II. *On a Theory of the Second Order Longitudinal Spherical Aberration for a Symmetrical Optical System.*

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§ 1. *Statement of the Problem and Historical References.*

If we consider a pencil of rays issuing from a point on the axis of a symmetrical optical system (*i.e.*, a system of refracting spherical surfaces, the centres of which lie on a straight line called the axis of the system), it is well known that, if the pencil be a thin one, of which the mean ray is along the axis, the first approximation to the emergent pencil is another punctual pencil, of which the rays pass through an image point, also situated on the axis. The general method of treatment of such image points, which are usually referred to as "geometrical" images, is due to Gauss, and is developed in any text book of Geometrical Optics.

When, however, the pencil considered is one of finite aperture, the outlying rays do not, after emergence, pass through the Gaussian image point, nor do they have the inclination assigned to them by the Gaussian calculation. The emergent rays lying in any one axial plane touch an envelope or caustic, which has one cusp at the Gaussian image, with the axis as proper tangent. The intercepts of any given emergent ray upon the axis and the image plane, measured from the Gaussian image, are known as the longitudinal and transverse spherical aberrations of that ray.

It is clear that if both these spherical aberrations, or either of them together with the inclination of the ray on emergence, be known for every possible position of object and image, and for every possible inclination of the incident ray, the whole complex of emergent rays lying in axial planes can be mapped out. The calculation of these aberrations is therefore of fundamental importance in practical optical design, where we do not deal with infinitely thin pencils.

The method employed hitherto for dealing with aberrations from the mathematical standpoint has been to develop the sines occurring in the refraction equations at each spherical surface in ascending power of some argument, which may be either the circular measure, or the sine, or the tangent, of one of the angles concerned, and to calculate, by the usual methods of successive approximation, the required aberrations as a series of ascending powers of such argument.

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When this is done it is found that the terms due to the first power of the argument lead to the Gaussian image point, so that the series begin with a term involving the second power of the argument in the case of the longitudinal aberration, and the third power in the case of the transverse aberration. These terms are the first order. The following terms next in sequence, which are of fourth and fifth power respectively, are spoken of as aberrations of the second order, and so on.

A considerable amount of theoretical work has been done on aberrations of the first order by SEIDEL, ABBE and others, and the treatment of these is fairly well known. Unfortunately, it is found in practice that the first order aberrations do not give a sufficient approximation for the optician's requirements. In fact for a certain range of object and image positions, they are so badly out that they cannot be said to constitute an approximation at all. This fact has long been recognised by optical designers, whose practice is invariably to calculate, using the exact trigonometrical equations which involve no approximation at all, the correct paths of a number of selected rays, from which they draw conclusions as to the efficiency, or otherwise, of the proposed system from the practical point of view.

The trigonometrical method, however, from the designer's point of view, has the radical defect that, while it gives partial information about the performance of a given system, it gives no direct intimation of the direction in which the elimination of various defects is to be looked for, and it entails a long and laborious process of seeking for the optimum by trial and error.

The object of the authors of the present paper has been to develop a method of expressing the aberrations, which, while carrying the algebraic development to a stage including the second order, should be free from certain grave troubles involved by failure of convergency, troubles which appear to have been hitherto neglected. In fact this method gives numerical results that, for a single lens, are considerably more accurate than the ordinary second order formulæ. Further, these methods enable one to deal, in a comparatively easier form, with the problem of the second order aberrations of combinations of surfaces and systems, a problem which, so far as we know, has never been attacked from any general standpoint. KOENIG and VON ROHR (VON ROHR, 'Theorie der Optischen Instrumente,' Cap. V.) give a development of a formula for the coefficients of first order and second order in the longitudinal spherical aberration, based on ABBE'S method of Invariants, but so far as can be seen, no definite results are obtained for the second order terms.

DENNIS TAYLOR ('System of Applied Optics,' p. 67) gives a formula for the spherical aberration, developed in powers of the intercept made by the ray on the first principal plane, which includes terms of second order. But his formula, a particular case of those dealt with in the present paper, is limited to the thin lens, and no attempt seems to be made at anything like a general treatment of such aberrations.

Another important object of the method to be described is to express the

aberrations in such a form that, in a combination of surfaces and lenses, the effect of a given surface or lens on the final result can be readily traced. This is fundamental for the designer, who usually proceeds to sketch out his system by Gaussian methods only, being guided therein by considerations of magnification, illumination, and field of view; and then goes on to eliminate the resulting image defects, so far as he can, by bending the lenses, *i.e.*, by altering their mean curvature without changing the focal length. In doing this he usually corrects one defect at a time, with the frequent result that, when, having corrected one defect by means of one lens, he proceeds to correct a second defect, he thereby causes the reappearance of the first.

If the effects of any given lens, however, are made apparent in the final formula, it becomes a more manageable problem to devise variations which will keep any one defect *invariant* whilst others are being dealt with.

§ 2. *Notation.*

There is no general agreement among mathematical writers as to the notation employed in dealing with optical problems, and it will be convenient to state here the symbols we have adopted. They are a modification of a system due to STEINHEIL.

The successive media, proceeding in the direction of travel of the light (from left to right in our figures), are denoted by even suffixes 0, 2, 4, &c., and the same suffixes affect the rays in these media, their inclinations, α_0, α_2, α_4, &c., to the axis, and their intersections I_0, I_2, I_4, &c., with that axis.

The successive geometrical images will be denoted by the letter J, thus J_0, J_2, J_4, &c. The successive surfaces of separation will be denoted by the odd suffixes 1, 3, 5, &c., and the same suffixes will affect the centres of curvature, the intersections of rays with the surfaces, and the points where the axis crosses the surfaces. The latter will be denoted by the letter A and the centres of curvature by the letter C.

Fig. 1 illustrates the use of this notation for two refracting surfaces.

The radii of curvature are r_1, r_3, r_5, &c., and are to be considered positive when $A_{2n+1}\,C_{2n+1}$ is measured from left to right.

The perpendicular from a centre of curvature on a ray is denoted by p and is affected by a double suffix, the first belonging to the centre of the curvature and the second to the ray. Thus p_{12} is the perpendicular from the centre of curvature C_1 of the first reflecting surface upon the ray in the second medium. Where there is no ambiguity the first suffix will usually be omitted.

The refractive index will be denoted by n and affected by the suffix of its medium.

Transverse magnifications will be denoted by M. The magnification produced by surface 1 will be denoted, as convenient, by M_1 or M_{02}; by surfaces 1, 3 combined either by M_{13} or M_{04}; by surfaces 1, 3, 5, combined either by M_{135}, or M_{06}, and so on.

The advantage of the double even suffix notation in this case is that we have a symbol, M_{40}, for the magnification when light passes backwards through the system, the order of the suffixes being material. Where odd suffixes are used, we have to use $1/M_1$, $1/M_{13}$, &c., for the reversed magnifications.

Ray magnifications will be denoted by $\mathbf{M}$. These are the limit of the sine-ratio for small inclinations, thus $\mathbf{M}_1 = \mathbf{M}_{02} = \underset{\alpha_0 \to 0}{\mathbf{L}} \sin \alpha_0/\sin \alpha_2$. $M = \mathbf{M}$ when the initial and final media are the same.

With regard to inclinations, they will be treated as positive when the rays *converge* to the axis, as in fig. 1. The inclinations of the rays calculated by GAUSS' process will be denoted by β. Thus $\beta_0 = \alpha_0$, $\tan \beta_2 = \tan \alpha_0/\mathbf{M}_{02}$, $\tan \beta_4 = \tan \alpha_4/\mathbf{M}_{04}$, &c.

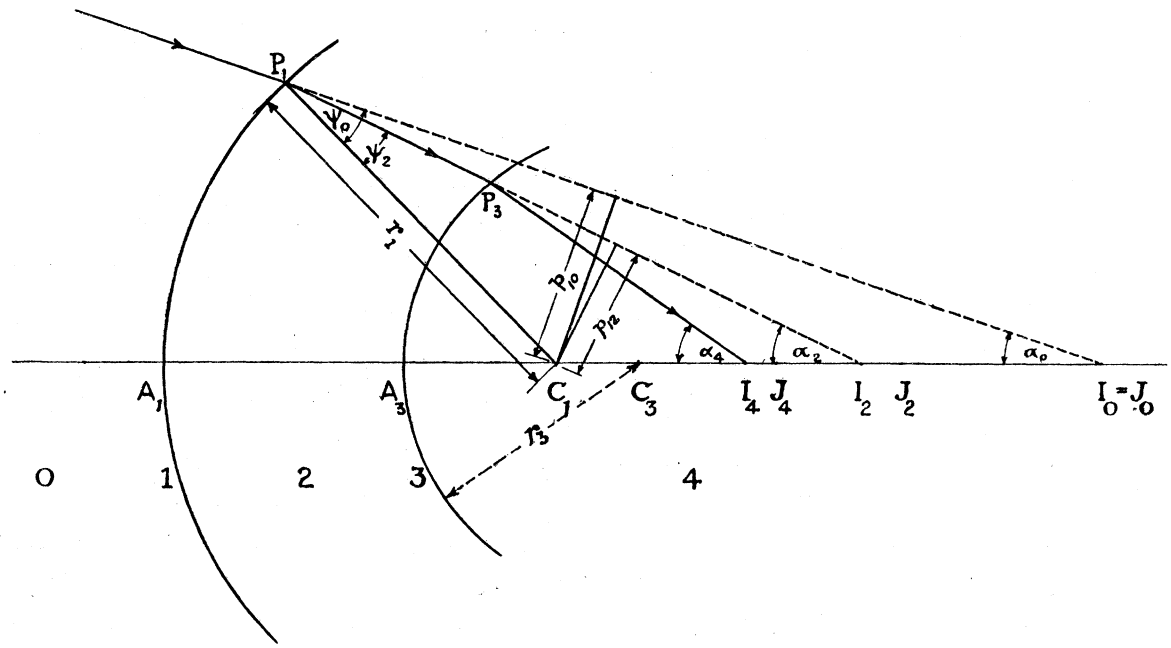

Fig. 1.

We may also use angles γ, calculated from a constant sine ratio, viz., $\gamma_0 = \alpha_0$, $\sin \gamma_2 = \sin \alpha_0/\mathbf{M}_{02}$, $\sin \gamma_4 = \sin \alpha_0/\mathbf{M}_{04}$, &c.

Throughout much of the work we shall use the same trigonometrical function (tangent or sine) of the angles α. If the tangent is used, we shall employ the following abbreviations:—

$$q_{2n} = \tan \alpha_{2n} \qquad t_{2n} = \tan \beta_{2n}.$$

If the sine is used, the meaning of q, t will be as follows:—

$$q_{2n} = \sin \alpha_{2n} \qquad t_{2n} = \sin \gamma_{2n}.$$

It will be found that many formulæ remain unaltered, whichever of the two interpretations for q and t is used.

All distances parallel to the axis will be denoted by x (the attribution of the symbol being indicated in each case) and will be measured positively from left to right.

The longitudinal aberration will be denoted by Δx, with the suffix of the medium, and will be reckoned positive when the point of intersection of the ray with the axis is to the right of the geometrical image. Thus $\Delta x_2 = J_2I_2$. This is opposite to the usual convention which is based on the fact that for positive or convergent lenses, I_2 is generally to the left of J_2, ; but, in the first place, this is not universally true, and, in the second place, the convention adopted by us was found more convenient in handling the algebra.

It is to be noted that, with the notation used, the well-known formula for a lens $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ becomes $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$, the distances u and v being measured in the same direction.

The distances between successive refracting surfaces we denote by c, with the suffix of the medium.

In dealing with a system, especially where the initial and final media are not the same, it is very convenient to use an "equivalent" Gaussian system, in which lengths parallel to the axis are measured in each medium in terms of a unit proportional to its absolute refractive index.

If we denote the corresponding points in the equivalent Gaussian system by accents, we find that

$$A'_1J'_0 = \frac{A_1J_0}{n_0}, \qquad A'_1J'_2 = \frac{A_1J_2}{n_2}, \qquad A'_1A'_3 = c'_2 = \frac{c_2}{n_2},$$

$$A'_3J'_2 = \frac{A_3J_2}{n_2}, \qquad A'_3J'_4 = \frac{A_3J_4}{n_4}, \text{ \&c.}$$

If then we denote the quantities $\frac{n_2-n_0}{r_1}$, $\frac{n_4-n_2}{r_3}$, &c., by $\frac{1}{f_1}$, $\frac{1}{f_3}$ ... ; f_1, f_3 ... may be called the focal lengths of the successive refracting surfaces. The equations connecting image and object in the equivalent Gaussian system are

$$\frac{1}{A'_1J'_2} - \frac{1}{A'_1J'_0} = \frac{1}{f_1},$$

which is of the same form as the equation connecting image and object for a thin lens at A'_1.

Now bearing in mind that $A_3J_2 = A_1J_2 - A_1A_3$ and therefore $A'_3J'_2 = A'_1J'_2 - A'_1A'_3$, it is easy to show that the effects of the successive refracting surfaces in the actual system can be obtained by compounding a corresponding set of thin lenses in the equivalent Gaussian system. By dealing with the latter, we get rid of the asymmetry introduced by the difference of initial and final index. Of course this applies only to the calculation of the geometrical images. We note that in the

equivalent Gaussian system the ray and transverse magnifications are identical and agree with the transverse magnifications in the actual system.

Finally the intercept of the incident ray on the leading principal plane of a system will usually be denoted by y. This is taken by some authors as the argument of the development in series, but differs only by a factor from $\tan \alpha_0$ or $\tan \beta_{2n}$.

In many cases it will be convenient, in order to avoid unnecessarily large suffixes, to *condense* a system of surfaces or lenses, affecting quantities referring to the system itself with suffix 1, and the initial and final media with suffixes 0 and 2, the paths in the intermediate media not being explicitly considered.

§ 3. *Singularities and Convergency.*

Consider any symmetrical optical system, of which PL and QM (fig. 2) are the initial and final refracting surfaces. Let F_0 be the front focus of the system and UF_0V the caustic for backward-travelling rays which are parallel in the final medium.

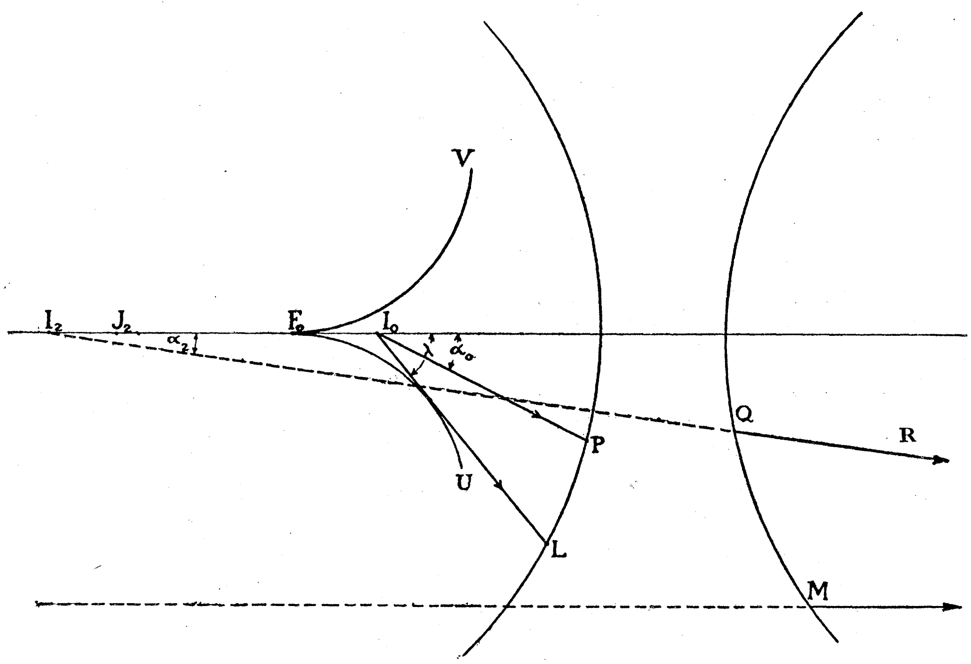

Fig. 2.

This caustic, as is well known, will usually be of the type shown in fig. 2, approximating to a semi-cubical parabola with a cusp at F_0, and, to fix ideas, we shall suppose the point of the cusp to be turned to the left. In the opposite case, an obvious modification of the argument will be found to lead to similar conclusions.

Any ray in the initial medium, which touches this caustic, must emerge parallel to the axis after passing through the system.

Let I_0 be an object point on the axis behind F_0 and sufficiently near to it for a real tangent to be drawn from I_0 to the caustic and yet go through the system. This

will involve for I_0 a positive ray magnification $\mathbf{M}$ exceeding some definite *finite* limit. The Gaussian image point J_2 will then lie a finite distance in front of F_0.

