VI. *An Account of a Book intituled,* A Treatise of Fluxions, in Two Books, *by* Colin Mc Laurin, *A. M. Professor of Mathematics in the University of* Edinburgh, *and Fellow of the* ROYAL SOCIETY, 4to. *in Two Volumes, Pages* 763.

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THE Author's first Design, in composing this Treatise, was to establish the Method of Fluxions on Principles equally evident and unexceptionable with those of the antient Geometricians, by Demonstrations deduced after their Manner, in the most rigid Form, and by illustrating the more abstruse Parts of the Doctrine, to vindicate it from the Imputation of Uncertainty or Obscurity. But he has likewise comprehended in this Work the Application of Fluxions to the most important geometrical and philosophical Inquiries. It consists of an Introduction, and Two Books. In the Introduction he gives an Abstract of the Discoveries of the Antients in the higher Parts of Geometry, with Observations on their Method, and those that first succeeded to it. The First Book treats of Fluxions in a geometrical Method, and the Second treats of the Computations.

In the Introduction we have an Abstract not only of the Discoveries of the Antients in the higher Parts of Geometry, but likewise of their Demonstrations. After an Account of the Propositions of this kind, that are to be found in the Twelfth Book of *Euclid,*

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there follows a Summary of what is moſt material in the Treatiſes of *Archimedes*, concerning the Sphere and Cylinder, Conoids and Spheroids, the Quadrature of the *Parabola* and the ſpiral Lines. The Demonſtrations are not preciſely in the ſame Form as thoſe of *Archimedes*, but are often illuſtrated from the elementary Propoſitions concerning the Cone, or Corollaries from them, after the Example of *Pappus*, (*Coll. Math.* Prop. 21ſt, Lib. 4.) from whom a Propoſition is demonſtrated, and rendered more general, concerning the Area of the Spiral that is generated on a ſpherical Surface by the Compoſition of Two uniform Motions analogous to thoſe by which the Spiral of *Archimedes* is deſcribed on a Plane. This Area, though a Portion of a curve Surface, is found to admit of a perfect Quadrature, and this Propoſition concludes the Abſtract. He takes occaſion from theſe Theorems to demonſtrate ſome Properties of the Conic Sections, that are not mentioned by the Writers on that Subject; and there are more of this kind deſcribed in the XIth and XIVth Chapters of the Firſt Book.

It is known, that if a Parallelogram, circumſcribed about a given Ellipſe, have its Sides parallel to the conjugate Diameters, then ſhall its Area be of an invariable or given Magnitude, and equal to the Rectangle contained by the Axes of the Figure; but this is only a Caſe of a more general Propoſition. For if, upon any Diameter produced without the Ellipſe, you take Two Points, one on each Side of the Centre at equal Diſtances from it, and the Four Tangents be drawn from theſe Points to the Ellipſe, thoſe Tangents ſhall form a Parallelogram, which is

always

always of a given or invariable Magnitude, when the Ellipse is given, if the *Ratio* of those Distances to the Diameter be given; and when the *Ratio* of those Distances to the Semidiameter is that of the Diagonal of a Square to the Side, (or of $\sqrt{2 \text{ to } 1}$) the Parallelogram has its Sides parallel to conjugate Diameters. It is likewise shown here, how the Triangles, *Trapezia*, or Polygons of any kind are determined, which, circumscribed about a given Ellipse, are always of a given Magnitude.

There is also a general Theorem concerning the *Frustum* of a Sphere, Cone, Spheroid, or Conoid, terminated by parallel Planes, when compared with a Cylinder of the same Altitude on a Base equal to the middle Section of the *Frustum* made by a parallel Plane. The Difference betwixt the *Frustum* and the Cylinder is always the same in different Parts of the same, or of similar Solids, when the Inclination of the Planes to the Axis, and the Altitude of the *Frustum*, are given. This Difference vanishes in the parabolic Conoid. It is the same in all Spheres; being equal to half the Content of a Sphere of a Diameter equal to the Altitude of the *Frustum*. In the Cone it is One-fourth of the Content of a similar Cone of the same Height with the *Frustum*; and in other Figures it is reduced to the Difference in the Cone.

In the Remarks on the Method of the Antients, the Author observes, that they established the higher Parts of their Geometry on the same Principles as the Elements of the Science, by Demonstrations of the same kind; that they seem to have been careful not to suppose any thing to be done, till by a previous Problem

blem they had shown how it was to be performed: Far less did they suppose any thing to be done, that cannot be conceived to be possible, as a Line or Series to be actually continued to Infinity, or a Magnitude to be diminished till it become infinitely less than it was. The Elements into which they resolved Magnitudes were always finite, and such as might be conceived to be real. Unbounded Liberties have been introduced of late, by which Geometry (wherein every thing ought to be clear) is filled with Mysteries, and Philosophy is likewise perplexed. Several Instances of this kind are mentioned. The Series 1, 2, 3, 4, 5, 6, 7, *&c.* is supposed by some to be actually continued to Infinity; and, after such a Supposition, we are puzzled with the Question, Whether the Number of finite Terms in such a Series is finite or infinite. In order to avoid such Suppositions, and their Consequences, the Author chose to follow the Antients in their Method of Demonstration as much as possible. Geometry has been always considered as our surest Bulwark against the Subtleties of the Sceptics, who are ready to make use of any Advantages that may be given them against it *; and it is important, not only that the Conclusions in Geometry be true, but likewise that their Evidence be unexceptionable. However, he is far from affirming, that the Method of Infinitesimals is without Foundation, and afterwards endeavours to justify a proper Application of it.

The Grounds of the Method of Fluxions are described in Chap. 1. Book I. and again in Chap. 1.

* See *Bayle*'s Dictionary, Article *Zeno*.

