for the suggestion of the theory in the Appendix; also to Prof. LARMOR, Prof. SAMPSON, and Dr. R. T. GLAZEBOOK for much valuable criticism.

APPENDIX.—*Properties of the Principal Modes of Vibration.**

In the theory of the approximation process we have assumed that—

- (i) An arbitrary function f can be expanded in the form $f = \sum A_k P_k$, where P_k is an integral of $\pm \mathfrak{D}'P_k = \lambda_k^2 P_k$, and satisfies the same boundary condition as f does.
- (ii) That the λ^2 are real, all of one sign, and lie in a finite range.
- (iii) That a certain one-signed function of position I can be found such that $SIP_k P_l = 0$, $SIP_k^2 = 1$, where S denotes a summation over the body points.

It will now first be shown that if the system of difference equations satisfies certain conditions, then the simultaneous equations

$$\mathfrak{D}'\psi_1 = 0, \mathfrak{D}'\psi_2 = 0, \dots \mathfrak{D}'\psi_n = 0 \quad (1)$$

are equivalent to

$$\frac{\partial V}{\partial \psi_1} = 0, \frac{\partial V}{\partial \psi_2} = 0, \dots \frac{\partial V}{\partial \psi_n} = 0 \quad (2),$$

where V is a one-signed homogeneous quadratic function of $\psi_1 \dots \psi_n$. Next, it will be shown that the desired properties (i), (ii), (iii) can be deduced from the existence of V .

No attempt will be made to determine whether the properties (i) and (ii) can hold under more general circumstances when V does not exist.† However, V exists for so wide a range of physical problems that it is well worth considering.

Conditions of Existence of V.—As equations (1) are to be equivalent to equations (2) it must be possible to find a set of numbers $i_1, i_2, \dots i_n$ such that

$$i_l \mathfrak{D}'\psi_l = \frac{\partial V}{\partial \psi_l} \quad \text{for } l = 1 \dots n \quad (3).$$

And then, since $\frac{\partial^2 V}{\partial \psi_l \partial \psi_k}$ is independent of the order of the differentiations,

$$\frac{\partial}{\partial \psi_k} (i_l \mathfrak{D}'\psi_l) = \frac{\partial}{\partial \psi_l} (i_k \mathfrak{D}'\psi_k) \quad (4)$$

* The dynamical analogy which is the basis of this section was pointed out to me by Prof. A. E. H. LOVE. It has been introduced by POCKELS in his book 'Über die Gleichung, $\Delta^2 u + k^2 u = 0$.' See also RAYLEIGH, 'Sound,' vol. I, chap. IV.

† The boundary values in equations (1) are supposed to have been expressed in terms of the body values by the boundary condition.

‡ As to the property (iii), note that by considering the coefficients of the body values in the sums $S[IP_j \mathfrak{D}'P_k]$, $S[IP_k \mathfrak{D}'P_j]$, it is easy to show that these sums are equal if, and only if, the reciprocal relation (4) holds. And if they are equal, then $\lambda_k^2 S[IP_j P_k] = \lambda_j^2 S[IP_k P_j]$, so that $SIP_j P_k = 0$ when λ_j^2 is not equal to λ_k^2 . But the reciprocal relation (4) is necessary to the existence of V .

for every pair kl . This is the condition that the body equations can be derived from a single function as in (2). The numbers i_1, i_2, \dots, i_n may be regarded as the body values of a function of position. It will be denoted by I and called the "unifying factor," because it allows all the body equations to be expressed in terms of a single function V .

For the sake of a certain transformation, which will be introduced later, V must be quadratic and homogeneous, and therefore both body and boundary equations must be linear and homogeneous.