Consider first a nearly paraxial incident ray I_0P. Such a ray will be refracted approximately according to the Gaussian law and will emerge at an inclination α_2, where α_2 is nearly equal to $\alpha_0/\mathbf{M}$ and both α_0 and $\mathbf{M}$ being finite and positive, α_2 is also finite and positive. The ray emerges as QR, passing through a point I_2, finitely different from J_2.

As α_0 increases, α_2 at first increases with it, but as α_0 reaches the value λ, corresponding to the inclination of the ray I_0L which touches the front focus caustic, α_2 is again zero. Hence between those two values α_2 has at least one *maximum*, and for a given value of α_2 there are at least two values of α_0.

Thus, *within the range of values which are of practical importance*, α_0 is a many-valued function of α_2 having one or more branch-points, of which the one of least modulus corresponds to the first maximum of α_2.

Now, within the same range of values, all the aberrations must be given as single valued functions of α_0, since clearly there can only be one physical emergent ray, corresponding to one given physical incident ray. This statement, as we shall see, needs to be qualified when we are dealing with purely geometrical rays, but this need not affect the present stage of the discussion.

In consequence, if any aberration be expressed in terms of α_2—or of any trigonometrical function of α_2—that aberration must, in general, be a many-valued function of α_2, having for its branch-point of least modulus the first maximum value of α_2 mentioned above. It follows by a well-known result in theory of functions, that no TAYLOR'S series in α_2, or in $\sin \alpha_2$, or $\tan \alpha_2$, can be valid for values of α_2 exceeding this modulus numerically. For such values the series will be definitely divergent.

It is interesting to consider what happens when I_0 is on the other side of F_0, so that we are dealing with a large negative magnification. In this case no real tangent can be drawn from I_0 to the front focus caustic and the value of α_0, for which $\alpha_2 = 0$, is a pure imaginary. But here again, although we are now dealing with imaginary values, we get two values of α_0 for a given (pure imaginary) value of α_2, and, although no such maximum of α_0 occurs in the purely real values, the modulus of the imaginary branch-point limits the validity of TAYLOR'S series in α_2 as before.

Thus there exists always a certain range, extending a finite distance (depending on the nature of the optical system) on either side of the front focus, within which no development of any aberration in powers of α_2 or of its trigonometrical functions (or, indeed, by similar reasoning, of any inclination of the ray, except in the original medium) is valid for the whole pencil of rays which actually traverse the system.

Indeed, as the object point I_0 approaches the front focus, it is clear that both λ and

the maximum α_2 tend to zero, so that only an infinitesimal portion of the rays can be dealt with by the method of successive aberrations, *i.e.*, by the TAYLOR's series.

That the range of failure is by no means an unimportant one is shown by an example given by the authors in a paper read before the Optical Society in December, 1918. In this example the system considered is a positive lens of unit focal length and thickness $\frac{1}{16}$, meniscus shaped, with curvatures 1 and 2·36, and its convex side towards the incoming light. For such a lens and magnification as low as 2, the critical value of α_4 is found to be about 4° 40′, corresponding to a value of α_0 of 13°, whilst the greatest practical value of α_0 is 26°, so that in this case only about $\frac{1}{4}$ of the light going through the lens could be dealt with by series in terms of the emergent angle. From $M = 2$ to $M = \infty$ the conditions are still worse.

As a matter of fact, it appears that in this case the range of magnifications, within which development in terms of the emergent inclinations is possible for all rays travelling through the lens, is restricted to a range lying somewhere between $M = -1$ and $M = 1·5$. This makes it clear that we cannot depend, in the calculation of the aberrations of an optical system, upon any series with the emergent inclination as argument. This is important, because from other considerations it would have been valuable to have been able to express the equation of the emergent ray in the form

$$y + qx = f(q)$$

where q is the inclination of the emergent ray, and to proceed to obtain successive approximations to the caustic by developing $f(q)$ in powers. It now appears that this is not, in general, legitimate.

We now come to the consideration of series proceeding by powers of α_0, or of its trigonometrical functions. Here the question of many-valuedness will not occur, except as follows.

If we consider a ray impinging upon a spherical refracting surface, this ray, if produced, will meet the surface at a second point. Treating the problem from the purely analytical standpoint, this second point is also one at which refraction takes place, and thus, for the same α_0, there will, in general, be two values of α_2, four of α_4, and so on. α_{2n} will therefore, in general, be a multiple-valued function of α_0, and the aberrations will also be multiple-valued functions, and the branch-points of these multiple-valued functions will, as before, limit the convergency of the TAYLOR series.

Now clearly two branches coincide whenever there occurs a grazing incidence; and, therefore, if the system be so arranged (as it almost necessarily is) so that no grazing incidence is reached, there will be no *real* branch-points within the range of practical values. But this does not mean that the TAYLOR's series will necessarily be valid, for there might be imaginary branch-points. A very simple example will show how such branch-points can occur.

If I_0 be a source of light placed in front of a plate of thickness c_2 and refractive index n, the perpendicular from I_0 on the plate being the axis of the system, it is easily verified that the longitudinal spherical aberration

$$J_4I_4 = c_2\left(\frac{1}{n} - \frac{\tan\alpha_2}{\tan\alpha_0}\right),$$

where $\sin\alpha_0 = n\sin\alpha_2$, so that

$$J_4I_4 = \frac{c_2}{n}\left(1 - \frac{\sqrt{1-\sin^2\alpha_0}}{\sqrt{1-\frac{\sin^2\alpha_0}{n^2}}}\right).$$

The branch-points here correspond to $\alpha_0 = \frac{1}{2}\pi$ or $\alpha_2 = \frac{1}{2}\pi$, *i.e.*, to grazing incidence at the first or second surface respectively.

Clearly if $n > 1$, then, since $\sin^2\alpha_0 \leq 1$, the second grazing incidence can never occur for real values of α_0.

But if we take as our argument $t_0 = \tan\alpha_0$, which removes the first branch-point to infinity, we find

$$J_4I_4 = \frac{c_2}{n}\left(1 - \frac{1}{\sqrt{1+t_0^2\left(1-\frac{1}{n^2}\right)}}\right),$$

and this has imaginary branch-points where $t_0 = \pm\dfrac{in}{\sqrt{(n^2-1)}}$. The radius of convergence of the TAYLOR'S series in t_0 is therefore given by $t_0 = \dfrac{n}{\sqrt{(n^2-1)}}$, a value which does not correspond to any physical limitation of the rays. This applies to both the longitudinal and the transverse spherical aberrations in this case.

The above example also brings out another important point; for if in it $\sin\alpha_0$ is taken as the argument, the branch-points are ± 1, $\pm n$; both of which correspond to definite physical limitations, viz., grazing incidence and total internal reflection, so that in this case the limitations of the TAYLOR'S series are also the limitations of the problem.

We see then that the validity even of the expansion in α_0 may be limited by the existence of branch-points, and that the choice of the particular trigonometrical function in which we expand may exercise a considerable influence on the result.

The limitation of the α_0 developments due to branch-points will not, however, as in the case of the α_2 developments, lead to vanishing radii of convergence. There is always a finite region within which these developments may be used. In what follows, therefore, we have exclusively used α_0 as argument.

In dealing with the longitudinal spherical aberration another limitation presents itself. We have seen that if $\alpha_0 = \lambda$ (fig. 2), $\alpha_2 = 0$. It follows that the intersection of the emergent ray with the axis is then at infinity, or the longitudinal aberration is

infinite and afterwards changes sign. Thus the values $\alpha_0 = \pm\lambda$ correspond to poles of the longitudinal aberration. These poles, being the singularities of least modulus, govern the convergence of the TAYLOR's series in this neighbourhood, and this radius of convergence tends to zero as the object approaches the front focus. It was a consideration of this difficulty which primarily led us to put the longitudinal aberration in a new form.

This difficulty does not arise with the transverse spherical aberration. The poles of the longitudinal spherical aberration are due to the zeros of α_0, and on multiplying by $\frac{\tan \alpha_0}{\mathbf{M}}$ to get the transverse aberration, these poles disappear.

§ 4. *Summary of Method and Results.*

The general principle of the method employed was suggested by an attempt to fit an empirical formula to the longitudinal aberration of a lens for a certain range of curvatures and object and image positions. This empirical formula was discussed by the authors in a paper recently read before the Optical Society* and was found to give, on the whole, a singularly good fit. Briefly stated, the formula is of the following type :—

$$\Delta x = \frac{\mathrm{A}t^2}{1+\mathrm{B}t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where t is the slope of the emergent "Gaussian" ray, so that $t = t_0/\mathbf{M}$. A is the (known) theoretical constant of the first-order aberration, which is a quartic in the magnification, and B is a cubic in the magnification, the coefficients in which are determined empirically. This formula was found to give a good approximation, even when the magnification was high and we were working well outside the limits of convergency of the TAYLOR's series for Δx.

If we consider any given object point, the longitudinal spherical aberration will be a function of t_0, that is, of t. Denoting it by $f(t)$, the reasoning of the preceding section shows that $f(t)$ is always one-valued for a finite (and generally quite considerable) range of t, but it is not regular, having poles at $t = \pm\tau$, where $\tau = \tan\lambda/\mathbf{M}$ and becomes rapidly small as the magnification increases numerically.

If, however, we write $(1-t^2/\tau^2)f(t) \equiv \phi(t)$, $\phi(t)$ is now limited only by the original branch-points of $f(t)$ and will, in general, have an adequate radius of convergence. We may therefore expand it in a TAYLOR's series, and we get for $f(t)$ the form

$$\Delta x = f(t) = \frac{at^2+bt^4+ct+\ldots}{1-t^2/\tau^2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* BAKER and FILON, "On an Empirical Formula for the Longitudinal Spherical Aberrations in a Thick Lens," 'Proceedings of the Optical Society,' December, 1918.

If the series in the numerator converges rapidly, it will be sufficient, provided a is not near zero, to stop at the first term, and we get as an approximate formula

$$\Delta x = \frac{at^2}{1-t^2/\tau^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is of the same form as (1).

We have necessarily $a = \mathrm{A}$, and if the two formulæ are to tally we should have in addition $\mathrm{B} = -1/\tau^2$.

The formula in the form (3), however, is not rigorously correct to the second order of aberrations inclusive, unless b happens to be small. If we wish to retain second order terms complete, we have to use

$$\Delta x = \frac{at^2+bt^4}{1-t^2/\tau^2} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and this can be written, to the same order of *algebraic* approximation, in the form

$$\Delta x = at^2/\{1-(1/\tau^2+b/a)\, t^2\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

provided again a is not zero.

If this form (5) is adopted, then the B of the empirical formula should be $1/\tau^2+b/a$. But if this is done, the formula suffers from two defects: (i) it fails whenever a is near zero; (ii) it does not give exact compensation for the poles in the critical range for M large.

The further question then arose: how far are formulæ of type (4) or (5) suitable for dealing with *combinations* of surfaces or lenses? An important guiding consideration, in all work of this kind, must be the relative simplicity of the formulæ in passing from a single surface or lens to a combination, and whether these formulæ are suitable for tracing the effect of individual surfaces or lenses upon the final result.

We have ultimately been led to the conclusion that no *single* formula can satisfy completely the three ideal requirements, viz.: (i) exact agreement with development as far as the second order inclusive; (ii) simplicity in dealing with combinations; (iii) exact compensation of the poles in the critical range of M.

The method finally adopted satisfies conditions (i) and (ii). It only satisfies (iii) approximately. Numerical calculations show that numerically the approximation is adequate in the case of a lens or a simple surface. In the case of more complicated systems we have, as yet, no numerical data.

The first part of the investigation deals with the single refraction. It is there shown that the longitudinal aberration can be put into the form (1), *i.e.*, $\Delta x = \mathrm{A}t^2/(1+\mathrm{B}t^2)$, where the formula is correct to the second order inclusive.

We also find, for the inclination of the emergent ray, the formula

$$q = t\,(1+\mathrm{B}t^2)/(1+\mathrm{C}t^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is correct to the first order when q and t are tangents and to the second order when q and t are sines.

In the above A, B, C are polynomials in M of degrees 4, 3, 2 respectively, so that the empirical formula is well justified for the simple refracting surface. In this case, too, it is possible to calculate τ directly, and, in fact, a simple geometrical construction is given for it. When this is followed for varying image-positions, it is found that outside a certain range of M, the τ so obtained becomes irrelevant, and that, in fact, if the correct factor $1-t^2/\tau^2$ is retained in the denominator of Δx, although it improves the fit by removing singularities in the range round $M = \infty$, it introduces entirely fictitious singularities in other and important parts of the range, and makes the formula worthless.

A good deal of light is thrown upon the problem when it is found that, if we develop $\frac{1}{\tau^2}$ in descending powers of M in the neighbourhood of $M = \infty$, the two leading terms are discovered to be identical with the two leading terms of the cubic B, previously obtained. This makes our B approximate more and more closely to $\frac{1}{\tau^2}$ precisely as the effect of the denominator term becomes more important, and it is this fact which is the key to the numerical value of the method.

We then proceed to show how the constants for a combination of the two systems can be obtained from the corresponding constants of the individual systems. In doing this it appears that, so soon as we pass from the single refracting surface to the lens, a new constant is introduced into the formula, which now takes the form

$$\Delta x = \frac{At^2 + Et^4}{1 + Bt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where A and B are of the same form as before, but E is now a polynomial of degree 6 in M. In the case of a lens the term Et^4 is found to be, in general, of small importance, which accounts for the good fit of the empirical formula.

The formula (7) *for a combination* holds good to the second order inclusive, and B agrees with $\frac{1}{\tau^2}$, when M is large, as far as the leading term only. For numerical purposes, however, a correction is discussed, which is very readily applied, and which makes the two leading terms in B agree with the two leading terms in $\frac{1}{\tau^2}$, as in the case of the single refracting surfaces.

The formulæ for combining two systems take comparatively simple forms; the A, B and C for the combination are expressed as linear functions of the A's, B's and C's of the components, and the E as a lineo-linear function of the A's, B's and C's of the components, each term involving a product of which one factor belongs to one

component of the combination and the other factor to the second. In addition the E for the combination involves linear terms in the E's of the components.

These results are found to hold good in the more general case of the combination of three or more systems. It will follow that if the constants A, B, C, E are tabulated for lenses of various curvatures, the effect of the combination can be traced relatively easily and the aberrations corrected, so far as possible, by suitably bending the lenses, while keeping the general arrangement and the magnifications the same.

Explicit values of the constants for the single refracting surface and a single thick or thin lens have been obtained and are tabulated for reference, so as to be available for eventual computation of the required tables. We have also given some numerical values for a single lens, and a numerical test of the accuracy in this case, which works out at about $\frac{1}{500}$ of the total aberration for the range of cases taken.

The corresponding formulæ with $\sin\gamma$ instead of $\tan\beta$ as argument are discussed, and it is shown that the equations of combination are of the same form as before.

Certain invariant relations between the coefficients in A, B, C, E are developed, which enable various calculations to be simplified and in particular to determine these constants for a system reversed, when they are known for the direct system. This will generally halve the work of tabulation.

§ 5. *The Single Refracting Surface.*

Using the general notation described in § 2, consider refractions at a single refracting surface.

Let ψ_0, ψ_2 denote the angles of incidence and refraction, so that $\psi_0 = C_1P_1I_0$, $\psi_2 = C_1P_1I_2$ (fig. 1).