Book II. In the former, Magnitudes are conceived to be generated by Motion, and the Velocity of the generating Motion is the Fluxion of the Magnitude. Lines are ſuppoſed to be generated by the Motion of Points. The Velocity of the Point that deſcribes the Line is its Fluxion, and meaſures the Rate of its Increaſe or Decreaſe. Other Magnitudes may be repreſented by Lines that increaſe or decreaſe in the ſame Proportion with them; and their Fluxions will be in the ſame Proportion as the Fluxions of thoſe Lines, or the Velocities of the Points that deſcribe them. When the Motion of a Point is uniform, its Velocity is conſtant, and is meaſured by the Space which is deſcribed by it in a given Time. When the Motion varies, the Velocity at any Term of the Time is meaſured by the Space which would be deſcribed in a given Time, if the Motion was to be continued uniformly from that Term without any Variation. In order to determine that Space, and conſequently the Velocity which is meaſured by it, Four Axioms are propoſed concerning variable Motions, Two concerning Motions that are accelerated, and Two concerning ſuch as are retarded. The Firſt is, That the Space deſcribed by an accelerated Motion is greater than the Space which would have been deſcribed in the ſame Time, if it had not been accelerated, but had continued uniform from the Beginning of the Time. The Second is, That the Space which is deſcribed by an accelerated Motion, is leſs than the Space which is deſcribed in an equal Time by the Motion which is acquired by that Acceleration continued afterwards uniformly. By theſe, and Two ſimilar Axioms concerning retarded Motions,

tions, the Theory of Motion is rendered applicable to this Doctrine with the greatest Evidence, without supposing Quantities infinitely little, or having recourse to prime or ultimate *Ratios*. The Author first demonstrates from them all the general Theorems concerning Motion, that are of Use in this Doctrine; as that when the Spaces described by Two variable Motions are always equal, or in a given *Ratio*, the Velocities are always equal, or in the same given *Ratio*; and conversely, when the Velocities of Two Motions are always equal to each other, or in a given *Ratio*, the Spaces described by those Motions in the same Time are always equal, or in that given *Ratio*; that when a Space is always equal to the Sum or Difference of the Spaces described by Two other Motions, the Velocity of the First Motion is always equal to the Sum or Difference of the Velocities of the other Motions; and conversely, that when a Velocity is always equal to the Sum or Difference of Two other Velocities, the Space described by the First Motion is always equal to the Sum or Difference of the Spaces described by these Two other Motions. In comparing Motions in this Doctrine, it is convenient and usual to suppose one of them uniform; and it is here demonstrated, that if the Relation of the Quantities be always determined by the same Rule or Equation, the *Ratio* of the Motions is determined in the same manner, when both are supposed variable. These Propositions are demonstrated strictly by the same Method which is carried on in the ensuing Chapters for determining the Fluxions of the Figures.

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In Chap. II. a Triangle that has Two of its Sides given in Position, is supposed to be generated by an Ordinate moving parallel to itself along the Base. When the Base increases uniformly, the Triangle increases with an accelerated Motion, because its successive Increments are *Trapezia*, that continually increase. Therefore, if the Motion with which the Triangle flows, was continued uniformly from any Term for a given Time, a less Space would be described by it than the Increment of the Triangle which is actually generated in that Time by Axiom I. but a greater Space than the Increment which was actually generated in an equal Time preceding that Term, by Axiom II. and hence it is demonstrated, that the Fluxion of the Triangle is accurately measured by the Rectangle contained by the corresponding Ordinate of the Triangle, and the right Line which measures the Fluxion of the Base. The Increment which the Triangle acquires in any Time, is resolved into Two Parts; that which is generated in consequence of the Motion with which the Triangle flows at the Beginning of the Time, and that which is generated in consequence of the Acceleration of this Motion for the same Time. The latter is justly neglected in measuring that Motion (or the Fluxion of the Triangle at that Term), but may serve for measuring its Acceleration, or the Second Fluxion of the Triangle. The Motion with which the Triangle flows, is similar to that of a Body descending in free Spaces by an uniform Gravity, the Velocity of which, at any Term of the Time, is not to be measured by the Space described by the Body in a given Time, either before or after that Term, because the

Motion continually increaſes, but by a Mean between theſe Spaces.

When the Sides of a Rectangle increaſe or decreaſe with uniform Motions, it may be always conſidered as the Sum or Difference of a Triangle and *Trapezium*; and its Fluxion is derived from the laſt Propoſition. If the Sides increaſe with uniform Motions, the Rectangle increaſes with an accelerated Motion; and in meaſuring this Motion at any Term of the Time, a Part of the Increment of the Rectangle, that is here determined, is rejected, as generated in conſequence of the Acceleration of that Motion.

The Fluxions of a curvilineal Area (whether it be generated by an Ordinate moving parallel to itſelf, or by a Ray revolving about a given Centre) and of the Solid, generated by the Area revolving about the Baſe, are determined by Demonſtrations of the ſame kind; and when the Ordinates of the Figure increaſe, the Increment of the Area is reſolved in like manner into Two Parts, one of which is only to be retained in meaſuring the Fluxion of the Area, the other being rejected as generated in conſequence of the Acceleration of the Motion with which the Figure flows. An Illuſtration of Second and Third Fluxions is given by reſolving the Increment of a Pyramid or Cone into the ſeveral reſpective Parts that are conceived to be generated in conſequence of the Firſt, Second, and Third Fluxions of the Solid, when the Axis is ſuppoſed to flow uniformly.

In Chap. V. a Series of Lines in Geometrical Progreſſion are repreſented by an eaſy Conſtruction. The Firſt Term being ſuppoſed invariable, and the Second to increaſe uniformly, all the ſubſequent

Terms

Terms increaſe with accelerated Motions. The Velocities of the Points that deſcribe thoſe Lines being compared, it is demonſtrated from the Axioms by common Geometry, that the Fluxions of any Two Terms are in a *Ratio* compounded of the *Ratio* of the Terms, and of the *Ratio* of the Numbers that expreſs how many Terms precede them in the Progreſſion.

In the VIth Chapter, the Nature and Properties of Logarithms are deſcribed after the celebrated Inventor; and it is obſerved, that he made uſe of the very Terms *Fluxus* and *Fluat* on this Occaſion. A Line is ſaid to increaſe or decreaſe *proportionally*, when the Velocity of the Point, that deſcribes it, is always as its Diſtance from a certain Term of the Line; and if in the mean time another Point deſcribes a Line with a certain uniform Motion, the Space deſcribed by the latter Point is always the Logarithm of the Diſtance of the former from the given Term. Hence the Fluxion of this Diſtance is to the Fluxion of its Logarithm as that Diſtance is to an invariable Line; and the Fluxions of the Quantities that have their Logarithms in an invariable *Ratio*, are to each other in a *Ratio* compounded of this invariable *Ratio*, and of the *Ratio* of the Quantities themſelves. Some Propoſitions are demonſtrated, that relate to the Computation of Logarithms; but this Subject is proſecuted farther in the Second Book. The Logarithmic Curve is here deſcribed, with the Analogy betwixt Logarithms and Hyperbolic *Ratios*.