Thus let

$$\left. \begin{aligned} \mathfrak{D}'\psi_k &= c_{k1}\psi_1 + c_{k2}\psi_2 + \dots + c_{kn}\psi_n \\ &+ e_{k1}\beta_1 + e_{k2}\beta_2 + \dots + e_{ks}\beta_s \end{aligned} \right\} \dots \dots \dots (5).$$

And let the boundary equations be

$$\beta_j = f_{j1}\psi_1 + f_{j2}\psi_2 + \dots + f_{jn}\psi_n \quad \dots \dots \dots (6),$$

where the (c) 's, (e) 's, and (f) 's are constants, most of which will commonly be zero. In many physical problems we have the integral ϕ_u given on the boundary. This corresponds to a term f_{j0} , independent of the (ψ) 's, in each of the equations represented by (6). However, it is not ϕ_u which we wish to expand in the form $\Sigma A_k P_k$, but the difference $(\phi_m - \phi_u)$ between the integral and an arbitrary function satisfying the same boundary conditions. In $\phi_m - \phi_u$ the f_{j0} terms cancel. Thus we are only here concerned with functions having boundary equations which are linear and homogeneous in the (ψ) 's.

Next, equation (4) leads to the following relations between the coefficients of (5) and (6):—

$$i_l c_{lk} + i_l \sum_{j=1}^{j=s} e_{lj} f_{jk} = i_k c_{kl} + i_k \sum_{j=1}^{j=s} e_{kj} f_{jl} \quad \dots \dots \dots (7)$$

for every pair k and l , making at most $\frac{1}{2}n(n-1)$ equations. Now it is customary in physical mathematics to treat body and boundary conditions separately. Let us adopt the same course here and treat only the case in which (7) splits into two parts, namely,

$$i_l c_{lk} = i_k c_{kl} \quad \dots \dots \dots (8)$$

to be satisfied within the body region, and

$$i_l \sum_{j=1}^{j=s} e_{lj} f_{jk} = i_k \sum_{j=1}^{j=s} e_{kj} f_{jl} \quad \dots \dots \dots (9)$$

to be satisfied on the boundary.

In the centre of a sufficiently large table (7) reduces to (8) even in the most general case, for there the (e) 's and (f) 's vanish.

Stated in words, $i_l c_{lk} = i_k c_{kl}$ means that the body equations must be able to be brought into forms such that if each pair of points in the table be taken in turn, then the integral ϕ at the first point of the pair, enters into the body equation $\mathfrak{D}'\phi$ at the

second, with the same coefficient that the value of ϕ at the second enters into the body equation at the first. For example, if $\psi_1, \psi_2, \dots \psi_n$ are spaced in order at equal intervals along the X axis, and if $\mathfrak{D} = \partial^2/\partial x^2$, then $\mathfrak{D}'\psi_k = \psi_{k-1} - 2\psi_k + \psi_{k+1}$ and (7) is satisfied when the (i) 's are simply unity. But if $\mathfrak{D} = \partial/\partial x$, then $\mathfrak{D}'\psi_k = \mu \delta\psi_k = \psi_{k-1} - \psi_{k+1}$, so that c_{ki} is ± 1 , while $c_{ik} = 0$, so it is impossible to find a unifying factor.

Next, as to $i_i \Sigma e_{ij} f_{jk} = i_k \Sigma e_{kj} f_{ji}$. A particular condition in which these sums become identical is if $f_{ji} = e_{ij} i_i \alpha_j$ where α_j is an arbitrary number, independent of the (ψ) 's.

By (6) this equation implies that

$$\beta_j = \alpha_j [e_{1j} i_1 \psi_1 + e_{2j} i_2 \psi_2 + \dots + e_{nj} i_n \psi_n]. \quad (10).$$

Equation (10) states that the boundary values will satisfy their part of (4) if every boundary number is formed as follows: Take each body-point. Multiply the value of ϕ there by the unifying factor and by the coefficient of the said boundary number in $\mathfrak{D}'\phi$ at this point. Sum this product for all body points and multiply the total by a number independent of the body values. This is not the most general way of satisfying (9), but it is a common one in physical problems. Thus, for example, if the equation to be solved is $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi_u = 0$, a boundary condition of frequent occurrence is to have ϕ_u given, and therefore $\phi_m - \phi_u = 0$. This corresponds to $\alpha_j = 0$.

Another common condition is $0 = \frac{\partial \phi_n}{\partial n} = \frac{\partial (\phi_m - \phi_u)}{\partial n}$, where ∂n is an element of the normal to the boundary. Then a boundary number will either be equal to a body-number or to a weighted mean of two neighbouring body-numbers; in either case $\alpha_j^{-1} = e_{1j} i_1 + e_{2j} i_2 + \dots + e_{nj} i_n$, in which all but one or two of the (e) 's will vanish.