Let $C_1I_0 = X_0 \doteq x_0$, $C_1I_2 = X_2 = x_2+\Delta x_2$, $C_1J_2 = x_2$ we then have the set of refraction equations

$$\sin\psi_0 = p_0/r_1 = x_0 \sin\alpha_0/r_1 \qquad (8)$$

$$\sin\psi_2 = p_2/r_1 = X_2 \sin\alpha_2/r_1 \qquad (9)$$

$$n_2 p_2 = n_0 p_0 \qquad (10)$$

$$\alpha_2-\alpha_0 = \psi_0-\psi_2 \qquad (11)$$

Let $\xi_0 = A'_1I'_0$, $\xi_2 = A'_1J'_2$ in the "equivalent" Gaussian system (see § 2). Then

$$n_0\xi_0 = x_0+r_1, \qquad n_2\xi_2 = x_2+r_1, \qquad (12)$$

and we find, using the first approximation when α_0, &c., are small

$$1/\xi_2-1/\xi_0 = 1/f_1. \qquad (13)$$

If $\mathbf{M}_1 = \underset{\alpha_0 \to 0}{\mathbf{L}} \alpha_0/\alpha_2$, which we shall call the *ray magnification*, we have

$$\mathbf{M}_1 = n_2\xi_2/n_0\xi_0 = n_2x_2/n_0x_0. \quad \text{(14)}$$

The transverse magnification M_1 is given by

$$\mathrm{M}_1 = \xi_2/\xi_0 = x_2/x_0 = n_0\mathbf{M}_1/n_2. \quad \text{(15)}$$

We can also express (13) in another well-known form, namely,

$$n_0/x_2 - n_2/x_0 = 1/f_1, \quad \text{(16)}$$

whence, using (14), we obtain

$$\left.\begin{aligned} x_0 &= n_2 f_1 (1-\mathbf{M}_1)/\mathbf{M}_1 \\ x_2 &= n_0 f_1 (1-\mathbf{M}_1) \end{aligned}\right\} \quad \text{(17)}$$

Again, from (9), (8) and (10)

$$\begin{aligned} \mathrm{X}_2 = p_2/\sin\alpha_2 = n_0x_0\sin\alpha_0/n_2\sin\alpha_2 &= n_0f_1(1-\mathbf{M}_1)\sin\alpha_0/\mathbf{M}_1\sin\alpha_2 \\ &= x_2\sin\alpha_0/\mathbf{M}_1\sin\alpha_2, \quad \text{(18)} \end{aligned}$$

and the longitudinal aberration

$$\Delta x_2 = \mathrm{X}_2 - x_2 = x_2(\sin\alpha_0/\mathbf{M}_1\sin\alpha_2 - 1). \quad \text{(19)}$$

The corresponding longitudinal aberration in the equivalent Gaussian system is found from

$$\Delta\xi_2 = \Delta x_2/n_2 = (n_0 f_1/n_2)(1-\mathbf{M}_1)(\sin\alpha_0/\mathbf{M}_1\sin\alpha_2 - 1). \quad \text{(20)}$$

Now from (11)

$$\begin{aligned} \sin\alpha_2 = {} & \sin\alpha_0\cos\psi_0\cos\psi_2 + \sin\psi_0\cos\alpha_0\cos\psi_2 - \sin\psi_2\cos\alpha_0\cos\psi_0 \\ & + \sin\alpha_0\sin\psi_0\sin\psi_2, \end{aligned}$$

whence, using (8), (9) and (10),

$$\begin{aligned} \sin\alpha_2/\sin\alpha_0 = {} & \{1-(x_0/r_1)^2\sin^2\alpha_0\}^{\frac{1}{2}}\{1-(n_0x_0/n_2r_1)^2\sin^2\alpha_0\}^{\frac{1}{2}} \\ & + (x_0/r_1)\{1-\sin^2\alpha_0\}^{\frac{1}{2}}\{1-(n_0x_0/n_2r_1)^2\sin^2\alpha_0\}^{\frac{1}{2}} \\ & - (n_0x_0/n_2r_1)\{1-\sin^2\alpha_0\}^{\frac{1}{2}}\{1-(x_0/r_1)^2\sin^2\alpha_0\}^{\frac{1}{2}} \\ & + (n_0x_0^2/n_2r_1^2)\sin^2\alpha_0, \end{aligned}$$

and developing this in ascending powers of $\sin\alpha_0$, we obtain, retaining only terms of fourth degree

$$\begin{aligned} \sin\alpha_2/\sin\alpha_0 = {} & 1 + x_0(n_2-n_0)/n_2r_1 - \tfrac{1}{2}\mathrm{P}\sin^2\alpha_0 \\ & - \tfrac{1}{8}\mathrm{P}\sin^4\alpha_0\{(n_2+n_0)^2x_0^2 + n_2(r_1-x_0)(n_2r_1+n_0x_0)\}/n_2^2r_1^2, \end{aligned}$$

where

$$\mathrm{P} = (1-n_0/n_2)(x_0/r_1)(1+x_0/r_1)(1-n_0x_0/n_2r_1),$$

and is the quantity whose vanishing gives the aplanatic points and must therefore be a factor of every coefficient after the first in the development of $\sin\alpha_2$ in powers of $\sin\alpha_0$.

If we write for shortness

$$Q \equiv (1+n_0/n_2)^2 x_0^2/r_1^2 + (1-x_0/r_1)(1+n_0 x_0/n_2 r_1),$$

and remember that

$$1+x_0(n_2-n_0)/n_2 r_1 = 1/\mathbf{M}_1,$$

we find

$$\begin{aligned}\sin\alpha_2/\sin\alpha_0 &= 1/\mathbf{M}_1 - \tfrac{1}{2}\mathrm{P}\sin^2\alpha_0/(1-\tfrac{1}{4}\mathrm{Q}\sin^2\alpha_0)\\ &= \mathbf{M}_1^{-1}\{1-(\tfrac{1}{2}\mathrm{P}\mathbf{M}_1+\tfrac{1}{4}\mathrm{Q})\sin^2\alpha_0\}/(1-\tfrac{1}{4}\mathrm{Q}\sin^2\alpha_0)\\ &= \mathbf{M}_1^{-1}\{1+\mathbf{B}\sin^2\alpha_0/\mathbf{M}_1^2\}/\{1+\mathbf{C}\sin^2\alpha_0/\mathbf{M}_1^2\} \quad \ldots \quad (21)\end{aligned}$$

correct as far as the second order inclusive, where

$$\mathbf{B} = -\tfrac{1}{2}\mathrm{P}\mathbf{M}_1^3 - \tfrac{1}{4}\mathrm{Q}\mathbf{M}_1^2, \qquad \mathbf{C} = -\tfrac{1}{4}\mathrm{Q}\mathbf{M}_1^2,$$

from which, after some reductions

$$\mathbf{C} = -\tfrac{1}{4}(n_2-n_0)^{-2}\{(n_2^2+n_0n_2+n_0^2) - 3(n_0^2+n_2^2)\mathbf{M}_1 + 3(n_2^2-n_0n_2+n_0^2)\mathbf{M}_1^2\}, \quad (22)$$

and

$$\begin{aligned}\mathbf{B} &= \tfrac{1}{2}(n_2-n_0)^{-2}(1-\mathbf{M}_1)(n_2-n_0\mathbf{M}_1)(n_0-n_2\mathbf{M}_1)+\mathbf{C}\\ &= \tfrac{1}{4}(n_2-n_0)^{-2}\{-(n_2^2-n_0n_2+n_0^2)+(n_2-n_0)^2\mathbf{M}_1-(n_2^2+n_0^2-5n_0n_2)\mathbf{M}_1^2-2n_0n_2\mathbf{M}_1^3\}. (23)\end{aligned}$$

Returning to equation (19) and using (21)

$$\begin{aligned}\Delta x_2 &= x_2(\mathbf{C}-\mathbf{B})\mathbf{M}_1^{-2}\sin^2\alpha_0/\{1+\mathbf{B}\mathbf{M}_1^{-2}\sin^2\alpha_0\}\\ &= n_2 f_1 \mathrm{A}\mathbf{M}_1^{-2}\sin^2\alpha_0/\{1+\mathbf{B}\mathbf{M}_1^{-2}\sin^2\alpha_0\}, \quad \ldots \quad (24)\end{aligned}$$

where

$$\begin{aligned}\mathrm{A} &= x_2(\mathbf{C}-\mathbf{B})/n_2 f_1\\ &= -\tfrac{1}{2}(n_2-n_0)^{-2}(n_0/n_2)(1-\mathbf{M}_1)^2(n_2-n_0\mathbf{M}_1)(n_0-n_2\mathbf{M}_1)\\ &= \tfrac{1}{2}(n_2-n_0)^{-2}(n_0/n_2)\left\{\begin{array}{l}-n_0n_2+(n_2+n_0)^2\mathbf{M}_1-2(n_2^2+n_0n_2+n_0^2)\mathbf{M}_1^2\\ \quad +(n_2+n_0)^2\mathbf{M}_1^3-n_0n_2\mathbf{M}_1^4\end{array}\right\}. \quad (25)\end{aligned}$$

All the above formulæ are correct to the second order of aberrations inclusive. We note that A, **B** and **C** are polynomials of degree 4, 3 and 2 in the magnification respectively.

If we express the aberration in terms of tangents instead of sines we have at once

$$\begin{aligned}\Delta x_2 &= n_2 f_1 \mathrm{A}\mathbf{M}_1^{-2}\tan^2\alpha_0/\{1+(\mathbf{B}+\mathbf{M}_1^2)\mathbf{M}_1^{-2}\tan^2\alpha_0\}\\ &= n_2 f_1 \mathrm{A}\tan^2\beta_2/\{1+\mathrm{B}\tan^2\beta_2\} \quad \ldots \quad (26)\end{aligned}$$

where

$$\begin{aligned} \mathrm{B} &= \mathbf{B}+\mathbf{M}_1^2 \\ &= \tfrac{1}{4}(n_2-n_0)^{-2}\{-(n_2^2-n_0n_2+n_0^2)+(n_2-n_0)^2\mathbf{M}_1+3(n_2^2-n_0n_2+n_0^2)\mathbf{M}_1^2-2n_0n_2\mathbf{M}_1^3\}. \quad (27) \end{aligned}$$

When, however, we come to develop a formula for tangents, similar to (21) for sines, it is found that, in the series for $\tan\alpha_2/\tan\alpha_0$, we do not have all the coefficients after the first vanishing together; for, even at the aplanatic points, the tangent ratio is not constant.

We can, indeed, write

$$\tan\alpha_2/\tan\alpha_0 = \mathbf{M}_1^{-1}(1+\mathrm{B}\mathbf{M}_1^{-2}\tan^2\alpha_0)/(1+\mathrm{C}\mathbf{M}_1^{-2}\tan^2\alpha_0), \quad . \quad . \quad . \quad (28)$$

and choose the coefficients B and C so that the developments shall agree as far as terms in $\tan^4\alpha_0$ inclusive. This can, in general, be done in one way only. But we then find that B and C are no longer integral functions of the magnification. Their infinities have to be taken into account, and generally the method becomes complicated and unsatisfactory.

From other considerations, however, it appears that since the zeroes of α_2 must be the same as the poles of Δx_2, the B in (28) must be the same as the B in (26), and this will fix C as follows:—

$$\begin{aligned} \tan^2\alpha_2 &= \sin^2\alpha_2/(1-\sin^2\alpha_2) \\ &= \frac{\mathbf{M}_1^{-2}\sin^2\alpha_0\{1+\mathbf{B}\sin^2\alpha_0/\mathbf{M}_1^2\}^2}{\{1+\mathbf{C}\sin^2\alpha_0/\mathbf{M}_1^2\}^2-\mathbf{M}_1^{-2}\sin^2\alpha_0\{1+\mathbf{B}\sin^2\alpha_0/\mathbf{M}_1^2\}^2} \\ &= \frac{\mathbf{M}_1^{-2}\tan^2\alpha_0\{1+(\mathbf{B}+\mathbf{M}_1^2)\mathbf{M}_1^{-2}\tan^2\alpha_0\}^2}{(1+\tan^2\alpha_0)\{1+(\mathbf{C}+\mathbf{M}_1^2)\mathbf{M}_1^{-2}\tan^2\alpha_0\}^2-\mathbf{M}_1^{-2}\tan^2\alpha_0\{1+(\mathbf{B}+\mathbf{M}_1^2)\mathbf{M}_1^{-2}\tan^2\alpha_0\}^2}, \end{aligned}$$

substituting from (21).

Hence, taking the square root and developing the denominator in powers of $\tan\beta_2$, *i.e.*, of $\mathbf{M}_1^{-1}\tan\alpha_0$,

$$\begin{aligned} \tan\alpha_2/\tan\beta_2 &= \frac{(1+\mathrm{B}\tan^2\beta_2)}{1+\tan^2\beta_2(\mathbf{C}+\frac{3}{2}\mathbf{M}_1^2-\frac{1}{2})+\frac{1}{2}\tan^4\beta_2\{\mathbf{C}(1+\mathbf{M}_1^2)+\frac{3}{4}\mathbf{M}_1^4+\frac{3}{2}\mathbf{M}_1^2-\frac{1}{4}-2\mathrm{B}\}} \\ &= (1+\mathrm{B}\tan^2\beta_2)/(1+\mathrm{C}\tan^2\beta_2+\mathrm{D}\tan^4\beta_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29) \end{aligned}$$

where

$$\left.\begin{aligned} \mathrm{C} &= \mathbf{C}+\tfrac{3}{2}\mathbf{M}_1^2-\tfrac{1}{2} \\ \mathrm{D} &= \mathbf{C}(1+\mathbf{M}_1^2)+\tfrac{3}{4}\mathbf{M}_1^4+\tfrac{3}{2}\mathbf{M}_1^2-\tfrac{1}{4}-2\mathrm{B} \\ &= \mathrm{C}(1+\mathbf{M}_1^2)+\tfrac{1}{4}(1-\mathbf{M}_1^2)(1+3\mathbf{M}_1^2)-2\mathrm{B} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

(29) is now correct as far as the second order inclusive.

If we only require the tangent ratio correct up to the first order of aberrations, we have the formula

$$\tan\alpha_2 = \tan\beta_2(1+\mathrm{B}\tan^2\beta_2)/(1+\mathrm{C}\tan^2\beta_2) \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The value of C, when written out fully, is given by

$$C = \tfrac{3}{4}(n_2-n_0)^{-2}\{-(n_2^2-n_0n_2+n_0^2)+(n_0^2+n_2^2)\mathbf{M}_1+(n_0^2-3n_0n_2+n_2^2)\mathbf{M}_1^2\}. \quad (32)$$

§ 6. *The Convergency Factor and the Singular Inclination for a Single Refracting Surface.*

Having now obtained expressions (24 and 26) for the longitudinal spherical aberration, which are correct to the second order of aberrations, when expansion in powers of $\sin\alpha_0$ or $\tan\alpha_0$ is legitimate and rapidly convergent, we have now to enquire how far the same expression remains valid as $\mathbf{M}$ increases, in which case we know that B or $\mathbf{B}$ increases without limit and the convergency fails, even for comparatively small values of α_0.

Here it will be convenient to introduce two definitions :—

I. We shall call *singular inclination* the value λ (see § 3) of α_0 for which the emergent ray is parallel to the axis.

II. The factor $1-\sin^2\alpha_0/\sin^2\lambda$ (or $1-\tan^2\alpha_0/\tan^2\lambda$ if we are dealing with tangents) we shall call the *convergency factor*. If we multiply Δx by the convergency factor we remove those singularities of Δx which are instrumental in causing critical failure of convergency.

To find the singular inclination and convergency factor for a single refracting surface, we have to find when $\alpha_2 = 0$.

Going back to the fundamental equations (8) to (11) we have $\alpha_2 = 0$ when $\alpha_0 = \lambda$, where

$$\lambda = \psi_2-\psi_0$$

which leads to

$$\begin{aligned}\sin\lambda &= \sin\psi_2\cos\psi_0-\sin\psi_0\cos\psi_2\\ &= (n_0x_0\sin\lambda/n_2r_1)\sqrt{\{1-(x_0\sin\lambda/r_1)^2\}}-(x_0\sin\lambda/r_1)\sqrt{\{1-(n_0x_0\sin\lambda/n_2r_1)^2\}}.\end{aligned}$$

Hence, either $\sin\lambda = 0$, which obviously refers to the axial ray, a trivial and (for our purpose) irrelevant solution, or

$$r_1/x_0 = (n_0/n_2)\sqrt{\{1-(x_0\sin\lambda/r_1)^2\}}-\sqrt{\{1-(n_0x_0\sin\lambda/n_2r_1)^2\}}. \quad (33)$$

On rationalising (33) leads to

$$\begin{aligned}4n_0^2\sin^2\lambda/n_2^2 &= 4r_1^2/x_0^2-(1+r_1^2/x_0^2-n_0^2/n_2^2)^2\\ &= -(1+r_1/x_0-n_0/n_2)(1+r_1/x_0+n_0/n_2)(1-r_1/x_0+n_0/n_2)(1-r_1/x_0-n_0/n_2). \quad (34)\end{aligned}$$

This gives the singular inclination.

If we write

$$R \equiv (1+r_1/x_0-n_0/n_2)(1+r_1/x_0+n_0/n_2)(1-r_1/x_0+n_0/n_2)(1-r_1/x_0-n_0/n_2), \quad (35)$$

it follows that

$$1+4n_0^2\sin^2\alpha_0/n_2^2\mathrm{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

is the required convergency factor.

If the formulæ of § 5 are to get accurately over the failure of convergency, this convergency factor should be identical with

$$1+\mathbf{B}\sin^2\alpha_0/\mathbf{M}_1^2,$$

that is, we should have

$$\mathbf{B}=4n_0^2\mathbf{M}_1^2/n_2^2\mathrm{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

which, when written out, becomes

$$\mathbf{B}=4n_0^2n_2^2\mathbf{M}_1^2(1-\mathbf{M}_1)^4/(n_2-n_0)^2(1-2\mathbf{M}_1)(n_2+n_0-2n_0\mathbf{M})(n_2+n_0-2n_2\mathbf{M}_1). \quad (38)$$

This does not agree with the previously found value for $\mathbf{B}$, being of fractional form in $\mathbf{M}_1$. It does lead to $\mathbf{B}$ becoming infinite of the order $\mathbf{M}_1^3$ when $\mathbf{M}_1$ tends to infinity, but it indicates an infinity of $\mathbf{B}$ (and therefore a critical failure of convergency) at three other places, namely when $\mathbf{M}_1=\frac{1}{2}$, $\frac{1}{2}(n_0+n_2)/n_0$, $\frac{1}{2}(n_0+n_2)/n_2$, at none of which does a failure of convergency really occur, as can readily be verified.