In the VIIth Chapter, after a general Definition of Tangents, it is demonſtrated, that the Fluxions of the Baſe, Ordinate, and Curve, are in the ſame Propor-

tion to each other, as the Sides of a Triangle respectively parallel to the Base, Ordinate, and Tangent. When the Base is supposed to flow uniformly, if the Curve be convex towards the Base, the Ordinate and Curve increase with accelerated Motions; but their Fluxions at any Term are the same as if the Point which describes the Curve had proceeded uniformly from that Term in the Tangent there. Any further Increment which the Ordinate or Curve acquires, is to be imputed to the Acceleration of the Motions with which they flow. A Ray that revolves about a given Centre, being supposed to meet any Curve and an Arc of a Circle described from the same Centre, the Fluxions of the Ray, Curve, and circular Arc, are compared together; and several other Propositions concerning Tangents are demonstrated from the Axioms. The next Chapter treats of the Fluxions of curve Surfaces in a similar manner.

The IXth Chapter treats chiefly of the greatest and least Ordinates of Figures, and of the Points of contrary Flexure and Cuspids. The Fluxion of the Base being given, when the Fluxion of the Ordinate vanishes, the Tangent becomes parallel to the Base, and the Ordinate most commonly is a *Maximum* or *Minimum*, according to the Rule given by Authors upon this Subject. But if the Second Fluxion of the Ordinate vanish at the same time, and the Third Fluxion be real, this Rule does not hold, for the Ordinate is in that Case neither a *Maximum* nor *Minimum*. If the First, Second, and Third Fluxions vanish, and the Fourth Fluxion be real, the Ordinate is a *Maximum* or *Minimum*. The general Rule demonstrated in this Chapter, and again in the last Chapter of the

Second

Second Book, is, that when the First Fluxion of the Ordinate, with its Fluxions of any subsequent successive Orders, vanish, and the Number of all these Fluxions that vanish is odd, then the Ordinate is a *Maximum* or *Minimum*, according as the Fluxion of the next Order to these is negative or positive. The Ordinate passes through a Point of contrary Flexure, when its Fluxion becomes a *Maximum* or *Minimum*, supposing the Curve to be continued on both Sides of the Ordinate. Hence the common Rule for finding the Points of contrary Flexure is corrected in a similar manner. Such a Point is not always formed when the Second Fluxion of the Ordinate vanishes; for if its Third Fluxion likewise vanishes, and its Fourth Fluxion be real, the Curve may have its Cavity turned all one Way. The same is to be said, when its Fluxions of the subsequent successive Orders vanish, if the Number of all those that vanish be even. Other Theorems are subjoined relating to this Subject.

The Xth Chapter treats of the Asymptotes of Lines, the Areas bounded by them and the Curves, the Solids generated by these Areas, of spiral Lines, and the Limits of the Sums of Progressions. The Analogy there is betwixt these Subjects, induced the Author to treat of them in one Chapter, and illustrate them by one another. He begins with Three of the most simple Instances of Figures that have Asymptotes. In the common Hyperbola, the Ordinate is reciprocally as the Base, and therefore decreases while the Base increases, but never vanishes, because the Rectangle contained by it and the Base is always a given Area, and it is assignable at any assignable Distance,

how

how great ſoever. The Points of the Conchoid are determined by drawing right Lines from a given Centre, and upon theſe produced from the Aſymptote, taking always a given right Line; ſo that the Curve never meets the Aſymptote, but continually approaches to it, becauſe of the greater and greater Obliquity of this right Line. The Third is the Logarithmic Curve, wherein the Ordinates, at equal Diſtances, decreaſe in Geometrical Proportion, but never vaniſh, becauſe each Ordinate is in a given *Ratio* to the preceding Ordinate. Geometrical Magnitude is always underſtood to conſiſt of Parts; and to have no Parts, or to have no Magnitude, are conſidered as equivalent in this Science *. There is, however, no Neceſſity for conſidering Magnitude as made up of an infinite Number of ſmall Parts; it is ſufficient, that no Quantity can be ſuppoſed to be ſo ſmall, but it may be conceived to be diminiſhed further; and it is obvious, that we are not to eſtimate the Number of Parts that may be conceived in a given Magnitude, by thoſe which in particular determinate Circumſtances may be actually perceived in it by Senſe; ſince a greater Number of Parts become viſible in it by varying the Circumſtances in which it is perceived.

It is hardly poſſible to give a tolerable Extract of this or the following Chapters, without Diagrams and Computations: We ſhall therefore obſerve only, that after giving ſome plain and obvious Inſtances, wherein a Quantity is always increaſing, and yet never

* See *Euclid*'s Elements, Def. I. Lib. I.

amounts

amounts to a certain finite Magnitude (as, while the Tangent increaſes, the Arc increaſes, but never amounts to a Quadrant); this is applied ſucceſſively to the ſeveral Subjects mentioned in the Title of the Chapter. Let the Figure be concave towards the Baſe, and ſuppoſe it to have an Aſymptote parallel to the Baſe; in this Caſe the Ordinate always increaſes while the Baſe is produced, but never amounts to the Diſtance between the Aſymptote and the Baſe. In like manner a curvilineal Area, in a Second Figure, may increaſe, while the Baſe is produced, and approach continually to a certain finite Space, but never amount to it: This is always the Caſe, when the Ordinate of this latter Figure is to a given right Line, as the Fluxion of the Ordinate of the former is to the Fluxion of the Baſe; and of this various Examples are given. A Solid may increaſe in the ſame manner, and yet never amount to a given Cube or Cylinder, when the Square of the Ordinate of the latter Figure is to a given Square, as the Fluxion of the Ordinate of the firſt Figure is to the Fluxion of the Baſe. A Spiral may in like manner approach to a Point continually, and yet in any Number of Revolutions never arrive at it; and there are Progreſſions of Fractions that may be continued at Pleaſure, and yet the Sum of the Terms may be always leſs than a given Number. Various Rules are demonſtrated, and illuſtrated by Examples, for determining when a Figure has an Aſymptote parallel or oblique to the Baſe; when the Area terminated by the Curve and the Aſymptote has a Limit which it never exceeds, or may be produced till it ſurpaſs any aſſignable Space; when the Solid generated by that Area, the

the Surface generated by the Perimeter of the Curve, the ſpiral Area generated by the revolving Ray, the ſpiral Line itſelf, or the Sum of the Terms of a Progreſſion, have ſuch Limits or not; and for meaſuring thoſe Limits. The Author inſiſts on theſe Subjects, the rather that they are commonly deſcribed in very myſterious Terms, and have been the moſt fertile of Paradoxes of any Parts of the higher Geometry. Theſe Paradoxes, however, amount to no more than this: That a Line or Number may be continually acquiring Increments, and thoſe Increments may decreaſe in ſuch a manner, that the whole Line or Number ſhall never amount to a given Line or Number. The Neceſſity of admitting this is obvious enough, and is here ſhewn from the Nature of the moſt common geometrical Figures in Art. 292, 293, *&c.* and from any Series of Fractions that decreaſe continually, in Art. 354, 355, *&c.*