Lastly, it will be necessary in what follows that V should be of one sign for all values of $\psi_1, \psi_2, \dots \psi_n$. This will be so if $\partial^2 V / \partial \psi_k^2$ is of the same sign for every ψ . Whence, by (3), (5), and (6), $i_k \{c_{kk} + \Sigma e_{kj} f_{jk}\}$ must be of the same sign for every k (11).

As before, we will treat only the case in which the boundary and body equations satisfy this separately, so that every $i_k c_{kk}$ and every $i_k \Sigma e_{kj} f_{jk}$ have all the same sign.

For simplicity we will take V as positive (12). Then i_k has the same sign as c_{kk} . Further, as the sign of the operator \mathfrak{D}' does not affect the solution of the equations $\mathfrak{D}'\psi_1 = 0 \dots \mathfrak{D}'\psi_n = 0$ we may alter it arbitrarily, and we will for simplicity suppose it chosen at each point so as to make c_{kk} positive . . . (13).

The $i_1 \dots i_n$ will then also be positive (14).

For convenience, a table is added giving the forms of I and V for some common forms of \mathfrak{D}' . These were taken by analogy from infinitesimals, and afterwards verified by trial. Thus for $\mathfrak{D} = \nabla^2$ we have $V = \frac{1}{2} \iiint [\nabla \phi]^2 d\tau$, where ∇ is the vector operator Nabla and $d\tau$ an element of volume. On expressing V in orthogonal curvilinear co-ordinates $q_1 q_2 q_3$ and varying ϕ , it is found that $\delta V = \iiint \delta \phi \cdot \nabla^2 \phi \cdot I \cdot d\tau +$ a surface

integral, where the unifying factor I is equal to minus the ratio of the volume of the elementary co-ordinate block to the product $dq_1 dq_2 dq_3$.*

\mathfrak{S} denotes a summation for every difference which involves a body-number directly (that is not by way of boundary equations (6)).

In the following table the co-ordinate difference is supposed constant for each co-ordinate separately:—

\mathfrak{D}' with sign to make c_{kk} plus	Unifying factor I.	V.	λ_L^2 .
$-\left(\frac{\mathfrak{X}^2}{\mathfrak{X}x^2} + \frac{\mathfrak{X}^2}{\mathfrak{X}y^2} + \frac{\mathfrak{X}^2}{\mathfrak{X}z^2}\right)$	Unity	$\frac{1}{2}\left[\mathfrak{S}\left(\frac{\mathfrak{X}\phi}{\mathfrak{X}x}\right)^2 + \mathfrak{S}\left(\frac{\mathfrak{X}\phi}{\mathfrak{X}y}\right)^2 + \mathfrak{S}\left(\frac{\mathfrak{X}\phi}{\mathfrak{X}z}\right)^2\right]$	$\frac{4}{\mathfrak{X}x^2} + \frac{4}{\mathfrak{X}y^2} + \frac{4}{\mathfrak{X}z^2}$
$-\left(\frac{\mathfrak{X}^2}{\mathfrak{X}r^2} + \frac{1}{r}\mu\frac{\mathfrak{X}}{\mathfrak{X}r} + \frac{\mathfrak{X}^2}{\mathfrak{X}z^2} + \frac{1}{r^2}\frac{\mathfrak{X}^2}{\mathfrak{X}\theta^2}\right)$	r	$\frac{1}{2}\left[\mathfrak{S}\left\{r\left(\frac{\mathfrak{X}\phi}{\mathfrak{X}z}\right)^2\right\} + \mathfrak{S}\left\{r\left(\frac{\mathfrak{X}\phi}{\mathfrak{X}r}\right)^2\right\} + \mathfrak{S}\left\{\frac{1}{r}\left(\frac{\mathfrak{X}\phi}{\mathfrak{X}\theta}\right)^2\right\}\right]$	
$-\left(r\frac{\mathfrak{X}^2}{\mathfrak{X}r^2} + \frac{\mu\mathfrak{X}}{\mathfrak{X}r}\right)$	Unity	$\frac{1}{2}\mathfrak{S}\left\{r\left(\frac{\mathfrak{X}\phi}{\mathfrak{X}r}\right)^2\right\}$	
$\frac{\mathfrak{X}^4}{\mathfrak{X}x^4} + 2\frac{\mathfrak{X}^4}{\mathfrak{X}x^2\mathfrak{X}y^2} + \frac{\mathfrak{X}^4}{\mathfrak{X}y^4}$	Unity	$\frac{1}{2}\mathfrak{S}\left(\frac{\mathfrak{X}^2\phi}{\mathfrak{X}x^2} + \frac{\mathfrak{X}^2\phi}{\mathfrak{X}y^2}\right)^2$	$\left(\frac{4}{\mathfrak{X}x^2} + \frac{4}{\mathfrak{X}y^2}\right)^2$