The reason for this is made clearer by geometrical reasoning as follows:—

Let $\mathrm{P_1Q_1}$ (fig. 3) be a ray which is parallel to the axis in medium 2. To make the figure easier and the quantities dealt with positive, the refraction has been taken from a denser to a rarer medium, so that $n_0/n_2>1$.

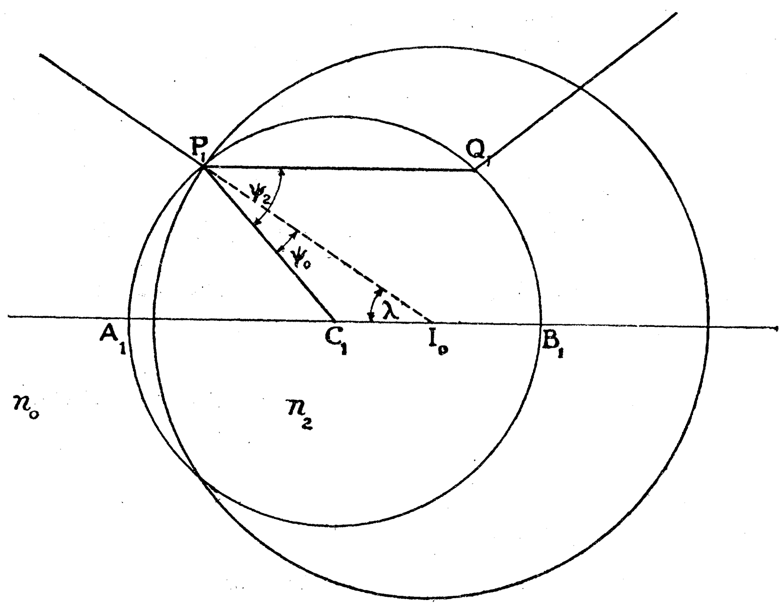

Fig. 3.

In the triangle $\mathrm{I_0C_1P_1}$ of fig. 3 we have $\sin\psi_2/\sin\psi_0=\mathrm{I_0P_1}/x_0$.
But

$$\sin\psi_2/\sin\psi_0=n_0/n_2\,;\quad\text{hence}\quad \mathrm{I_0P_1}=n_0x_0/n_2.$$

The point P_1 and singular inclination λ can therefore be constructed geometrically as follows.

With I_0 as centre and radius $n_0 . C_1I_0/n_2$ describe a circle meeting the refracting surface at P_1; $C_1I_0P_1$ is the angle λ required.

This angle λ approaches zero, that is, we get a critical failure of convergency, when the two circles approach contact at A_1. The limiting case is, therefore, when $A_1I_0 = n_0 . C_1I_0/n_2$ or I_0 divides A_1C_1 *externally* in the ratio $n_0 : n_2$.

When this happens $r_1+x_0 = n_0x_0/n_2$ leading to $\mathbf{M}_1 = \infty$, a case of true critical failure. But clearly, by symmetry, we get a precisely similar result when P_1Q_1 is due to a ray entering the surface at Q_1, and travelling backward through medium 2. In this case the limiting position of I_0 divides B_1C_1 externally in the ratio $n_0 : n_2$ and is defined by $r_1-x_0 = -n_0x_0/n_2$ or $\mathbf{M}_1 = \frac{1}{2}$.

This case would correspond, analytically, to $\alpha_2 = \pi$, and the corresponding equation (33) would become

$$-r_1/x_0 = (n_0/n_2) \sqrt{\{1-(x_0 \sin \lambda/r_1)^2\}} - \sqrt{\{1-(n_0x_0 \sin \lambda/n_2r_1)^2\}}.$$

Now if we examine (34) we find that in the process of clearing roots, r_1/x_0 appears squared in the final result, which accordingly includes both $\alpha_2 = 0$ and $\alpha_2 = \pi$. If we write $r_1^2/x_0^2 = u$, then we should really write equation (33) in the form

$$\sqrt{u} = (n_0/n_2) \sqrt{(1-\sin^2 \lambda/u)} - \sqrt{(1-n_0^2 \sin^2 \lambda/n_2^2u)}, \quad . \quad . \quad . \quad . \quad (39)$$

and the two cases are discriminated by assigning to $\sqrt{u}$ one or the other sign. One of these cases is necessarily irrelevant since refraction at the posterior surface of the sphere is physically excluded.

Further, if we consider the other two values which make $R = 0$, viz.,

$$\mathbf{M}_1 = (n_0+n_2)/2n_0 \text{ and } \mathbf{M}_1 = (n_0+n_2)/2n_2,$$

they correspond to

$$r_1+x_0 = -n_0x_0/n_2$$

and

$$r_1-x_0 = n_0x_0/n_2,$$

i.e., to positions of I_0 in which it divides A_1C_1 and B_1C_1 *internally* in the ratio $n_0 : n_2$. But these belong geometrically to the limit of cases in which the incident and refracted rays lie on opposite sides of the normal, *i.e.*, to a negative refractive index. And indeed they are obtained from the two previous points by reversing the sign of n_0/n_2.

Here again, examination of (34) shows that n_0/n_2 appears squared in it. Therefore (34) includes the cases in question. These, however, may be obtained analytically by changing ψ_0, or ψ_2, into its supplement, *i.e.*, by reversing the sign of $\cos \psi_0$ or $\cos \psi_2$ or by changing the determination of the sign of one or other of the square roots on the right-hand side of (39).

The cases are discriminated by the vanishing of these square roots, which occurs when $\cos\psi_0$ or $\cos\psi_2 = 0$.

It appears, therefore, that the vanishing of the factors in the denominator of (38) is wholly irrelevant, and, if we adopted for **B** the value given on the right-hand side of that equation, we should thereby be introducing, in the neighbourhood of $\mathbf{M}_1 = \frac{1}{2}$, $(n_0+n_2)/2n_0$, $(n_0+n_2)/2n_2$ entirely irrelevant singularities, which would make the formula worthless.

The question arises, what is the range of values of $\mathbf{M}_1$ for which the equation

$$4n_0^2 \sin^2\lambda/n_2^2 = -\mathrm{R}$$

is valid and legitimate?

If we start from $\mathbf{M}_1 = \infty$, which corresponds to a real case, the signs of the square roots in (39) are well determined, and the correspondence between $\sin^2\lambda$ and $\mathbf{M}_1$ is unique and definite and can be continued until we reach a point where one case passes into another. These cases we have found to be the branch-points of the three square roots, namely :—

$$u = 0, \qquad \cos\psi_0 = 0 \quad \text{and} \quad \cos\psi_2 = 0.$$

$$u = 0 \quad \text{leads to} \quad x_0 = \infty \text{ or } \mathbf{M}_1 = 0 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(A)}$$

$\cos\psi_0 = 0$ leads to $\sin^2\lambda = u$, or, using the first form of (34)

$$4n_0^2u/n_2^2 = 4u-(1+u-n_0^2/n_2^2)^2,$$

i.e., $(1-u-n_0^2/n_2^2)^2 = 0$, that is $u = 1-n_0^2/n_2^2$, leading to

$$x_0 = \pm r_1\,(1-n_0^2/n_2^2)^{-\frac{1}{2}} \text{ and } \mathbf{M}_1 = \{1\pm(n_2-n_0)/\sqrt{(n_2^2-n_0^2)}\}^{-1} \quad . \quad . \quad . \qquad \text{(B)}$$

$\cos\psi_2 = 0$ leads to $\sin^2\lambda = n_2^2u/n_0^2$, that is, to

$$u = n_0^2/n_2^2-1,\; x_0 = \pm r_1\,(n_0^2/n_2^2-1)^{-\frac{1}{2}},\; \mathbf{M}_1 = \{1\pm(n_2-n_0)/\sqrt{(n_0^2-n_2^2)}\}^{-1} . \quad . \qquad \text{(C)}$$

If $n_0 > n_2$, both values of $\mathbf{M}_1$ given by (B) are imaginary. The values given by (C) are both positive, $\mathbf{M}_1 = \{1+(n_2-n_0)/\sqrt{(n_0^2-n_2^2)}\}^{-1}$ being the greater.

The range over which we can travel without ambiguity is, therefore, from $\mathbf{M}_1 = +\infty$ to $\mathbf{M}_1 = \{1+(n_2-n_0)/\sqrt{(n_0^2-n_2^2)}\}^{-1}$ and from $\mathbf{M}_1 = -\infty$ to $\mathbf{M}_1 = 0$.

If $n_0 < n_2$, the values of $\mathbf{M}_1$ given by (C) are imaginary, those given by (B) are positive, and $\mathbf{M}_1 = \{1-(n_2-n_0)/\sqrt{(n_2^2-n_0^2)}\}^{-1}$ is the greater, so that the range of validity is from $\mathbf{M}_1 = +\infty$ to $\mathbf{M}_1 = \{1-(n_2-n_0)/\sqrt{(n_2^2-n_0^2)}\}^{-1}$ and from $\mathbf{M}_1 = -\infty$ to $\mathbf{M}_1 = 0$.

Within this range $(1+4n_0^2\sin^2\alpha_0/n_2^2\mathrm{R})$ is the correct convergency factor; outside this range it is irrelevant.

It is clear, then, that we cannot find a single formula for the convergency factor, which will hold for all values of the magnification.

Further, if the factor $(1+4n_0^2 \sin^2 \alpha_0/n_2^2\mathrm{R})$ is introduced into the denominator of Δx_2, we no longer obtain expressions of the simple type (24) and (26), and endless complications are introduced when we come to consider a compound system.

Can we make our expression $\mathbf{B}$ given by (23) give a tolerable approximation to $(4n_0^2\mathbf{M}_1^2/n_2^2\mathrm{R})$ for those regions where the denominator factor is really needed, namely for $\mathbf{M}_1$ large, positively or negatively?

To get the answer to this question we develop $(4n_0^2\mathbf{M}_1^2/n_2^2\mathrm{R})$ in descending powers of $\mathbf{M}_1$.

This is found to be (the most rapid method is to break up first into partial fractions)

$$\tfrac{1}{4}(n_2-n_0)^{-2}\left\{\begin{array}{l} -2n_0n_2\mathbf{M}_1^3-(n_0^2-5n_0n_2+n_2^2)\mathbf{M}_1^2 \\ -[(n_2-n_0)^4+n_0^2n_2^2]\mathbf{M}_1/2n_0n_2-[(n_2-n_0)^3(n_2^3-n_0^3)+n_2^3n_0^3]/4n_0^2n_2^2 \\ \quad +\text{terms in } 1/\mathbf{M}_1, \text{ \&c.} \end{array}\right\}. \quad (40)$$

If we now compare (40) with (23) we find that the most important terms when $\mathbf{M}_1$ is large, namely those in $\mathbf{M}_1^3$ and $\mathbf{M}_1^2$ agree in the two expressions.

We may, therefore, take it that the approximations (21) and (24) which we have seen hold good to the second order when expansion in series is convergent, will probably not be numerically very far out when $\mathbf{M}_1$ has a large value, in which case the normal method of development cannot be used.

It is important, at this stage, and to justify the above assertion, to consider a few numerical examples.

Tables I. and II. give the values of Δx_2 and $\sin\alpha_2$ for a single refracting surface, calculated for a number of values of $\mathbf{M}_1$ and two inclinations in each case. The inclinations are fixed from the perpendicular distance ϖ of A_1 from the incident ray. This, for moderate inclinations, is sensibly the same as the intercept made by the incident ray on the principal plane. ϖ has been given the two values 0·5 and 0·25 in every case, except for $\mathbf{M}_1 = 2$ where $\varpi = 0{\cdot}5$ leads to a physically impossible value. In this case $\varpi = 0{\cdot}25$ and $\varpi = 0{\cdot}125$ have been used to define the ray.

In each case four values have been computed (1) the correct one, from trigonometrical calculation; (2) the values given by formulæ (21) and (24)—these are shown in the column headed "fractional formula"; (3) the values obtained by expansion in series, up to the optician's first order of aberrations inclusive, that is including $\sin^3\alpha_0$ in the development of Δx_2 and $\sin\alpha_2$—these are shown in the column headed "first order"; (4) the same series carried to the second order of aberrations inclusive, *i.e.*, to the terms involving $\sin^5\alpha_0$—these are shown in the column headed "second order." It should be noted that these first and second order approximations are the most accurate that can be obtained, much more so than more usual ones, proceeding in powers of $\sin\alpha_2$ or $\tan\alpha_2$.

TABLE I.—Values of Δx_2 for Single Refracting Surface.

M.	ϖ.	First order.	Second order.	Fractional formula.	True.
10	0·5	− 33·35293	− 107·6252	+ 27·18563	+ 26·66794
10	0·25	− 8·33823	− 12·98025	− 18·81008	− 18·85273
2	0·25	− 0·250000	− 0·378906	− 0·516129	− 0·527526
2	0·125	− 0·062500	− 0·070557	− 0·071749	− 0·071781
0·5	0·5	− 0·015625	− 0·016541	− 0·016598	− 0·016611
0·5	0·25	− 0·003906	− 0·003963	− 0·003964	− 0·003963
0	0·5	− 0·166667	− 0·174769	− 0·175183	− 0·175809
0	0·25	− 0·041667	− 0·042173	− 0·042179	− 0·042189
− 1	0·5	− 1·000000	− 0·947500	− 0·950119	− 0·956680
− 1	0·25	− 0·250000	− 0·246719	− 0·246761	− 0·246838

TABLE II.—Values of Sin α_2 for Single Refracting Surface.

M.	ϖ.	First order.	Second order.	Fractional formula.	True.
10	0·5	+ 0·0250865	+ 0·0454644	+ 0·0576348	+ 0·0610770
10	0·25	− 0·0078936	− 0·0072568	− 0·0071911	− 0·0071828
2	0·25	− 0·2187500	− 0·2065430	− 0·1987180	− 0·1978219
2	0·125	− 0·1210938	− 0·1207123	− 0·1206710	− 0·1206691
0·5	0·5	0·2539063	0·2541962	0·2542195	0·2542230
0·5	0·25	0·1254883	0·1254974	0·1254975	0·1254974
0	0·5	0·1805556	0·1817130	0·1826667	0·1827294
0	0·25	0·0850694	0·0851267	0·0851286	0·0851291
− 1	0·5	0·1250000	0·1299375	0·1311526	0·1314354
− 1	0·25	0·0531250	0·0532793	0·0532873	0·0532884

It appears from the above that the fractional formulæ are not merely equal, but appreciably superior to the second order formulæ, and this not merely in cases such as those of the three first entries in Table I., in which the convergency of the series for Δx_2 is either absent or slow, but in *every case* where the fractional or second order formulæ differ sensibly from the true value. (Clearly a divergence of 1 in the last place cannot be claimed as significant, for the last figure in Tables I. and II. is probably not correct within ± 2, in some cases.) An estimate of the range of the formula can be obtained from the fact that in the cases, $\varpi = 0·5$, $\mathbf{M}_1 = 10$ and 2, the angles of incidence were 52° 34′ and 48° 35′ respectively, and, for the other values, angles of incidence of 20° and 30° are quite common.

In view of this the accuracy of the results is surprising and, from the point of view of the further applications of the method, most encouraging.

§ 7. *Combination of Two Systems.*

Call the systems 1 and 3, and the initial, intermediate and final media 0, 2, 4.

f_1, f_3 are the focal lengths of the systems, as defined in § 2. M_1, $\mathbf{M}_1$ are the transverse and ray magnifications in the first system, M_3, $\mathbf{M}_3$ in the second system.

$\mathrm{M}_3+\Delta\mathrm{M}_3$, $\mathbf{M}_3+\Delta\mathbf{M}_3$ refer to the transverse and ray magnifications in the second system when I_2, the true intersection of ray 2 with the axis, is taken as the object point for the second refraction (instead of J_2, which refers to transverse and ray magnifications M_3, $\mathbf{M}_3$).

Using the notation of §§ 2, 5, we assume

$$\Delta x_2 = n_2 f_1 (\mathrm{A}_1 t_2^2+\mathrm{E}_1 t_2^4)/(1+\mathrm{B}_1 t_2^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

$$q_2 = t_2 (1+\mathrm{B}_1 t_2^2)/(1+\mathrm{C}_1 t_2^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

where $q_2 = \frac{\sin}{\tan}\{\alpha_2\}$, $t_2 = \begin{Bmatrix}\sin\gamma_2\\ \tan\beta_2\end{Bmatrix}$, and B_1, C_1 have suitable forms according as sines or tangents are considered. The constant E_1 is zero if the systems reduce to single refracting surfaces. Its form in the more general case will be discussed later.