The XIth Chapter treats of the Curvature of Lines, its Variation, the Degrees of Contact of the Curve and Circle of Curvature, and of various Problems that depend on the Curvature of Lines. This Subject is treated fully, becauſe of its extenſive Uſefulneſs, and becauſe in this conſiſts one of the greateſt Advantages of the modern Geometry above that of the Antients. The Author on this, as former Occaſions, begins by premiſing the neceſſary Definitions. Curve Lines touch each other in a Point, when the ſame right Line is their common Tangent at that Point; and that which has the cloſeſt Contact with the Tangent, or paſſes betwixt it and the other Curve through the Angle of Contact formed by them, being leſs inflected from the Tangent, is therefore leſs curve. Thus a greater

greater Circle has a leſs Curvature than a leſſer Circle; and ſince the Curvature of Circles may be varied indefinitely, by inlarging or diminiſhing their Diameters, they afford a Scale by which the Curvature of other Lines may be meaſured. As the Tangent is the right Line which touches the Arc ſo cloſely, that no other right Line can be drawn between them; ſo the Circle of Curvature is that which touches the Curve ſo cloſely, that no other Circle can be drawn through the Point of Contact between them. As the Curve is ſeparated from its Tangent in conſequence of its Flexure or Curvature, ſo it is ſeparated from the Circle of Curvature in conſequence of the Variation of its Curvature; which is greater or leſs, according as its Flexure from that Circle is greater or leſs.

The Tangent of the Figure being conſidered as the Baſe, a new Figure is imagined, whoſe Ordinate is a Third Proportional to the Ordinate and Baſe of the Firſt. This new Figure determines the Chord of the Circle of Curvature by its Interſection with the Ordinate at the Point of Contact, and by the Tangent of the Angle in which it cuts that Circle, meaſures the Variation of Curvature. The leſs this Angle is, the cloſer is the Contact of the Curve and Circle of Curvature, of which there may be indefinite Degrees. When the Figure propoſed is a conic Section, the new Figure is likewiſe a conic Section; and it is a right Line when the Firſt Figure is a *Parabola*, and the Ordinates are parallel to the Axis; or when the Firſt Figure is an *Hyperbola*, and the Ordinates are parallel to either Aſymptote. Hence the Curvature and its Variation in a conic Section are determined by ſeveral Conſtructions; and, amongſt other Theorems,

orems, it is ſhewn, that the Variation of Curvature at any Point of a conic Section is as the Tangent of the Angle contained by the Diameter which paſſes through that Point, and by the Perpendicular to the Curve.

When the Ordinate at the Point of Contact is an Aſymptote to the new Figure, the Curvature is leſs than in any Circle; and this is the Caſe in which it is ſaid to be infinitely little, or the Ray of Curvature is ſaid to be infinitely great. Of this kind is the Curvature at the Points of contrary Flexure in the Lines of the Third Order. When the new Figure paſſes through the Point of Contact, the Curvature is greater than in any Circle, or the Ray of Curvature vaniſhes; and in this Caſe the Curvature is ſaid to be infinitely great. Of this kind is the Curvature at the Cuſpids of the Lines of the Third Order.

As Lines which paſs through the ſame Point have the ſame Tangent when the Firſt Fluxions of the Ordinate are equal, ſo they have the ſame Curvature when the Second Fluxions of the Ordinate are likewiſe equal; and half the Chord of the Circle of Curvature that is intercepted between the Points wherein it interſects the Ordinate, is a Third Proportional to the right Lines that meaſure the Second Fluxion of the Ordinate and Firſt Fluxion of the Curve, the Baſe being ſuppoſed to flow uniformly. When a Ray revolving about a given Point, and terminated by the Curve, becomes perpendicular to it, the Firſt Fluxion of the Ray vaniſhes; and if its Second Fluxion vaniſhes at the ſame time, that Point muſt be the Centre of Curvature. The ſame is to be ſaid when the angular Motion of the Ray about that Point

Point is equal to the angular Motion of the Tangent of the Curve; as the angular Motion of the *Radius* of a Circle about its Centre is always equal to the angular Motion of the Tangent of the Circle. Thus the various Properties of the Circle suggest various Theorems for determining the Centre of the Curvature.

Because Figures are often supposed to be described by the Intersections of right Lines revolving about given Poles, Three Theorems are given in Prop. 18. 26. and 35. for determining the Tangents, Asymptotes, and Curvature of such Lines, from the Description, which are illustrated by Examples. A new Property of Lines of the Third Order is subjoined to Prop. 35. The Evolution of Lines is considered in Prop. 36. The Tangents of the *Evoluta* are the Rays of Curvature of the Line which is described by its Evolution; and the Variation of Curvature in the latter is measured by the *Ratio* of the Ray of Curvature of the former to the Ray of Curvature of the latter.

Sir *Isaac Newton*, in a Treatise lately published, measures the Variation of the Curvature by the *Ratio* of the Fluxion of the Ray of Curvature to the Fluxion of the Curve; and is followed by the Author, to avoid the Perplexity which a Difference in Definitions occasions to Readers, though he hints (in Art. 386.) that this *Ratio* gives rather the Variation of the Ray of Curvature, and that it might have been proper to have measured the Variation of Curvature rather by the *Ratio* of the Fluxion of the Curvature itself to the Fluxion of the Curve; so that the Curvature being inversely as the Ray of Curvature, and

consequently its Fluxion as the Fluxion of the Ray itself directly, and the Square of the Ray inversely, its Variation would have been directly as the Measure of it, according to Sir *Isaac Newton*'s Definition, and inversely as the Square of the Ray of Curvature: According to this Explication, it would have been measured by the Angle of Contact contained by the Curve and Circle of Curvature, in the same manner as the Curvature itself is measured by the Angle of Contact contained by the Curve and Tangent. The Ground of this Remark will better appear from an Example: According to Sir *Isaac Newton*'s Explication, the Variation of Curvature is uniform in the Logarithmic Spiral, the Fluxion of the Ray of Curvature in this Figure being always in the same *Ratio* to the Fluxion of the Curve; and yet while the Spiral is produced, though its Curvature decreases, it never vanishes; which must appear strange to such as do not attend to the Import of his Definition.—— It is easy, however, to derive one of these Measures of this Variation from the other, and because Sir *Isaac Newton*'s is (generally speaking) assigned by more simple Expressions, the Author has the rather conformed to it in this Treatise, but thought it necessary to give the Caution we have mentioned.