Note that the averager μ occurs in \mathfrak{D}' but not in V. Having now set out the conditions of existence of V in a form in which they can easily be applied to test any operator \mathfrak{D}' given with unifying factor and boundary conditions, let us pass on to deduce the properties of the principal modes of vibration from the existence of V.

Let

$$2T = i_1\psi_1^2 + i_2\psi_2^2 + \dots i_n\psi_n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

then

$$\partial T / \partial \psi_i = i_i \psi_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

and by (14) T is essentially positive.

Now V and T being real quadratic forms in the same variables, one of which, T, is

* Simply putting \mathfrak{X} for ∂ in the infinitesimal V does not give the finite-difference V except in special cases.—April, 1910.

it follows that when $f = P_k$ then $V/T = \lambda_k^2$, and when f differs slightly from P_k then V/T will differ from λ_k^2 by a small quantity of the second order (RAYLEIGH, 'Theory of Sound,' § 88). In § 3·2·3 a guess is made at P_k , and λ_k^2 is then calculated roughly as V/T .

The special values $\lambda_1^2 \dots \lambda_n^2$ for any region are included between the greatest and least of the λ^2 pertaining to any region, which includes the former region when the boundary values of both regions vanish. For let P_k , the integral of $(\mathfrak{D}' + \lambda_k^2) P_k = 0$ for the smaller region, be set round about with noughts until a larger boundary is reached. The conditions for the existence of a function V will then be satisfied for the larger region. Let quantities belonging to the larger region be distinguished by dashes. Then by (28) P_k may be expanded in the form

$$P_k = \Sigma B_i P'_i, \text{ where the } (B)'s \text{ are constants.}$$

Now by (31) $\lambda_k^2 = V/T$, when the body values have the values given by $\phi = A_k P_k$.

Also, by (15), $T' \equiv T$ since the added squares are zero. Also, by (3), $i_k \mathfrak{D}' \psi_k$ is equal to both $\partial V / \partial \psi_k$ and $\partial V' / \partial \psi_k$ so that V and V' can only differ by terms independent of $\psi_1, \psi_2, \dots \psi_n$. But there are no other variables in V' . Therefore $V' \equiv V$ and $\lambda_k^2 = V'/T'$.

Now, expanding V' and T' by (17) and (18),

$$\lambda_k^2 = \frac{\lambda_1'^2 G_1^2 + \lambda_2'^2 G_2^2 + \dots + \lambda_r'^2 G_r^2}{G_1^2 + G_2^2 + \dots + G_r^2}.$$

And therefore λ_k^2 lies between the greatest and least of the λ'^2 . If, however, the boundary values do not vanish then $\partial V' / \partial \psi_k = +i_k \mathfrak{D}' \psi_k$ when the $(\beta)'s$ are regarded as independent of the $(\psi)'s$ during the differentiation, whereas $\partial V / \partial \psi_k =$ the same quantity when V is expressed as a function of the $(\psi)'s$ entirely, and a more detailed examination shows that λ_k^2 need not lie between the greatest and least values of λ'^2 for an enclosing boundary with zero boundary values. However, λ_k^2 is necessarily finite, for it is the root of a rational integral function with finite coefficients.