If we denote by $\Delta_1 x_4$ that part of Δx_4 which is due to Δx_2 and by $\Delta_3 x_4$ that which is introduced by the aberrations proper to the system 3,

$$\Delta_1 x_4 = -n_4 f_3 \, \Delta\mathrm{M}_3$$

where $\Delta\mathrm{M}_3$ is obtained from Δx_2 by means of

$$\Delta x_2 = n_2 f_3 \{1/(\mathrm{M}_3+\Delta\mathrm{M}_3)-1/\mathrm{M}_3\},$$

leading to

$$\Delta\mathrm{M}_3 = -\mathrm{M}_3^2 \, \Delta x_2/(n_2 f_3+\mathrm{M}_3 \, \Delta x_2).$$

Thus

$$\Delta_1 x_4 = n_4 f_1 \mathrm{M}_3^2 (\mathrm{A}_1 t_2^2+\mathrm{E}_1 t_2^4)/\{1+t_2^2 (\mathrm{B}_1+\mathrm{M}_3 \mathrm{A}_1 f_1/f_3)\} \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

Again

$$\begin{aligned}\Delta_3 x_4 &= \frac{n_4 f_3 \{(\mathrm{A}_3+\Delta\mathrm{A}_3)\, q_2^2 (\mathbf{M}_3+\Delta\mathbf{M}_3)^{-2}+(\mathrm{E}_3+\Delta\mathrm{E}_3)\, q_2^4 (\mathbf{M}_3+\Delta\mathbf{M}_3)^{-4}\}}{1+(\mathrm{B}_3+\Delta\mathrm{B}_3)\, q_2^2 (\mathbf{M}_3+\Delta\mathbf{M}_3)^{-2}} \\ &= n_4 f_3 [\{\mathrm{A}_3\mathbf{M}_3^{-2}+\Delta\mathrm{M}_3 \, d(\mathrm{A}_3\mathbf{M}_3^{-2})/d\mathrm{M}_3\} q_2^2+\mathrm{E}_3 t_2^4 \mathbf{M}_3^{-4}]/(1+\mathrm{B}_3 t_2^2 \mathbf{M}_3^{-2}), \qquad (44)\end{aligned}$$

retaining only terms of second order in t_2^2. Writing now for q_2^2 in the above its "first order" equivalent $t_2^2+2(\mathrm{B}_1-\mathrm{C}_1)\, t_2^4$, we have

$$\begin{aligned}\Delta x_4/n_4 &= \Delta_1 x_4/n_4+\Delta_3 x_4/n_4 \\ &= \frac{\begin{aligned}&\{f_1\mathrm{M}_3^2\mathrm{A}_1+f_3\mathrm{A}_3/\mathbf{M}_3^2\}\, t_2^2+t_2^4 [f_1\{\mathrm{B}_3\mathrm{A}_1\mathrm{M}_3^2\mathbf{M}_3^{-2}+\mathrm{E}_1\mathrm{M}_3^2-\mathrm{A}_1\mathrm{M}_3^2 \, d(\mathrm{A}_3\mathbf{M}_3^{-2})/d\mathrm{M}_3 \\ &+\mathrm{A}_1\mathrm{A}_3\mathrm{M}_3\mathbf{M}_3^{-2}\}+f_3\{\mathrm{B}_1\mathrm{A}_3\mathbf{M}_3^{-2}+2\mathrm{A}_3(\mathrm{B}_1-\mathrm{C}_1)\,\mathbf{M}_3^{-2}+\mathrm{E}_3\mathbf{M}_3^{-4}\}]\end{aligned}}{\{1+t_2^2(\mathrm{B}_1+\mathrm{A}_1\mathrm{M}_3 f_1/f_3)\}\{1+\mathrm{B}_3 t_2^2/\mathbf{M}_3^2\}} \quad . \quad . \quad . \quad . \quad . \qquad (45)\end{aligned}$$

Remembering that $t_2 = \mathbf{M}_3 t_4$ and retaining only the first two terms of the denominator product

$$\Delta x_4/n_4 = (f_{13}A_{13}t_4^2 + f_{13}E_{13}t_4^4)/(1+B_{13}t_4^2) \quad \ldots \quad (46)$$

where

$$f_{13}A_{13} = f_3A_3 + f_1A_1M_3^2\mathbf{M}_3^2 \quad \ldots \quad (47)$$

$$B_{13} = B_3 + B_1\mathbf{M}_3^2 + A_1M_3\mathbf{M}_3^2 f_1/f_3 \quad \ldots \quad (48)$$

$$\begin{aligned} f_{13}E_{13} &= f_3E_3 + f_1E_1M_3^2\mathbf{M}_3^4 \\ &\quad + f_1A_1\mathbf{M}_3^2\{B_3M_3^2 + A_3M_3 - M_3^2\mathbf{M}_3^2\, d\,(A_3\mathbf{M}_3^{-2})/dM_3\} \\ &\quad + f_3A_3\mathbf{M}_3^2\{3B_1' - 2C_1\} \quad \ldots \quad (49) \end{aligned}$$

Again

$$q_4 = \frac{q_2(\mathbf{M}_3+\Delta\mathbf{M}_3)^{-1}\{1+(B_3+\Delta B_3)\,q_2^2(\mathbf{M}_3+\Delta\mathbf{M}_3)^{-2}\}}{1+(C_3+\Delta C_3)\,q_2^2(\mathbf{M}_3+\Delta\mathbf{M}_3)^{-2}},$$

and retaining only terms of order t_2^3

$$\begin{aligned} q_4 &= q_2(\mathbf{M}_3+\Delta\mathbf{M}_3)^{-1}(1+B_3t_2^2\mathbf{M}_3^{-2})/(1+C_3t_2^2\mathbf{M}_3^{-2}) \\ &= t_2\mathbf{M}_3^{-1}(1-\Delta\mathbf{M}_3/\mathbf{M}_3)(1+B_1t_2^2)(1+B_3t_2^2\mathbf{M}_3^{-2})/\{(1+C_1t_2^2)(1+C_3t_2^2\mathbf{M}_3^{-2})\} \\ &= t_2\mathbf{M}_3^{-1}(1+B_1t_2^2+B_3t_2^2\mathbf{M}_3^{-2}+t_2^2A_1M_3f_1/f_3)/(1+C_1t_2^2+C_3t_2^2\mathbf{M}_3^{-2}) \\ &= t_4(1+B_{13}t_4^2)/(1+C_{13}t_4^2), \quad \ldots \quad (50) \end{aligned}$$

where B_{13} has the value given by (48) and

$$C_{13} = C_3 + C_1\mathbf{M}_3^2. \quad \ldots \quad (51)$$

The equations (47), (48), (49), (51), give the constants for the combined system in terms of those for the components. It may appear at first sight as if the choice of the constants B_{13} and E_{13} had been arbitrary, for clearly, if λ be any quantity,

$$\{f_{13}A_{13}t_4^2 + f_{13}(E_{13}+\lambda A_{13})\,t_4^4\}/\{1+(B_{13}+\lambda)\,t_4^2\}$$

will give a development equally valid to the second order. But, if we do this, and we wish to preserve the simple character of the relation (48) giving the B for the combination, λ will have to be a linear function of A_1, A_3, B_1, B_3. λA_{13} must then necessarily contain terms of one or other of the forms A_1B_1, A_1^2, A_3B_3, A_3^2. Thus the new E will contain such terms and will no longer be of type (49) which is linear in the aberration coefficients of each system taken separately. Thus the lineo-linear type of equation for E_{13} requires $\lambda = 0$.

We note also that the equations of combination are identical in form, whether we are dealing with the sine or the tangent of the inclination as argument.

§ 8. *Nature of the Quantities* A, B, C, E *in the General Case of any System.*

In the case of the single refracting surface we found that A, B, C were polynomials of degrees 4, 3, 2 in $\mathbf{M}$, and that E was identically zero.

In addition, for such a surface, equations (23), (25) and (27) show that the coefficient of $\mathbf{M}_1^4$ in A is n_0/n_2 times the coefficient of $\mathbf{M}_1^3$ in $\mathbf{B}$ or B. This may be otherwise stated in the form :—

$$A_1 - n_0\mathbf{M}_1B_1/n_2, \quad i.e., \quad A_1 - M_1B_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(I)}$$

is of the fourth degree only in appearance and reduces to an expression of the third degree in $\mathbf{M}_1$ or M_1.

The same holds good for $A_1 - M_1\mathbf{B}_1$, so this result is independent of whether the sine or tangent is taken as argument. The same remark applies to all the results of the present section and to the other invariant relations shortly to be proved. We may therefore conveniently state it here once for all.

If we now refer to the equations (47), (48), and remember that in any combination—

$$M_{13}/f_3 = M_3/f_{13} - 1/f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

and

$$1/f_1M_{13} = 1/f_{13}M_1 - 1/f_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

with the corresponding equations

$$n_0\mathbf{M}_{13}/f_3 = n_2\mathbf{M}_3/f_{13} - n_4/f_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

$$n_4/f_1\,\mathbf{M}_{13} = n_2/f_{13}\,\mathbf{M}_1 - n_0/f_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

and the obvious conditions

$$\mathbf{M}_{13} = \mathbf{M}_1\mathbf{M}_3\,; \qquad M_{13} = M_1M_3,$$

we note first that, if A_3 is a quartic in $\mathbf{M}_3$ it is also a quartic in M_{13} or $\mathbf{M}_{13}$, and that if A_1 is a quartic in $\mathbf{M}_1$, it becomes, on multiplication by $M_3^2\,\mathbf{M}_3^2$, a quartic in M_{13} or $\mathbf{M}_{13}$, since $M_3 = n_2\mathbf{M}_3/n_4$, and therefore $\mathbf{M}_1^rM_3^2\mathbf{M}_3^2 = n_2^2\mathbf{M}_{13}^r\mathbf{M}_3^{4-r}/n_4^2$, which makes every term in $A_1M_3^2\mathbf{M}_3^2$ a quartic in $\mathbf{M}_{13}$, because $\mathbf{M}_3$ is a linear function of $\mathbf{M}_{13}$ and $4-r$ is here zero or positive.

(47) then shows that A_{13} will be a quartic function of $\mathbf{M}_{13}$, if A_1 and A_3 are quartic functions of $\mathbf{M}_1$ and $\mathbf{M}_3$ respectively. But we know this to be the case for a single refracting surface. Hence it holds good of any system compounded of such surfaces.

Now consider (48). If B_3 is a cubic in $\mathbf{M}_3$ it becomes a cubic in $\mathbf{M}_{13}$.

Again, if $A_1 - M_1B_1 =$ a cubic U_1 in $\mathbf{M}_1$

$$\begin{aligned}\mathbf{M}_3^2\,(B_1 + A_1M_3f_1/f_3) &= \mathbf{M}_3^2\,(B_1\{1 + M_{13}f_1/f_3\} + U_1M_3f_1/f_3) \\ &= \mathbf{M}_3^2M_3f_1\,(B_1/f_{13} + U_1/f_3),\end{aligned}$$

using (52), and by reasoning similar to the one given for A_{13} the last expression is a cubic in $\mathbf{M}_{13}$.

Hence B_{13} is a cubic in $\mathbf{M}_{13}$.

Consider now $A_{13}-M_{13}B_{13}$. This is found, after some reductions, and using (52), to be

$$(f_3/f_{13})(A_3-M_3B_3)+f_3B_3/f_1+M_3\mathbf{M}_3^2(A_1-M_1B_1).$$

Of the above terms, $A_3-M_3B_3$ is a cubic in $\mathbf{M}_3$ and therefore also a cubic in $\mathbf{M}_{13}$. B_3 is a cubic in $\mathbf{M}_{13}$. $A_1-M_1B_1$ is a cubic in $\mathbf{M}_1$ and when multiplied by $M_3\mathbf{M}_3^2$ becomes a cubic in $\mathbf{M}_{13}$.

Hence if the condition (I) holds good for the components, it also holds good for the resultant system. But we have seen that it holds for a single refracting system; thus it holds for any combination. Also B will be a cubic in M for any system.

As regards C, examination of (51), remembering that for a single surface C_3 and C_1 are quadratics in $\mathbf{M}_3$, $\mathbf{M}_1$ respectively, leads immediately to the conclusion that C is a quadratic in $\mathbf{M}$ for any system.

We now come to the coefficient E. Here the single refracting surface gives no precedent for E_1 and E_3. Let us examine the other terms in E_{13}. These can be written in the form $f_1A_1\mathbf{M}_3^2M_3^2(B_3-dA_3/dM_3)+3A_3\mathbf{M}_3^2(f_3B_1+f_1M_3A_1)-2f_3A_3\mathbf{M}_3^2C_1$, and, using $A_1-B_1M_1=U_1$; $A_3-B_3M_3=U_3$, where U_1, U_3 are then cubics in M_1, M_3 respectively, this is found to reduce to

$$-f_1A_1\mathbf{M}_3^2M_3^2(M_3\,dB_3/dM_3+dU_3/dM_3)+3A_3\mathbf{M}_3^2M_3f_1(U_1+f_3B_1/f_{13})-2f_3A_3\mathbf{M}_3^2C_1. \quad (56)$$

Now

$$A_1\mathbf{M}_3^2M_3^2 = \text{quartic in } M_{13}.$$

$$M_3\,dB_3/dM_3+dU_3/dM_3 = \text{cubic in } M_3 = \text{cubic in } M_{13}.$$

$$\mathbf{M}_3^2M_3(U_1+f_3\mathbf{B}_1/f_{13}) = \text{cubic in } M_3 = \text{cubic in } M_{13}.$$

$$\mathbf{A}_3 = \text{quartic in } M_{13}.$$

$$\mathbf{M}_3^2C_1 = \text{quadratic in } M_{13}.$$

Hence the three terms in (56) are of form

$$(\text{quartic})(\text{cubic})+(\text{quartic})(\text{cubic})+(\text{quartic})(\text{quadratic}),$$

and this leads to a rational integral polynomial of degree 7 in M_{13}.

Further consideration, however, shows that it is of degree 7 only in appearance, for the terms which can lead to expressions of degree 7 in M_{13} are clearly

$$-f_1\mathbf{M}_3^2M_3^3A_1\,dB_3/dM_3+3A_3f_1\mathbf{M}_3^2M_3(U_1+f_3B_1/f_{13}),$$

or, dropping the factor f_1

$$\begin{aligned}\mathbf{M}_3^2M_3\{3A_3\,(U_1+f_3B_1/f_{13})-M_3^2A_1dB_3/dM_3\}\\ =\mathbf{M}_3^2M_3\,\{3B_3M_3\,(U_1+f_3B_1/f_{13})+3U_3\,(U_1+f_3B_1/f_{13})\\ -M_3^2M_1B_1\,dB_3/dM_3-M_3^2U_1\,dB_3/dM_3\}.\end{aligned}$$

The term

$$3U_3\mathbf{M}_3^2M_3\,(U_1+f_3B_1/f_{13})$$

is clearly of the form cubic × cubic and the terms leading to expressions of 7th degree reduce to

$$\mathbf{M}_3^2M_3\,\{M_3U_1\,(3B_3-M_3\,dB_3/dM_3)+M_3B_1\,(3B_3f_3/f_{13}-M_{13}\,dB_3/dM_3)\},$$

and since $dM_{13}/dM_3=f_3/f_{13}$, this can be written

$$\mathbf{M}_3^2M_3\,\{M_3U_1\,(3B_3-M_3\,dB_3/dM_3)+(M_3B_1f_3/f_{13})\,(3B_3-M_{13}\,dB_3/dM_{13})\}.$$

But since B_3 is a cubic, in either M_3 or M_{13}, $3B_3-M_3dB_3/dM_3$ and $3B_3-M_{13}\,dB_3/dM_{13}$ are both quadratics in M_3 or M_{13}.

The above expression therefore reduces to M_3 × sum of two quantities each of form: cubic in M_{13} × quadratic in M_{13}, that is, to a sextic in M_{13}.

We see, therefore, that those terms in E_{13} which do not involve E_1 or E_3 are a polynomial of sixth degree in M_{13} or $\mathbf{M}_{13}$.

It follows that for a *lens* E is necessarily a sextic in the magnification.

Suppose now that E_1 and E_3 are both sextics in $\mathbf{M}_1$, $\mathbf{M}_3$ respectively. Then E_3 is a sextic in M_{13} and $E_1\mathbf{M}_3^4M_3^2$ will also be a sextic in M_{13}, that is E_{13} will again be a sextic in $\mathbf{M}_{13}$.

Hence, since any system is built up of combinations of lenses or single refracting surfaces, we find that E is a sextic polynomial in $\mathbf{M}$ for any system.

Examination of particular cases shows that E is not, in general, divisible by A, so that the vanishing of the latter does not usually involve the disappearance of the second order terms.

§9. *Invariant Relations.*

Certain relations exist between the coefficients A, B, C, E which remain the same in form, whatever the number of refracting surfaces. One of these we have already dealt with, namely the fact that

$$A-MB$$

reduces to an expression of the third degree, *i.e.*, the coefficients of highest degree in M in A and B are the same.