The greatest Part of this Chapter is imployed in treating of useful Problems, that have a Dependence on the Curvature of Lines. First, the Properties of the Cycloid are briefly demonstrated, with the Application of this Doctrine to the Motion of Pendulums, by shewing that when the Motion of the generating Circle along the Base is uniform, and therefore may measure the Time, the Motion of the Point that

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describes the Cycloid, is such as would be acquired by a heavy Body descending along the cycloidal Arc, the Axis of the Figure being supposed perpendicular to the Horizon. In the next place, the Caustics, by Reflexion and Refraction, are determined. If Perpendiculars be always drawn from the radiating Point to the Tangents of the Curve, and a new Curve be supposed to be the *Locus* of the Intersections of the Perpendiculars and Tangents, then the Line, by the Evolution of which that new Curve can be described, is similar and similarly situated to the Caustic by Reflexion. The Doctrine of centripetal Forces is treated at length from Art. 416. to 493.

First, a Body is supposed to descend freely by its Gravity in a vertical Line; and because the Gravity is the Power which accelerates the Motion of the Body, it must be measured by the Fluxion of its Velocity, or the Second Fluxion of the Space described by it. When the vertical Line is supposed to move parallel to itself with an uniform Motion, the Body will descend in it in the same manner as before; and the Gravity will be still measured by the Second Fluxion of the Descent, or the Second Fluxion of the Ordinate of the Curve that is traced in this Case by the Body on an immoveable Plain, and therefore is as the Square of the Velocity (which is measured by the Fluxion of the Curve) directly, and the Chord of the Circle of Curvature that is in the Direction of the Gravity inversely, by a Proposition mentioned above. When the Gravity acts uniformly, and in parallel Lines, the Projectile, in describing any Arc, falls below the Tangent drawn at the Beginning of the Arc, as much as if it had fallen perpendicu-

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larly in the Vertical; and the Time being given, the Gravity may be meaſured by the Space which is the Subtenſe of the Angle of Contact. In other Caſes, when the Gravity varies, or its Direction changes, it may be meaſured at any Point by the Subtenſe of the Angle of Contact, that would have been generated in a given Time, if the Gravity had continued to act uniformly in parallel Lines from that Term, that is, by the Subtenſe of the Angle of Contact in the Parabola that has its Diameter in the Direction of the Force, and has the cloſeſt Contact with the Curve; which leads us to the ſame Theorem as before.

In general, let the Gravity (that reſults from the Compoſition of any Number of centripetal Forces, which are ſuppoſed to act on the Body in one Plane) be reſolved into a Force parallel to the Baſe; then the former ſhall be meaſured by the Second Fluxion of the Ordinate, and the latter by the Second Fluxion of the Baſe, the Time being ſuppoſed to flow uniformly, ſo that the Velocity of the Body may be meaſured by the Fluxion of the Curve. When the Trajectory is not in one Plane, the Force is reſolved in a ſimilar manner into Three Forces, which are meaſured by Three Second Fluxions analogous to them.

Whether the Body move in a Void, or in a Medium that reſiſts its Motion; the Gravity that reſults from the Compoſition of the centripetal Forces which act upon the Body, is always as the Square of its Velocity directly, and the Chord of the Circle of Curvature that is in the Direction of the Gravity inverſely.

When

When a Body deſcribes any Trajectory in a Void or in a Medium, by a Force directed to One given Centre, the Velocity at any Point of the Trajectory is to the Velocity by which a Circle could be deſcribed in a Void about the ſame Centre, at the ſame Diſtance, by the ſame Gravity, in the ſubduplicate *Ratio* of the angular Motion of the Ray drawn always from the Body to the Centre, to the angular Motion of the Tangent of the Trajectory: And, if there be no Reſiſtance, the Velocity in the Trajectory at any Point, is the ſame that would be acquired by the Body, if it was to fall from that Point through One-fourth of the Chord of the Circle of Curvature that is in the Direction of the Gravity, and the Gravity at that Point was to be continued uniformly during its Deſcent.

If the centripetal Force be inverſely as any Power of the Diſtance whoſe Exponent is any Number m greater than Unit, there is a certain Velocity (*viz.* that which is to the Velocity in a Circle at the ſame Diſtance as $\sqrt{2}$ to $\sqrt{m-1}$) which would be juſt ſufficient to carry off the Body upwards in a vertical Line, ſo as that it ſhould continue to aſcend for ever, and never return towards the Centre. If the Body be projected in any other Direction with the ſame Velocity, it will deſcribe a Trajectory which is here conſtructed: It is a *Parabola* when $m=2$, a Logarithmic Spiral when $m=3$, an Epicycloid when $m=4$, a Circle that paſſes through the Centre of the Forces when $m=5$, and the *Lemniſcata* when $m=7$. In general, it is conſtructed by drawing a Perpendicular from the Centre of the Forces to a right Line given in Poſition, and any other Ray to the ſame right Line, then

then increasing or diminishing the Angle contained by this Ray and the Perpendicular in the given *Ratio* of 2 to the Difference between 3 and *m*, and increasing or diminishing the Logarithm of the Ray in the same given *Ratio*. The Trajectories described in analogous Cases by centrifugal Forces, are constructed in a similar manner. These are the Figures in which the Perpendicular, from a given Centre on the Tangent, is always as some Power of the Ray drawn from the same Centre to the Point of Contact, which are afterwards found to arise in the Resolution of the most simple Cases of Problems of various kinds.