This we shall refer to as the *first* invariant relation (I.).

A second invariant relation takes the form

$$B-C=\tfrac{3}{8}\,(1-\mathbf{M}^2)+\tfrac{1}{4}\,dA/dM, \qquad \text{(II)}$$

when we use $\tan \beta_{2n}$ as argument, and

$$\mathrm{B}-\mathrm{C} = -\tfrac{1}{8}(1-\mathbf{M}^2)+\tfrac{1}{4}\, d\mathrm{A}/d\mathrm{M}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\mathrm{II}')$$

when we use $\sin \gamma_{2n}$ as argument.

That these relations hold good for the single refracting surface is readily verified from equations (22), (23), (25), (27) and (32). Suppose now that, for systems 1 and 3 separately, the relation

$$\mathrm{B}-\mathrm{C} = \sigma(1-\mathbf{M}^2)+\tfrac{1}{4}\, d\mathrm{A}/d\mathrm{M}$$

holds, where $\sigma = \frac{3}{8}$ or $-\frac{1}{8}$ according to the nature of the argument, then from (48) and (51)

$$\begin{aligned}\mathrm{B}_{13}-\mathrm{C}_{13} &= \mathrm{B}_3-\mathrm{C}_3+\mathbf{M}_3^{\,2}(\mathrm{B}_1-\mathrm{C}_1)+\mathrm{A}_1\mathrm{M}_3\mathbf{M}_3^{\,2} f_1/f_3 \\ &= \sigma(1-\mathbf{M}_{13}^{\,2})+\tfrac{1}{4}\, d\mathrm{A}_3/d\mathrm{M}_3+\tfrac{1}{4}\mathbf{M}_3^{\,2}(d\mathrm{A}_1/d\mathrm{M}_1+4\mathrm{A}_1\mathrm{M}_3 f_1/f_3), \quad . \quad . \qquad (57)\end{aligned}$$

and from (47)

$$f_{13}\, d\mathrm{A}_{13}/d\mathrm{M}_{13} = f_3\, d\mathrm{A}_3/d\mathrm{M}_{13}+f_1\{\mathrm{M}_3^{\,2}\mathbf{M}_3^{\,2}\, d\mathrm{A}_1/d\mathrm{M}_{13}+\mathrm{A}_1\, d(\mathrm{M}_3^{\,2}\mathbf{M}_3^{\,2})/d\mathrm{M}_{13}\}.$$

But

$$d\mathrm{M}_{13} = (f_3/f_{13})\, d\mathrm{M}_3 = (f_1/f_{13})\, \mathrm{M}_3^{\,2}\, d\mathrm{M}_1.$$

Hence

$$d\mathrm{A}_{13}/d\mathrm{M}_{13} = d\mathrm{A}_3/d\mathrm{M}_3+\mathbf{M}_3^{\,2}\, d\mathrm{A}_1/d\mathrm{M}_1+(f_1\mathrm{A}_1/f_3)\, d(\mathrm{M}_3^{\,2}\mathbf{M}_3^{\,2})/d\mathrm{M}_3,$$

and since $\mathbf{M}_3/\mathrm{M}_3 =$ const., the last differential coefficient is $4\mathrm{M}_3\mathbf{M}_3^{\,2}$.

Using this result (57) becomes

$$\mathrm{B}_{13}-\mathrm{C}_{13} = \sigma(1-\mathbf{M}_{13}^{\,2})+\tfrac{1}{4}\, d\mathrm{A}_{13}/d\mathrm{M}_{13},$$

which is of the same form as the equation we started from. Hence, if the two components of the compound system satisfy the second invariant relation, the resultant system also satisfies it. But we have seen that the relation holds good for single refracting surfaces—hence it holds good universally.

It should be noted that the second invariant relation is really a *first order* relation and connects the first order aberration of the inclination of a ray, with the first order longitudinal spherical aberration.

§ 10. *The Constants* A, B, C, E *for an Optical System Reversed and for Negative Lenses.*

Certain important general relations are found to hold between the constants A, B, C, E for rays going through an optical system and the corresponding constants A′, B′, C′, E′, for the same system reversed, and by making use of them we can obtain either set from the other.

We arrive most simply at these relations as follows:—If after traversing the system we retrace our steps, the result is equivalent to compounding the system with

itself reversed, with the difference that, in the second set of refractions, the measurement of length parallel to the axis is reversed in direction. An examination of the equations (41) *et seq.*, § 7, on which the formulæ of combination are based, shows that this is analytically equivalent to changes in the sign of the focal length in the second set of refractions.

We have therefore

$$f_1 = f, \quad f_3 = -f, \quad \mathrm{M}_1 = \mathrm{M}, \quad \mathrm{M}_3 = 1/\mathrm{M}, \quad \mathbf{M}_1 = \mathbf{M}, \quad \mathbf{M}_3 = 1/\mathbf{M}, \quad \mathbf{M}_{13} = \mathrm{M}_{13} = 1,$$

and we also find that $f_{13} = \infty$. But $f_{13}\mathrm{A}_{13}$, and $f_{13}\mathrm{E}_{13}$ have definite limiting values, and as f_{13} does not otherwise explicitly enter into the equations of combination, no difficulty arises on that account.

Now, after retracing our steps in this way, we necessarily arrive at a perfect image, so that $\Delta x_4 \equiv 0$ and $\tan \alpha_4 = \tan \beta_4$, leading to

$$f_{13}\mathrm{A}_{13} \equiv 0,$$
$$f_{13}\mathrm{E}_{13} \equiv 0,$$

and

$$\mathrm{B}_{13} - \mathrm{C}_{13} \equiv 0.$$

These lead to the following identical relations

$$\mathrm{A}(\mathrm{M})/\mathrm{M}^2\mathbf{M}^2 - \mathrm{A}'(\mathrm{M}^{-1}) \equiv 0, \quad \ldots \ldots \quad (58)$$

$$\mathrm{E}(\mathrm{M})/\mathrm{M}^2\mathbf{M}^4 - \mathrm{E}'(\mathrm{M}^{-1}) + \mathrm{A}(\mathrm{M})\,\mathbf{M}^{-2}\{\mathrm{B}'(\mathrm{M}^{-1})\,\mathrm{M}^{-2} + 3\mathrm{A}'(\mathrm{M}^{-1})\,\mathrm{M}^{-1} + d\mathrm{A}'(\mathrm{M}^{-1})/d\mathrm{M}\}$$
$$- \mathrm{A}'(\mathrm{M}^{-1})\,\mathbf{M}^{-2}\{3\mathrm{B}(\mathrm{M}) - 2\mathrm{C}(\mathrm{M})\} \equiv 0, \quad \ldots \ldots \quad (59)$$

$$\mathrm{B}'(\mathrm{M}^{-1}) - \mathrm{C}'(\mathrm{M}^{-1}) + \mathbf{M}^{-2}\{\mathrm{B}(\mathrm{M}) - \mathrm{C}(\mathrm{M})\} - \mathrm{A}(\mathrm{M})\,\mathrm{M}^{-1}\mathbf{M}^{-2} \equiv 0. \quad \ldots \quad (60)$$

Equation (58) may be written in either of the two forms

$$\mathrm{A}(\mathrm{M}) \equiv \mathrm{M}^2\mathbf{M}^2\mathrm{A}'(\mathrm{M}^{-1})$$
$$\mathrm{A}(\mathbf{M})/n_0^2 \equiv \mathbf{M}^4\mathrm{A}'(\mathbf{M}^{-1})/n_2^2. \quad \ldots \ldots \quad (\mathrm{III})$$

This we shall refer to as the third invariant relation. It shows that, if we divide A by the square of the initial refractive index, the coefficients of powers of $\mathbf{M}$ equidistant from the beginning and end of the development are interchanged by reversing the system. Equation (59) becomes on multiplying up by $\mathrm{M}^2\mathbf{M}^4$, using (58) and simplifying

$$\mathrm{E} - \mathrm{M}^2\mathbf{M}^4\mathrm{E}' + \mathrm{A}\{\mathbf{M}^2\mathrm{B}' - \mathrm{A}/\mathrm{M} - 3\mathrm{B} + 2\mathrm{C} + d\mathrm{A}/d\mathrm{M}\} \equiv 0, \quad \ldots \quad (61)$$

omitting the arguments M, 1/M of A, B, B′, &c., since no confusion can occur.

Use now the second invariant relation

$$d\mathrm{A}/d\mathrm{M} \equiv 4\mathrm{B} - 4\mathrm{C} - 4\sigma(1 - \mathbf{M}^2),$$

(61) becomes

$$\mathrm{E}-\mathrm{M}^2\mathbf{M}^4\mathrm{E}'+\mathrm{A}\{\mathrm{B}+\mathbf{M}^2\mathrm{B}'-2\mathrm{C}-\mathrm{A}/\mathrm{M}-4\sigma(1-\mathbf{M}^2)\}\equiv 0. \quad . \quad . \quad . \quad (62)$$

Now substitute from (60) for $\mathrm{B}+\mathbf{M}^2\mathrm{B}'$ the value

$$\mathrm{C}+\mathbf{M}^2\mathrm{C}'+\mathrm{A}/\mathrm{M},$$

and (61) leads to

$$\mathrm{E}-\mathrm{M}^2\mathbf{M}^4\mathrm{E}'+\mathrm{A}\{\mathbf{M}^2\mathrm{C}'-\mathrm{C}-4\sigma(1-\mathbf{M}^2)\}\equiv 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

For a single refracting surface, where E, E′ are identically zero, this must lead to

$$\mathbf{M}^2\mathrm{C}'-\mathrm{C} = 4\sigma(1-\mathbf{M}^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{IV})$$

a result which is easily verified from equation (32).

Now consider a system compounded of two systems. For the system direct, we have

$$\mathrm{C}_{13} = \mathrm{C}_3+\mathbf{M}_3{}^2\mathrm{C}_1.$$

Similarly, for the system reversed, change C_3 into C'_1, C_1 into C'_3, $\mathbf{M}_3$ into $1/\mathbf{M}_1$.

$$\begin{aligned}\mathrm{C}_{31} &= \mathrm{C}'_1+\mathbf{M}_1{}^{-2}\mathrm{C}'_3\\ \mathbf{M}_{13}{}^2\mathrm{C}_{31}-\mathrm{C}_{13} &= \mathbf{M}_1{}^2\mathbf{M}_3{}^2\mathrm{C}'_1-\mathbf{M}_3{}^2\mathrm{C}_1+\mathbf{M}_3{}^2\mathrm{C}'_3-\mathrm{C}_3\\ &= \mathbf{M}_3{}^2(\mathbf{M}_1{}^2\mathrm{C}'_1-\mathrm{C}_1)+\mathbf{M}_3{}^2\mathrm{C}'_3-\mathrm{C}_3\end{aligned}$$

and using (IV) which we know to be true for a single refracting surface

$$\begin{aligned}\mathbf{M}_{13}{}^2\mathrm{C}_{31}-\mathrm{C}_{13} &= 4\sigma[\mathbf{M}_3{}^2(1-\mathbf{M}_1{}^2)+1-\mathbf{M}_3{}^2]\\ &= 4\sigma(1-\mathbf{M}_{13}{}^2).\end{aligned}$$

In other words (IV) will hold for the resultant system if it holds for the components, and therefore as in previous similar cases, it holds for any system.

We shall call (IV) the *fourth* invariant relation.

Equation (63) then shows that there exists a *fifth* invariant relation

$$\mathrm{E} = \mathrm{M}^2\mathbf{M}^4\mathrm{E}'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{V})$$

or

$$\mathrm{E}(\mathbf{M})/n_0{}^2 = \mathbf{M}^6\mathrm{E}'(\mathbf{M}^{-1})/n_2{}^2,$$

so that E possesses a property similar to that of A, previously noticed, viz., if we divide it by the square of the initial refractive index, the coefficients of powers of $\mathbf{M}$ equidistant from the beginning and end of the development in $\mathbf{M}$ are interchanged.

Equation (59) has therefore led us to *two* independent invariant relations.

On the other hand it will be found that (60) leads to no new relation. For if we substitute into it for $\mathrm{B}'-\mathrm{C}'$ and $\mathrm{B}-\mathrm{C}$ in virtue of the second invariant relation, and then use the third relation, it becomes an identity.

It gives, however, on using (IV) to eliminate C′,

$$\mathbf{M}^2 B' + B = A/M + 2C + 4\sigma (1 - \mathbf{M}^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (64)$$

which is a convenient form for calculating B′.

If now A, B, C, E are known for any system, the corresponding quantities are immediately obtainable for the reversed system, A′, B′, C′, E′ being given by equations (III), (64), (IV) and (V) respectively.

This will generally halve the labour of calculations, if it is found desirable to tabulate these constants for a complete set of lenses. It will then be sufficient to start from the equi-convex lens and vary the curvatures in one sense only.

Incidentally we note also that the second invariant relation enables us to find C, so soon as A and B are known, so that only A, B and E require to be calculated.

The aberration constants for a reversed system have a further important application in the case of lenses. Consider a positive lens (fig. 4), the initial ray converging to I_0 and the final ray to I_4. If now we interchange the full and dotted portions of the initial and final rays in fig. 4, we obtain, since here the initial and final media are the

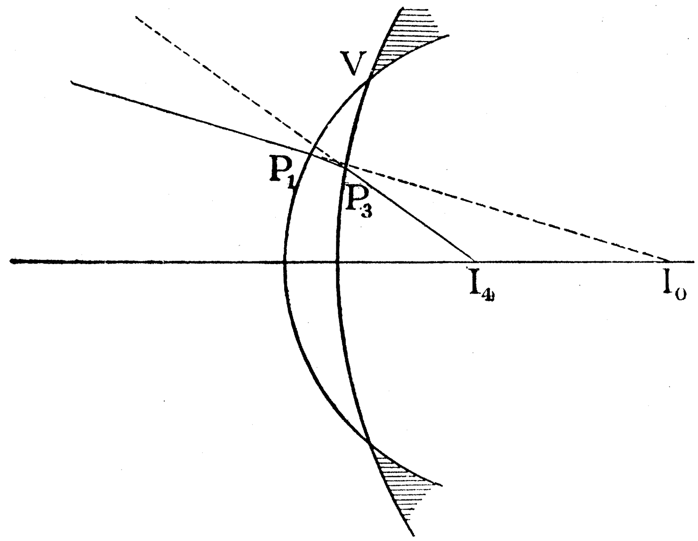

Fig. 4.

same, the case of a ray going through a lens in which the front and back character of the two surfaces have been interchanged. In fact r_1 and r_3 have been interchanged and the sign of the thickness c_2 has been reversed. This leads to a negative lens, of the same *numerical* power as the original positive lens, and with the same mean curvature, but a negative thickness. Such a lens, of course, is not physically realisable, although a part of it can be physically obtained by rotating the wedge *beyond* the intersection V of the two surfaces.

But, in the case of the ideally *thin* lenses, where the thickness is zero, the ideally thin positive and negative lenses, having the same mean curvature and numerical power, correspond in this way.

Now I_4 and I_0 are also interchanged. If we consider I_4 as the initial point, then we are really considering a set of rays starting from I_4 in the last medium and travelling backwards through the original positive lens. In other words, the aberration constants for the corresponding negative lens are identical with those for the original positive lens reversed, and the equations (III), (64), (IV) and (V) are applicable to calculate them.

This, again, will greatly diminish the work of calculation. In the case of ideally thin negative lenses, we see that A, B, C, E are directly obtained from the corresponding thin positive lenses. In the case of *thick* negative lenses the corresponding positive lens has a negative thickness.

Now for various reasons it will probably be convenient, in calculating A, B, C, E for lenses, to express them in the form

$$A_c = A_0 + c\,(dA/dc)_0,$$

&c., where c, the thickness, is small, as it usually is in practice, and A_0 refers to an ideally thin lens.

When the formulæ are put in this form, it is perfectly simple to calculate A_{-c}, B_{-c}, &c., and then to obtain the corresponding results for the negative lens with a positive thickness.

§ 11. *Explicit Values of* A, B, C, E *for a Thick Lens* (*Tangent Formula*).

For the purposes of numerical calculation and comparison with correct trigonometrically found values, we have worked out explicitly the form of the expressions A, B, C, E for a thick lens, when we use $\tan\beta_4$ as the argument; the formulæ are expressed in terms of the focal lengths of each surface and of the combination and the thickness does not appear explicitly. The initial and final media being the same $n_0 = n_4$ and we have written $n = n_2/n_0$.

The work of algebraic calculation has been straightforward but extremely heavy, and we therefore omit it here entirely, the object being to publish the results for reference, in case other workers desire to use them for tabulation purposes, but it is hardly to be expected that designing opticians should work direct from the algebraic expressions as they stand.

For the purpose of this section we shall write

$$A = A_0 + A_1M + A_2M^2 + A_3M^3 + A_4M^4.$$

$$B = B_0 + B_1M + B_2M^2 + B_3M^3.$$

$$C = C_0 + C_1M + C_2M^2.$$

$$E = E_0 + E_1M + E_2M^2 + E_3M^3 + E_4M^4 + E_5M^5 + E_6M^6.$$

The suffixes here have a different meaning to that which has been previously ascribed to them, but no confusion is likely to arise on this account.

The values are as follows, f denoting the focal length of the lens, f_1, f_3 the focal lengths of the surfaces, so that

$$f_1 = r_1/(n-1), \qquad f_3 = -r_3/(n-1)$$

$$A_0 = \frac{1}{2n^2(n-1)^2}\left[-\frac{f_1+f_3}{f_1^4}f^3+\frac{(n+1)^2f_3f^2}{f_1^3}-2n(1+n+n^2)\frac{f_3f}{f_1^2}+n^2(n+1)^2\frac{f_3}{f_1}-\frac{n^4f_3}{f}\right]$$

$$A_1 = \frac{1}{2n^2(n-1)^2}\left[-\frac{4(f_1+f_3)}{f_1^3f_3}f^3+4(n+1)^2\frac{f^2}{f_1^2}-4n(1+n+n^2)\frac{f}{f_1}+n^2(n+1)^2\right]$$

$$A_2 = \frac{1}{2n^2(n-1)^2}\left[-\frac{6(f_1+f_3)}{f_1^2f_3^2}f^3+6(n+1)^2\frac{f^2}{f_1f_3}+2n(1+n+n^2)f\left(\frac{1}{f_1}+\frac{1}{f_3}\right)\right]$$

$A_3 = A_1$ with f_1 and f_3 interchanged

$A_4 = A_0$,, ,, ,,

$$B_0 = \frac{1}{4n^2(n-1)^2}\left[-\frac{2f^3}{f_1^2}\left(\frac{1}{f_1}+\frac{1}{f_3}\right)+2(n^2-n+1)\frac{f^2}{f_1^2}+n(n-1)^2\frac{f}{f_1}-n^2(n^2-n+1)\right]$$

$$B_1 = \frac{1}{4n^2(n-1)^2}\left[-\frac{6f^3}{f_1f_3}\left(\frac{1}{f_1}+\frac{1}{f_3}\right)+6(n^2+1)\frac{f^2}{f_1f_3}+n(n-1)^2f\left(\frac{1}{f_1}+\frac{1}{f_3}\right)\right]$$

$$B_2 = \frac{1}{4n^2(n-1)^2}\left[-\frac{6f^3}{f_3^2}\left(\frac{1}{f_1}+\frac{1}{f_3}\right)+6(n^2+n+1)\frac{f^2}{f_3^2}-3n(n+1)^2\frac{f}{f_3}+3n^2(n^2-n+1)\right]$$

$B_3 = A_4$

$C_0 = B_0 - \frac{3}{8} - \frac{1}{4}A_1$

$C_1 = B_1 - \frac{1}{2}A_2$

$C_2 = B_2 + \frac{3}{8} - \frac{3}{4}A_3$

$$E_0 = \frac{3}{8(n-1)^4}\left[\begin{aligned}&-\frac{(n^2-n+1)}{n^4}\frac{(f_1+f_3)f^5}{f_1^6}+\frac{(n+1)^2}{n^4}f^4\left\{\frac{n}{f_1^4}+\frac{(n^2-n+1)f_3}{f_1^5}\right\}\\&-\frac{f^3}{n^3}\left\{\frac{n^3+3n^2+n}{f_1^3}+\frac{2(n^4+n^2+1)f_3}{f_1^4}\right\}+\frac{f^2}{n^2}\left\{\frac{2n^2}{f_1^2}+(n+1)^2(n^2-n+1)\frac{f_3}{f_1^3}\right\}\\&\qquad -f(n^2-n+1)\frac{f_3}{f_1^2}\end{aligned}\right]$$

$$E_1 = \frac{3}{8(n-1)^4}\left[\begin{aligned}&-\frac{6(n^2-n+1)}{n^4}\frac{(f_1+f_3)f^5}{f_3f_1^5}+\frac{(n+1)^2}{n^4}f^4\left\{\frac{5n}{f_3f_1^3}+\frac{6(n^2-n+1)}{f_1^4}+\frac{nf_3}{f_1^5}\right\}\\&-\frac{f^3}{n^3}\left\{\frac{4(n^3+3n^2+n)}{f_3f_1^2}+\frac{(n+1)^4+8(n^4+n^2+1)}{f_1^3}+(n+1)^4\frac{f_3}{f_1^4}\right\}\\&+\frac{f^2}{n^2}\left\{\frac{6n^2}{f_3f_1}+\frac{4(n+1)^2(n^2+1)}{f_1^2}+\frac{2(n-1)^2(n^2+n+1)f_3}{f_1^3}\right\}\\&-\frac{f}{n}\left\{\frac{2n(2n^2+n+2)}{f_1}+\frac{(n+1)^4f_3}{f_1^2}\right\}+\frac{nf_3}{f_1}(n+1)^2\end{aligned}\right]$$

$$
\begin{aligned}
E_2 = \frac{3}{8(n-1)^4}\Bigg[& -\frac{15(n^2-n+1)}{n^4}\frac{f^5(f_1+f_3)}{f_1^4 f_3^2} + \frac{(n+1)^2}{n^4} f^4 \left\{ \frac{10n}{f_1^2 f_3^2} + \frac{15(n^2-n+1)}{f_1^3 f_3} + \frac{5n}{f_1^4} \right\} \\
& - \frac{f^3}{n^3}\left\{ \frac{6(n^3+3n^2+n)}{f_1 f_3^2} + \frac{12(n^4+n^2+1)+4(n+1)^4}{f_1^2 f_3} \right. \\
& \qquad \left. + \frac{4(n+1)^4+2(n^4+n^2+1)}{f_1^3} + \frac{(n^3+3n^2+n)f_3}{f_1^4} \right\} \\
& + \frac{f^2}{n^2}\left\{ \frac{6n^2}{f_3^2} + \frac{6(n+1)^2(n^2+n+1)}{f_1 f_3} + \frac{8(n+1)^2(n^2+n+1)}{f_1^2} \right. \\
& \qquad \left. + \frac{(n+1)^2(n^2+3n+1)f_3}{f_1^3} \right\} \\
& - \frac{f}{n}\left\{ \frac{4n(n+1)^2+n^3-n^2+n}{f_3} + \frac{2(n+1)^4+2(n^2+n+1)(n^2+3n+1)}{f_1} \right. \\
& \qquad \left. + \frac{2(n^2+n+1)(n^2+3n+1)f_3}{f_1^2} \right\} \\
& + n\{(n+1)^2+4(n^2+n+1)\} + (n+1)^2(n^2+3n+1)\frac{f_3}{f_1} \\
& \qquad - n^2(n^2+3n+1)\frac{f_3}{f_1} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
E_3 = \frac{3}{8(n-1)^4}\Bigg[& -\frac{20(n^2-n+1)}{n^4}\frac{f^5(f_1+f_3)}{f_1^3 f_3^3} + \frac{(n+1)^2}{n^4} f^4 \left\{ \frac{10n}{f_3 f_1}\left(\frac{1}{f_1^2}+\frac{1}{f_3^2}\right) \right. \\
& \qquad \left. + \frac{20(n^2-n+1)}{f_1^2 f_3^2} \right\} \\
& - \frac{f^3}{n^3}\left\{ 4(n^3+3n^2+n)\left(\frac{1}{f_1^3}+\frac{1}{f_3^3}\right) + \frac{8(n^4+n^2+1)+6(n+1)^4}{f_1 f_3}\left(\frac{1}{f_1}+\frac{1}{f_3}\right) \right\} \\
& + \frac{f^2}{n^2}\left\{ 2n^2\left(\frac{f_1}{f_3^3}+\frac{f_3}{f_1^3}\right) + 4(n+1)^4\left(\frac{1}{f_1^2}+\frac{1}{f_3^2}\right) + \frac{12(n+1)^2(n^2+n+1)}{f_1 f_3} \right\} \\
& - \frac{f}{n}\left\{ 2n(n+1)^2\left(\frac{f_1}{f_3^2}+\frac{f_3}{f_1^2}\right) + [(n+1)^4 \right. \\
& \qquad \left. + 4(n^2+n+1)(n^2+3n+1)]\left(\frac{1}{f_1}+\frac{1}{f_3}\right) \right\} \\
& + 4n(n^2+n+1)\left(\frac{f_1}{f_3}+\frac{f_3}{f_1}\right) + 2(n+1)^2(n^2+3n+1) \\
& - 2n^2(n+1)^2\frac{(f_1+f_3)}{f} + 2n^4\frac{f_1 f_3}{f^2} \Bigg]
\end{aligned}
$$

$E_4 = E_2$ with f_1 and f_3 interchanged.

$E_5 = E_1$,, ,, ,, ,,

$E_6 = E_0$,, ,, ,, ,,

It should be noted carefully that all the above refer to expressions in terms of the tangent of the Gaussian inclination, this being the argument we have used in the numerical work.

§ 12. *Values of* A, B, C, E *for a Thin Lens.*

When the lens is thin, we have the relation

$$\frac{1}{f_1}+\frac{1}{f_3}=\frac{1}{f},$$

which enables the values of § 11 to be considerably simplified.

In this case it is useful to introduce a quantity K such that

$$\mathrm{K}=\frac{\text{mean curvature of the lens}}{\text{power of the lens}}=\frac{f}{2}\left(\frac{1}{r_1}+\frac{1}{r_3}\right).$$

When this is done the constants A, B, C, E take the following forms :—

$$\begin{aligned}\mathrm{A} = -\{(1-\mathrm{M})^2/2n\}\ \{&(n+2)\,[(1-\mathrm{M})\,\mathrm{K}-(1+\mathrm{M})\,(n+1)/(n+2)]^2 \\ &+n^3\,(1-\mathrm{M})^2/4\,(n-1)^2-n^2\,(1+\mathrm{M})^2/4\,(n+2)\}.\end{aligned}$$

$$\begin{aligned}\mathrm{B} = (1-\mathrm{M})^2\,\mathrm{K}^2\,&\{n-1-(n+2)\,\mathrm{M}\}/2n+(1-\mathrm{M})\,\mathrm{K}\,(1+\mathrm{M}+4\mathrm{M}^2)\,(n+1)/4n \\ &+(1-\mathrm{M})\,\{\mathrm{M}\,(1+\mathrm{M})/4n+(1-\mathrm{M})\,[3n-2n^2-3+\mathrm{M}\,(6n-4n^2-3)]/8\,(n-1)^2\}.\end{aligned}$$

$$\mathrm{C} = -3\,(1-\mathrm{M})^2\,\mathrm{K}^2/2n+3\,(1-\mathrm{M}^2)\,\mathrm{K}\,(n+1)/4n-\tfrac{3}{4}\,(1-\mathrm{M}^2)-\tfrac{3}{8}n\,(1-\mathrm{M})^2/(n-1)^2.$$

$$\mathrm{E} = \frac{3\,(1-\mathrm{M})^2}{128\,(n-1)^4\,n^3}\left[\begin{aligned}&(1+\mathrm{M}^4)\,(-4n^5+8n^4-n^3-4n^2+3n-1)\\ &\quad+\mathrm{M}\,(1+\mathrm{M}^2)\,(8n^6-16n^5+4n^4+4n^3-12n^2+12n-4)\\ &\quad+\mathrm{M}^2\,(-16n^7+32n^6-8n^5-8n^4+10n^3-16n^2+18n-6)\\ &\quad+8\,(n-1)\,(1-\mathrm{M})\,(1+\mathrm{M})\,\mathrm{K}\,\{(1+\mathrm{M})^2\\ &\qquad\qquad(-2n^5+5n^4-2n^3-3n^2+3n-1)\\ &\qquad\qquad\qquad+\mathrm{M}\,(2n^6-8n^4+4n^3+2n^2)\}\\ &\quad+8\,(n-1)^2\,(1-\mathrm{M})^2\,\mathrm{K}^2\,\{(1+\mathrm{M})^2\\ &\qquad\qquad(-2n^5+8n^4-7n^3-6n^2+9n-3)\\ &\qquad\qquad\qquad-2\mathrm{M}\,n^2\,(n^2-2n-1)\}\\ &\quad+16\,(n-1)^3\,(1+\mathrm{M})\,(1-\mathrm{M})^3\,\mathrm{K}^3\,\{2n^4-4n^3-2n^2+6n-2\}\\ &\quad+16\,(n-1)^4\,(1-\mathrm{M})^4\,\mathrm{K}^4\,(-n^3+3n-1)\end{aligned}\right]$$

We notice that when M = 1 (which gives one of the zeros of A) B, C and E all vanish with it, and also E/A remains finite. Hence in this case, the term $\mathrm{E}t_4{}^4$ will not rise in importance, even when A = 0. But in this case A may have two other real zeros, and these are not zeros of E, so that E plays an important part in the neighbourhood of such zeros.

§ 13. *Numerical Test for a Single Lens.*

To test the formulæ, a number of longitudinal aberrations were calculated trigonometrically for five positive lenses of refractive index 1·52, unit focal length and thickness $\frac{1}{16}$. The first was a meniscus-shaped lens for which $r_1 = 0·349418$, $r_3 = 1$.

The second was a plano-convex lens, of which the convex side is towards the incoming light. The third was an equi-convex lens. The fourth and fifth were the second and first reversed. If K has the meaning defined in § 12, the values of K for these five lenses are 1·93094, 0·96154, 0, −0·96154 and − 1·93094 respectively, so that they proceed by approximately equal steps of K.

The constants A, B, E were calculated for these five lenses and the longitudinal aberrations computed from the formula. The rays selected met the first principal plane at a distance 0·15 from the axis, corresponding to an aperture $f/3·4$, nearly.

The results are shown in Table III.

TABLE III.—Longitudinal Aberrations of Five Selected Lenses.

M.	Lens 1.			Lens 2.			Lens 3.		
	Formula.	True.	Percentage error.	Formula.	True.	Percentage error.	Formula.	True.	Percentage error.
3	−7·98683	−7·87422	1·41	−1·12450	−1·12814	0·32	−0·32378	−0·32439	0·19
2	−1·55287	−1·52334	1·94	−0·40167	−0·40283	0·29	−0·14240	−0·14282	0·29
0·5	−0·003527	−0·003530	0·09	−0·011697	−0·011705	0·07	−0·033008	−0·033096	0·27
0	−0·06028	−0·06043	0·24	−0·024728	−0·024749	0·08	−0·036194	−0·036213	0·05
−0·5	−0·20902	−0·20946	0·21	−0·083863	−0·083950	0·10	−0·057067	−0·057101	0·06
−1	−0·42692	−0·42772	0·19	−0·18428	−0·18448	0·11	−0·094736	−0·094787	0·05
−2	−1·00383	−1·00531	0·15	−0·48874	−0·48930	0·12	−0·21707	−0·21724	0·08

M.	Lens 4.			Lens 5.		
	Formula.	True.	Percentage error.	Formula.	True.	Percentage error.
3	−0·098969	−0·099086	0·12	−0·10401	−0·10444	0·41
2	−0·046089	−0·046128	0·30	−0·013671	−0·013686	0·11
0·5	−0·073959	−0·074535	0·77	−0·14762	−0·15068	2·03
0	−0·099843	−0·100044	0·20	−0·22172	−0·22324	0·68
−0·5	−0·13967	−0·13984	0·12	−0·32828	−0·32966	0·42
−1	−0·19084	−0·19101	0·09	−0·45753	−0·45887	0·29
−2	−0·32454	−0·32476	0·07	−0·77417	−0·77511	0·12

The mean percentage error of these results is 0·34, so that the formula determines, on the average, the longitudinal aberration correct to 1 part in 300. The bulk of

this, however, is contributed by three cases, namely: $M = 3$ and 2 for lens 1 where the aberrations are very large and differ very widely from the usual first and second order approximations, so that, although the error of the formula approaches 2 per cent., it nevertheless represents a great improvement upon these approximations; and $M = 0{\cdot}5$ for lens 5, which corresponds to extreme curvature and highest inclination, so that one of the angles of refraction is as great as $48\frac{1}{2}$ degrees. Even here the table below shows that the formula is an appreciable improvement on the usual second-order approximation. If these three cases are omitted, the mean percentage error works out to be about 0·21, so that in general the formula determines the longitudinal aberration correct to about 1 part in 500.

It is interesting to note what the usual first and second order approximations lead to in a few cases.

	First order approximation.	Percentage error.	Second order approximation.	Percentage error.
Lens 1, $M = 3$ . . .	$-1{\cdot}34012$	83·0	$-2{\cdot}45432$	68·8
,, 1, $M = 2$ . . .	$-0{\cdot}48545$	68·1	$-0{\cdot}82104$	46·1
,, 1, $M = -2$ . . .	$-1{\cdot}348142$	34·1	$-0{\cdot}88456$	12·0
,, 2, $M = 3$ . . .	$-0{\cdot}67356$	40·3	$-0{\cdot}94436$	16·3
,, 5, $M = 0{\cdot}5$ . . .	$-0{\cdot}12136$	19·5	$-0{\cdot}14307$	5·1

This gives a measure of the numerical improvement effected by the fractional formula whenever the usual method of approximation is seriously out, even though in none of the cases above does the convergency of the series actually fail.