When the Area described about the Centre of an Ellipse is given, the Subtense of the Angle of Contact, drawn through one Extremity of the Arc parallel to the Semidiameter drawn to the other Extremity, is in a given *Ratio* to this Semidiameter; and therefore, when an Ellipse is described by a Force directed towards the Centre, that Force is always as the Distance from the Centre. When the Force is directed toward the *Focus*, it is inversely as the Square of the Distance. And these Two Cases are considered particularly, because of their Usefulness in the true Theory of Gravity. To illustrate which, the Laws of centripetal Forces that would cause a Body to descend continually toward the Centre, or ascend from it, are distinguished from those which cause the Body to approach towards the Centre, and recede from it by turns. A Body approaches from the higher Apsid toward the Centre, when its Velocity is less than what is requisite to carry it in a Circle; and if its Velocity increase, while it descends, in a higher

higher Proportion than the Velocities requisite to carry Bodies in Circles about the same Centre, the Velocity in the lower Part of the Curve may exceed the Velocity in a Circle at the same Distance, and thereby become sufficient to carry off the Body again. But while the Distance decreases, if the Velocities in Circles increase in the same or in a higher Proportion, than the Velocity in a Trajectory can increase, the Body must either continually approach toward the Centre, if it once begin to approach to it, or recede continually from the Centre, if it once begin to ascend from it; and this is the Case, when the centripetal Force increases as the Cube of the Distance decreases, or in a higher Proportion. But though, in such Cases, the Body approach continually towards the Centre, we are not to conclude, that it will always approach to it till it fall into it, or come within any given Distance; for it is demonstrated afterwards in Art. 879 and 880. that it may approach to the Centre for ever, in a Spiral that never descends to a given Circle described in the same Plane, and that it may recede from it for ever in a Spiral that never arises to a given Altitude. An Example of each Case is given when the centripetal Force is inversely as the Fifth Power of the Distance.

When the Trajectory is described in a *Medium*, let z be to a given Magnitude as the centripetal Force is to the Force by which the same Trajectory could be described in a Void; and if the Area be supposed to flow uniformly, the Resistance will be in the compound *Ratio* of the Fluxion of z, and of the Fluxion of the Curve; and the Density of the Medium (supposing the Resistance to be in the compound *Ratio*

of the Denſity and of the Square of the Velocity) ſhall be as the Fluxion of the Logarithm of z directly, and the Fluxion of the Curve inverſely. Hence, when any Figure that can be deſcribed in a Void by a Force that varies according to any Power of the Diſtance from the Centre, is deſcribed in a Medium, the Denſity of the Medium muſt be inverſely as the Tangent of the Figure bounded by a Perpendicular at the Centre to the Ray drawn from it to the Point of Contact.

After giving ſome Properties of the Trajectories, that are deſcribed by a Body when it gravitates in right Lines perpendicular to a given Surface, and their Application to optical Uſes, the Author proceeds to conſider the Motion of a Body that gravitates towards ſeveral Centres. In ſuch Caſes, that Surface is ſaid to be horizontal, which is always perpendicular to the Direction of the Gravity that reſults from the Compoſition of the ſeveral Forces; and it is ſhewn, that the Velocity which is acquired by deſcending from one horizontal Surface to another, is always the ſame (whether the Body move in right Lines, or in any Curves); the Square of which is meaſured by the Aggregate of ſeveral Areas which have the Diſtances from the reſpective Centres for their Baſes, and right Lines proportional to the Forces at theſe Diſtances for their Ordinates.

The Force which acts upon the Moon is reſolved into a Force perpendicular to the Plane of the Ecliptic, and a Force parallel to it. This laſt is again reſolved into that which is parallel to the Line of the *Syzigies*, and that which is parallel to the Line joining the Quadratures. The Firſt meaſures the Se-

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cond Fluxion of the Diſtance of the Moon from that Plane, the Second and Third meaſure the Second Fluxions of her Diſtances from the Line of the Quadratures, and from the Line of the *Syzigies*, reſpectively. Hence a Conſtruction is derived of the Trajectory which would be deſcribed by the Moon about the Earth, in conſequence of their unequal Gravitation towards the Sun, if the Gravity of the Moon towards the Earth was as her Diſtance from it. From this a Computation is deduced of the Motion of the Nodes of the Moon, and of the Variation of the Inclination of the Plane of her Orbit, which we cannot deſcribe here. It is ſufficient to obſerve, that theſe Motions are found to agree nearly with thoſe which have been deduced from other Theories, and from Aſtronomical Obſervations.

A Fluid being ſuppoſed to gravitate towards two given Centres with equal and invariable Forces, it is ſhewn, that the Figure of the Fluid muſt be that of an oblong Spheroid, and that thoſe two Centres muſt be the *Foci* of the generating Ellipſe. The Nature of the Figure is alſo ſhewn, when the Fluid gravitates towards ſeveral Centres, or when it revolves on its Axis; but theſe are mentioned briefly, becauſe ſuch Theories are of little or no Uſe for diſcovering the Figures of the Planets.

In the 12th Chapter, the Author proceeds to conſider the more conciſe Methods, by which the Fluxions of Quantities are uſually determined, and to deduce general Theorems more immediately applicable to the Reſolution of Geometrical and Philoſophical Problems. In the Method of Infiniteſimals, the Element, by which any Quantity increaſes or decreaſes,

is ſuppoſed to become infinitely ſmall, and is generally expreſſed by Two or more Terms, ſome of which become infinitely leſs than the reſt, and therefore being neglected as of no Importance, the remaining Terms form, what is called the *Difference* of the Quantity propoſed. The Terms that are neglected in this manner are the very ſame which ariſe in conſequence of the Acceleration or Retardation of the generating Motion, during the infinitely ſmall Time in which the Element is generated; and therefore theſe Differences are in the ſame *Ratio* to each other as the generating Motions or Fluxions. Hence the Concluſions in this Method are accurately true, without even an infinitely ſmall Error, and agree with thoſe that are deduced by the Method of Fluxions.

It is uſual in this Method to conſider a Curve as a Polygon of an infinite Number of Sides, which, being produced, give the Tangents of the Curve, and, by their Inclination to each other, meaſure its Curvature. But it is neceſſary in ſome Caſes, if we would avoid Error, to reſolve the Element of the Curve into ſeveral infinitely ſmall Parts, or even ſometimes into Infiniteſimals of the Second Order; and Errors that might otherwiſe ariſe in its Application, may, with due Care, be corrected by a proper Uſe of this Method itſelf, of which ſome Inſtances are given. If we were to ſuppoſe, for Example, the leaſt Arc that can be deſcribed by a Pendulum to coincide with its Chord, the Time of the Vibration derived from this Suppoſition will be found erroneous; but by reſolving that Arc into more and more infinitely ſmall Parts, we approach to the true
Time

Time in which it is described. By supposing the Tangent of the Curve to be the Production of the rectilineal Element of the Curve, the Subtense of the Angle of Contact is found equal to the Second Difference or Fluxion of the Ordinate; but in this Inquiry, the Tangent ought to be supposed to be equally inclined to the two Elements of the Curve that terminate at the Point of Contact; and then the Subtense of the Angle of Contact will be found equal to half the Second Difference of the Ordinate, which is its true Value.

Sir *Isaac Newton*, however, investigates the Fluxions of Quantities in a more unexceptionable manner. He first determines the finite simultaneous Increments of the Fluents, and, by comparing them, investigates the *Ratio* that is the Limit of the various Proportions which they bear to each other, while he supposes them to decrease together till they vanish. When the generating Motions are variable, the *Ratio* of the simultaneous Increments that are generated from any Term, is expressed by several Quantities, some of which arise from the *Ratio* of the generating Motions at that Term, and others from the subsequent Acceleration or Retardation of these Motions. While the Increments are supposed to be diminished, the former remain invariable, but the latter decrease continually, and vanish with the Increments; and hence the Limit of the variable *Ratio* of the Increments (or their ultimate *Ratio)* gives the precise *Ratio* of the generating Motions or Fluxions. Most of the Propositions in the preceding Chapters may be more briefly demonstrated by this Method, (of which

several

ſeveral Examples are given) and the Author makes always uſe of it in the Sequel of this Book.

It is one of the great Advantages of this Method, that it ſuggeſts general Theorems for the Reſolution of Problems, which may be readily applied as there is occaſion for them. Our Author proceeds to treat of theſe, and firſt of ſuch as relate to the Centre of Gravity and its Motion. In any Syſtem of Bodies, the Sum of their Motions, eſtimated in a given Direction, is the ſame as if all the Bodies were united in their common Centre of Gravity. If the Motion of all the Bodies is uniform and rectilineal, the Centre of Gravity is either quieſcent, or its Motion is uniform and rectilineal. When Action is equal to Reaction, the State of the Centre of Gravity is never affected by the Colliſions of the Bodies, or by their attracting or repelling each other mutually. It is not, however, the Sum of the abſolute Motions of the Bodies that is preſerved invariable in conſequence of the Equality of the Action and Reaction, as they ſeem to imagine, who tell us, that this Sum is unalterable by the Colliſions of Bodies, and that this follows ſo evidently from the Equality of Action and Reaction, that to endeavour to demonſtrate it would ſerve only to render it more obſcure. On this Occaſion the Author illuſtrates an Argument which he had propoſed in a Piece that obtained the Prize propoſed by the *Royal Academy of Sciences* at *Paris* in 1724. againſt the Menſuration of the Forces of Bodies by the Square of the Velocities, ſhewing that if this Doctrine was admitted, the ſame Power or Agent, exerting the ſame Effort, would produce more Force in the ſame Body when in a Space carried uniformly forwards,

forwards, than if the Space was at Reſt; or that Springs acting equally on Two equal Bodies in ſuch a Space, would produce unequal Changes in the Forces of thoſe Bodies.

Various Problems concerning the Colliſion of Bodies are reſolved in a more general manner than uſual. Mr. *Bernouilli* had determined the Motions when the Elaſticity is perfect, and One Body ſtrikes Two equal Bodies in Directions that form equal Angles with its Direction; or when there are any Number of Bodies impelled by it on one Side in various Directions, providing equal Bodies be impelled by it on the other Side, in Directions equally inclined to its own Direction. But the Problem is reſolved here without theſe Limitations; ſome others of this kind are ſubjoined, and this Doctrine is applied for determining the Motions of Bodies that act upon each other while they deſcend by their Gravity.

The general Principle derived from theſe Inquiries is, that if there be no Colliſion, or ſudden Communication of Motion from one Body to another, while they deſcend together, and in any caſe, if the Elaſticity be perfect, the Sum of the Products, when each Body is multiplied by the Square of the Velocity acquired by it, is the ſame as if all the Bodies had deſcended freely from the ſame reſpective Altitudes to their ſeveral Places; only in collecting that Sum, if any Body is made to aſcend, the Product of it multiplied by the Square of its Velocity is to be ſubducted: And if the Bodies be ſuppoſed to aſcend from their Places with the reſpective Velocities acquired by them, then their common Centre of Gravity will riſe to the

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ſame Level from which it deſcended. In other Caſes, however, the Aſcent of the Centre of Gravity will be leſs than its Deſcent, but is never greater.

After demonſtrating the uſual Rule for finding the Centre of Oſcillation, the Author treats of the Motion of Water iſſuing from a cylindric Veſſel. The Effect of the Gravitation of the whole Maſs of Water is conſidered as Threefold. It accelerates, for ſome time at leaſt, the Motion with which the Water in the Veſſel deſcends; it generates the Exceſs of the Motion with which the Water iſſues at the Orifice above the Motion which it had in common with the reſt of the Water; and it acts on the Bottom of the Veſſel at the ſame time. Then ſuppoſing the laſt Two Parts of the Force to be in any invariable *Ratio* to each other, when the Diameters of the Baſe and Orifice are given, he determines by Logarithms the Velocity with which the Water iſſues at the Orifice; and ſhews that this Velocity will approach very near to its utmoſt Limit in an exceeding ſmall Time. When the Water is ſuppoſed to be ſupplied in a Cylinder, ſo as to ſtand always at the ſame Altitude above the Orifice, there is an Analogy between the Acceleration of the Motion of the Water that iſſues at the Orifice, and the Acceleration of a Body that deſcends by its Gravity in a Medium which reſiſts in the duplicate *Ratio* of the Velocity. For when the utmoſt Velocities, or Limits, are equal in thoſe two Caſes, the Time in which the iſſuing Water acquires any leſſer Velocity, is to the Time in which the deſcending Body acquires the ſame Velocity as the Area of the Orifice to the Area of the Baſe; and if a cylindric Column be ſuppoſed to be

be erected on the Orifice equal to the Quantity of Water that issues at the Orifice in the former of those Times, the Height of this Column will be to the Space described by the descending Body in the latter Time, in the same *Ratio* as the Orifice to the Area of the Base. The *Ratio* of the Force that acts on the Bottom of the Vessel to the Force that generates the Motion of the Water issuing at the Orifice, is deduced from Sir *Isaac Newton's* Cataract, and is the same that follows from the Principle concerning the Equality of the Ascent and Descent of the Centre of Gravity, which was first applied to this Inquiry by Mr. *Daniel Bernouilli* Comment. *Acad. Petrop. Tom.* 2. But there are several Precautions to be taken in applying this Doctrine.