In the above the series are in powers of $\tan \beta_2$. Had they been taken in powers of $\tan \alpha_2$, as is frequently done, the first and second order approximations would have been far worse.

One interesting outcome of these calculations relates to the relative importance of the terms in Et_4^4 and At_4^2. The ratio Et_4^2/A is small in every case taken (of course these exclude the neighbourhood of points where $A = 0$, where naturally E becomes of great importance). But for the set of magnifications taken, the greatest ratio of the second term to the first is less than 0·03 and the mean value of this ratio is only 0·0082, so that, in fact, the E term—although so complicated algebraically—does not exercise any great influence numerically.

This is important, as it shows that, at any rate for lenses, it does not require to be computed with anything like the same order of accuracy which is needed for A and B.

§ 14. *The Singular Inclination and Convergency Factor for any System.*

Referring again to fig. 2 we see that $\lambda = \alpha_2$ and $F_0I_0 = -\Delta x_2$ for rays proceeding through the system reversed and initially parallel.

Thus using accents, as before, to denote the coefficients and inclinations for

the system reversed, and noting that the accented coefficients all refer to zero magnification, we have, using *tangents*

$$\tan\lambda = \tan\beta'_2\,(1+B'_0\tan^2\beta'_2)/(1+C'_0\tan^2\beta'_2)$$

and

$$-F_0I_0 = fn_0\,(A'_0\tan^2\beta'_2+E'_0\tan^4\beta'_2)/(1+B'_0\tan^2\beta'_2).$$

Here the suffixes in the A, B, C, &c., A′, B′, C′, &c., have the same meaning as in § 11.

Thus

$$-1/M = (A'_0\tan^2\beta'_2+E'_0\tan^4\beta'_2)/(1+B'_0\tan^2\beta'_2),$$

whence, developing $\cot^2\beta'_2$ in descending powers of M and stopping at the second term

$$\cot^2\beta'_2 = -A'_0M-B'_0+E'_0/A'_0.$$

Substituting into

$$\cot^2\lambda = \cot^2\beta'_2+2\,(C'_0-B'_0)$$

which is valid to the same order of approximation, we obtain

$$\cot^2\lambda = -A'_0M+2\,(C'_0-B'_0)-B'_0+E'_0/A'_0 \quad . \quad . \quad . \quad . \quad . \quad (67)$$

as the second approximation for the singular inclination when M is large, the first approximation being $\cot^2\lambda = -A'_0M$.

To the same order the convergency factor is

$$1+\tan^2\alpha_0\,(A'_0M+\{2\,(B'_0-C'_0)+B'_0-(E'_0/A'_0)\}),$$

i.e.,

$$1+(n_2^2\tan^2\beta_2/n_0^2)\,\{A'_0M^3+(3B'_0-2C'_0-E'_0/A'_0)\,M^2\}, \quad . \quad . \quad . \quad . \quad (68)$$

β_2 referring to a ray passing through the system in the standard sense.

Now, using the equations (III), (64), (IV) and (V) of § 10 and equating suitable coefficients, we find that

$$A'_0 = (n_0^2/n_2^2)\,A_4, \qquad E'_0 = (n_0^4/n_2^4)\,E_6,$$

$$C'_0 = (n_0^2/n_2^2)\,C_2-4\sigma, \qquad B'_0 = (n_0^2/n_2^2)\,(A_3+2C_2-B_2)-4\sigma$$

$$3A_3 = 4B_2-4C_2+4\,(n_2^2/n_0^2)\,\sigma, \qquad A_4 = B_3,$$

whence, after substitution, (68) becomes

$$1+\tan^2\beta_2\,(B_3M^3+\{B_2-E_6/A_4\}\,M^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Now, if our B leads to a sound approximation to the convergency factor for M large, this should be

$$1+B\tan^2\beta_2,$$

or, to the same approximation which we have been using

$$1+\tan^2\beta_2\,(B_3M^3+B_2M^2)\,. \qquad (70)$$

We see, therefore, that the development of the correct convergency factor in descending powers of M will give a result which always agrees with our B, so far as the highest term in M is concerned, but makes the term in M^2 in general different.

In the case of a lens E_6/A_4 is in general small, compared with B_2, so that this discrepancy makes little difference, but it may well be that, when we come to deal with more complicated systems, this will not be the case.

A little consideration, however, shows that, when this is so, our formula is very readily corrected so as to take this difficulty into account, without involving any lengthy numerical computation.

If we consider the formula

$$\Delta x_2 = n_2 f\,\{At_2^2 + (E - AE_6M^2/A_4)\,t_2^4\}/\{1 + (B - E_6M^2/A_4)\,t_2^2\}$$

it is clear that it leaves the development of Δx_2 in powers of t_2 unaltered as far as the second order inclusive. It alters the coefficient B_2 of B so as to make the two leading terms agree with (69). It also alters E in such a way as to remove the term in M^6 and reduce E to a quintic. In fact it gives for the new E the remainder obtained after the first step in the division of E by A, according to the usual process.

In practice, the terms in $(E_6/A_4)\,M^2$ are very readily added as follows :—

$$\Delta x_2 = n_2 f\,\frac{At_2^2 + Et_2^4/(1 - E_6M^2t_2^2/A_4)}{1 + Bt_2^2/(1 - E_6M^2t_2^2/A_4)}$$

and this amounts to applying the same corrective factor $1/(1 - E_6M^2t_2^2/A_4)$ or $1/(1 - En_0^2t_0^2/n_2^2A_4)$ to the second terms in both numerator and denominator. This factor, expressed in terms of the inclination of the incident ray, is independent of the magnification, and a short table will enable it to be found in any given case without difficulty.

A similar correction has then to be made in C; in order to keep the development of $\tan\alpha_2$ the same we must have

$$\tan\alpha_2 = t_2\,\{1 + (B - E_6M^2/A_4)\,t_2^2\}\,/\,\{1 + (C - E_6M^2/A_4)\,t_2^2\},$$

and writing this as

$$\frac{t_2\,\{1 + Bt_2^2/(1 - E_6M^2t_2^2/A_4)\}}{1 + Ct_2^2/(1 - E_6M^2t_2^2/A_4)}$$

we see that the same corrective factor has to be applied to all the second terms in the formulæ.

In the above we have used $\tan\beta_2$ as our argument, but the formulæ and the correction take precisely the same form if $\sin\gamma_2$ is the argument.

If we apply this corrective factor to the first two entries of Table III., which give a large percentage error—these correspond to cases approaching the failure of convergency and are therefore critical, we find, for lens (1)

M.	True aberration.	Formula uncorrected.	Percentage error.	Formula corrected.	Percentage error.
3	$-7 \cdot 87422$	$-7 \cdot 98683$	$1 \cdot 41$	$-7 \cdot 90811$	$0 \cdot 43$
2	$-1 \cdot 52334$	$-1 \cdot 55287$	$1 \cdot 94$	$-1 \cdot 54089$	$1 \cdot 16$

which shows a very sensible improvement.

The significance of this alteration is brought out more clearly when we consider the limiting case $M = \infty$ that is, rays actually issuing from the front focus (the case of an eye-piece). In this case, the geometrical image being at infinity, it is inconvenient to define the emergent ray by means of either longitudinal or transverse aberrations.

Let us consider the intercept of the ray on the back focal plane.

$$\text{This} = (-n_2 f\text{M}+\Delta x_2)\tan\alpha_2$$
$$= n_2 f[(\text{A}\tan^2\beta_2+e\tan^4\beta_2)/(1+b\tan^2\beta_2)-\text{M}]\tan\alpha_2.$$

Also

$$\tan\alpha_2 = (\tan\alpha_0/\mathbf{M})(1+b\mathbf{M}^{-2}\tan^2\alpha_0)/(1+c\mathbf{M}^{-2}\tan^2\alpha_0),$$

where

$$b = \text{B}-\text{E}_6\text{M}^2/\text{A}_4,\quad c = \text{C}-\text{E}_6\text{M}^2/\text{A}_4,\quad e = \text{E}-\text{E}_6\text{AM}^2/\text{A}_4,$$

and it is clear that, in the limit, where M (and $\mathbf{M}$) $= \infty$, $\tan\alpha_2$ must be *finite.*

This requires that b and c shall be of order M^3 and M^2 respectively, which is right, and leads to

$$\tan\alpha_2 = nb_3\tan^3\alpha_0/n\,(1+c_2\tan^2\alpha_0),$$

b_3 and c_2 being the coefficients of M^3 and M^2 in b and c respectively.

On the other hand, it is equally obvious that the intercept on the back focal plane must also approach a definite limit. Hence the factor

$$(\text{A}\tan^2\beta_2+e\tan^4\beta_2)/(1+b\tan^2\beta_2)-\text{M},$$

i.e.,

$$\frac{-1+(\text{A}-\text{M}b)\tan^2\beta_2/\text{M}+e\tan^4\beta_2/\text{M}}{1/\text{M}+b\tan^2\beta_2/\text{M}},$$

or

$$\frac{-1+(\text{A}-\text{M}b)\tan^2\alpha_0/\text{M}\mathbf{M}^2+e\tan^4\alpha_0/\text{M}\mathbf{M}^4}{1/\text{M}+b\tan^2\alpha_0/\text{M}\mathbf{M}^2}$$

must tend to a finite limit as M approaches ∞.

This necessarily involves that (1) $\text{A}-\text{M}b$ has M^3 in its leading term—a result already established, and (2) that e involves M^5 (and not M^6) in its leading term.

Hence, whenever we deal with incident rays actually passing through the front focus—and this necessarily occurs as soon as the results of the present paper are applied to aberrations off the axis—the *modified* B, C and E have to be used.

§ 15. *Combination Formulæ for More than Two Systems.*

The combination formulæ (47), (48), (51) are capable of explicit generalisation for any number of systems.

In the case of A and C successive applications of equations (47) and (51) lead at once to the results

$$f_{135\ldots 2n+1}A_{135\ldots 2n+1} = (f_1A_1)\,M^2_{35\ldots 2n+1}\mathbf{M}^2_{35\ldots 2n+1} + (f_3A_3)\,M^2_{5\ldots 2n+1}\mathbf{M}^2_{5\ldots 2n+1} + \ldots + (f_{2n-1}A_{2n-1})\,M^2_{2n+1}\mathbf{M}^2_{2n+1} + f_{2n+1}A_{2n+1}. \quad (71)$$

$$C_{135\ldots 2n+1} = C_1\mathbf{M}^2_{35\ldots 2n+1} + C_3\mathbf{M}^2_{5\ldots 2n+1} + \ldots + C_{2n-1}\mathbf{M}^2_{2n+1} + C_{2n+1}. \quad (72)$$

in which we have reverted to the notation of §§ 2, 7.

In the case of B, we have for three systems

$$B_{135} = B_{35} + B_1\mathbf{M}_{35}{}^2 + A_1M_{35}\mathbf{M}_{35}{}^2 f_1/f_{35}$$
$$= B_5 + B_3\mathbf{M}_5{}^2 + B_1\mathbf{M}_{35}{}^2 + (f_1A_1)\,M_{35}\mathbf{M}_{35}{}^2/f_{35} + (f_3A_3)\,M_5\mathbf{M}_5{}^2/f_5,$$

and the general law of combination is at once obvious.

We have

$$B_{135\ldots 2n+1} = B_1\mathbf{M}^2_{35\ldots 2n+1} + B_3\mathbf{M}^2_{5\ldots 2n+1} + \ldots + B_{2n-1}\mathbf{M}^2_{2n+1} + B_{2n+1} + (f_1A_1)\,(M\mathbf{M}^2/f)_{35\ldots 2n+1} + (f_3A_3)\,(M\mathbf{M}^2/f)_{5\ldots 2n+1} + \ldots + (f_{2n-1}A_{2n-1})\,(M\mathbf{M}^2/f)_{2n+1}. \quad (73)$$

The case of E is more difficult and we have not been able to obtain a form in which it can be written down for the combination of n systems.

But we can deal with it as follows, by determining the contribution of any one component to the whole :—

Consider three systems 1, 3, 5. It will be convenient to look upon 3 as a single lens, forming part of a larger system. 1 will then be the system of lenses preceding 3, and 5 the system following.

Applying equation (49), simplified by the use of the second invariant relation, we find

$$f_{135}E_{135} = f_5E_5 + f_{13}E_{13}M_5{}^2\mathbf{M}_5{}^4 + f_{13}A_{13}\mathbf{M}_5{}^2M_5{}^2\,(3A_5/M_5 - 3B_5 + 4C_5 + 4\sigma\{1 - \mathbf{M}_5{}^2\}) + 3B_{13}f_5A_5\mathbf{M}_5{}^2 - 2C_{13}f_5A_5\mathbf{M}_5{}^2.$$

Applying the equations of combination a second time, and picking out the terms involving A_3, B_3, C_3, E_3, we find these to be

$$f_3E_3M_5^2\mathbf{M}_5^4+M_5^2\mathbf{M}_5^4\left[f_1A_1\mathbf{M}_3^2M_3^2(3A_3/M_3-3B_3+4C_3)+f_3A_3\mathbf{M}_3^2(3B_1-2C_1)\right]$$
$$+f_3A_3\mathbf{M}_5^2M_5^2\left[3A_5/M_5-3B_5+4C_5+4\sigma(1-\mathbf{M}_5^2)\right]$$
$$+3B_3f_5A_5\mathbf{M}_5^2-2C_3f_5A_5\mathbf{M}_5^2.$$

Hence the lens 3 contributes to the final E

$$f_3A_3\{3M_5\mathbf{M}_5^2\left[A_5+(f_1/f_3)A_1M_{35}\mathbf{M}_{35}^2\right]+4\sigma(1-\mathbf{M}_5^2)\mathbf{M}_5^2M_5^2$$
$$+M_5^2\mathbf{M}_5^2\left[-3B_5+4C_5+(3B_1-2C_1)\mathbf{M}^2_{35}\right]\}$$
$$+B_3\mathbf{M}_5^2(3f_5A_5-3f_1A_1\mathbf{M}_{35}^2M_{35}^2)$$
$$+C_3\mathbf{M}_5^2(-2f_5A_5+4f_1A_1\mathbf{M}_{35}^2M_{35}^2)$$
$$+f_3E_3M_5^2\mathbf{M}_5^4,$$

and the A's, B's and C's in the curled brackets can be expressed in terms of the individual lenses of the system by means of equations (71), (72), (73).

If we denote the coefficients of A_3, B_3, C_3, E_3, in the above by l_3, m_3, p_3, q_3, then the contribution of the individual lens to E

$$= l_3A_3+m_3B_3+p_3C_3+q_3E_3.$$

Hence, if we vary K_3 for this lens, keeping focal length and magnifications unaltered

$$\Delta E = \Delta K_3\,(l_3\,\partial A_3/\partial K_3+m_3\,\partial B_3/\partial K_3+p_3\,\partial C_3/\partial K_3+q_3\,\partial E_3/\partial K_3).$$

If all the lenses are simultaneously varied, then we have

$$\Delta E = \Sigma\,\Delta K\,(l\,\partial A/\partial K+m\,\partial B/\partial K+p\,\partial C/\partial K+q\,\partial E/\partial K).$$

We have similar equations for ΔA, ΔB, ΔC, but they take a much simpler form.

Using these, we can, if we have enough lenses, vary the K's so that, *between limits*, we can make our four constants A, B, C, E take up any assigned values, or, if we wish to keep any one constant whilst slightly varying the others, we have a linear relation between the ΔK's.

§ 16. *Conclusion.*

We have now established a formula of fractional type for the longitudinal aberration of a symmetrical system which, while algebraically correct as far as the second order, does in fact, give results beyond this order in those numerical cases which have been tried, and largely overcomes the difficulties of slow convergency in critical regions.

We have further obtained a method for calculating the coefficients of this formula for any symmetrical optical system in terms of the coefficients for the components, in such a way that the effect of any single component upon the whole combination is immediately obtained.

In considering the convergency of the series usually employed, we have found that the value of the approximation depends upon the particular variable employed, and that if we wish to avoid trouble owing to lack of convergency we must use sin α_0 or tan α_0 (or a suitable multiple of these) as argument, where α_0 is the inclination of the original incident ray.

The numerical success of the new formula appears to suggest that progress in the algebraic treatment of symmetrical instruments is to be sought, not so much along the lines of developments in series, but in other mathematical directions such as continued products, or possibly continued fractions.

The next step would be to develop the method so as to cover the second order approximations to the emergent inclination. This will enable us to deal with aberrations off the axis of the system.

Some progress has already been made by the authors in this direction and the results will, it is hoped, form the subject of a later communication.