After some other Theorems concerning the Centre of Gravity, and several Observations concerning the Curvature of Lines, and the Angles of Contact; the Author represents four general Propositions in one View, that the Analogy between them may appear. The First gives the Property of the Trajectories that are described by any centripetal Forces, how variable soever these Forces, or their Directions, may be. The Second gives a like general Property of the Lines of swiftest Descent. The Third gives the Property of the Line that is described in less time than any other of an equal Perimeter. And the Fourth gives the Property of the Figure that is assumed by a flexible Line or Chain, in consequence of any such Forces acting upon it. If we suppose a Body to set out from any Point in the Trajectory, or in the Line of swiftest Descent, with the Velocity which it has acquired there, and to move in the right Line which is the

Direction of the Gravity, that results from the Composition of the centripetal Forces, then shall its Velocity, and its Distance from the Point where the Perpendicular from the Centre of Curvature meets that right Line, flow *proportionally*, *i. e.* the Fluxion of the Velocity (or of the right Line that measures it) shall be to the Velocity as the Fluxion of that Distance is to the Distance. When the Velocity and Direction of the Motion is the same in the Line of swiftest Descent as in the Trajectory, their Curvature is the same. Thus in the common Hypothesis of Gravity, the Curvature in the Cycloid, the Line of swiftest Descent, is the same as the *Parabola* described by a Projectile, if the Velocities in those Lines be equal, and their Tangents be equally inclined to the Horizon. In order to find the Nature of the *Catenaria* in any Hypothesis of Gravity, suppose the Gravity to be increased or diminished in the same Proportion as the Thickness of the Chain varies, and to have its Direction changed into the opposite Direction; then imagine a Body to set out with a just Velocity from a given Point in the Chain, and to describe the Curve. The Tension of the Chain at any Point will be always as the Square of the Velocity acquired at that Point, and if a Body be projected with this Velocity in the Direction of the Tangent, the Curvature of the Trajectory described by it will be one Half of the Curvature of the Chain at that Point. We must refer to the Book for a fuller Account of these and of other Theorems.

In the XIIIth Chapter, the Problems concerning the Lines of swiftest Descent, the Figures which amongst all those that have equal Perimeters produce

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Maxima or *Minima*, and the Solid of leaſt Reſiſtance, are reſolved without Computations, from the firſt Fluxions only. There are alſo eaſy ſynthetic Demonſtrations ſubjoined, becauſe this Theory is commonly eſteemed of an abſtruſe Nature, and Miſtakes have been more frequently committed in the Proſecution of it, than of any other relating to Fluxions. To give ſome Idea of the Author's Method, ſuppoſe the Gravity to act in parallel Lines, *a* to denote the Velocity acquired at the lowermoſt Point of the Curve, and *u* the Velocity acquired at any other Point of the Curve. Suppoſe the Element of the Curve to be deſcribed by this Velocity *u*, but the Element of the Baſe to be always deſcribed by the conſtant Velocity *a*. Then it is eaſily demonſtrated without any Computation, that the Element of the Ordinate being given, the Difference of the Times in which the Elements of the Curve and Baſe are thus deſcribed is a *Minimum*, when the *Ratio* of thoſe Elements is that of *a* to *u*; *i.e.* when the Sine of the Angle, in which the Ordinate interſects the Curve, is to the *Radius* in this *Ratio*. Suppoſing therefore this Property to take place over all the Curve, the Exceſs of the Time in which it is deſcribed by the Body deſcending alongſt it, above the Time in which the Baſe is deſcribed uniformly with the Velocity *a*, muſt be a *Minimum*; and this latter Time being given, it follows that the Time of Deſcent in this Curve is a *Minimum*. When the Gravity tends to a given Centre, ſubſtitute an Arc of a Circle deſcribed from that Centre through the lowermoſt Point of the Curve in the Place of the Baſe in the former Caſe; and the Property of the Line of ſwifteſt Deſcent will be diſcovered in the

 ſame

ſame manner. The Nature of the Line that among all thoſe of the ſame Perimeter is deſcribed in the leaſt Time, is diſcovered with great Facility, by determining from the former Caſe the Property of the Figure when the Sum or Difference of the Time in which it is deſcribed by the deſcending Body, and of the Time in which it would be deſcribed by any given uniform Motion, is a *Minimum*; for the latter Time being the ſame in all Curves of the ſame Length, it follows that the Figure, which has this Property, muſt be deſcribed in leſs Time than any other of an equal Perimeter. The general Iſoperimetrical Problems are reſolved, and the Solutions are rendered more general, with like Facility by the ſame Method; which is alſo applied for determining the Property of the Solid of leaſt Reſiſtance, and ſerves for reſolving the Problem, when Limitations are added concerning the Capacity of the Solid, or the Surface that bounds it.

The laſt Chapter of the Firſt Book treats chiefly of Gravitation towards Spheroids, of the Figure of the Planets, and of the Tides. The Author, having Occaſion in thoſe Inquiries for ſeveral new Properties of the Ellipſe, begins this Chapter by deriving its Properties from thoſe of the Circle, by conſidering it as the oblique Section of a Cylinder, or as the Projection of the Circle by parallel Rays upon a Plane oblique to the Circle. In this manner the Properties are briefly transferred from the one to the other, becauſe by this Projection the Centre of the Circle gives the Centre of the Ellipſe; Diameters perpendicular to each other in the Circle with their Ordinates, and the circumſcribed Square, give conjugate Diameters of the

the Ellipſe with their Ordinates, and the circumſcribed Parallelogram; parallel Lines in the Plane of this Circle are projected by Parallels in the Plane of the Ellipſe that are in the ſame *Ratio*; any Area in the former is projected by an Area in the latter, which is in an invariable *Ratio* to it; and concentric Circles give ſimilar concentric Ellipſes. It is likewiſe ſhewn how Properties of a certain kind are briefly transferred from the Circle to any conic Section with the ſame Facility.

After demonſtrating the Properties of the Ellipſe, it is ſhewn, that if the Gravity of any Particle of a Spheroid being reſolved into two Forces, one perpendicular to the Axis of the Solid, the other perpendicular to the Plain of its Equator, then all Particles, equally diſtant from the Axis, muſt tend towards it with equal Forces; and all Particles at equal Diſtances from the Plain of the Equator, gravitate equally towards this Plain; but that the Forces with which Particles at different Diſtances from the Axis tend towards it, are as the Diſtances; and that the ſame is to be ſaid of the Forces with which they tend towards the Plain of the Equator.

From this it is demonſtrated, that when the Particles of a fluid Spheroid of an uniform Denſity gravitate towards each other with Forces that are inverſely as the Squares of their Diſtances, and at the ſame time any other Powers act on the Particles, either in right Lines perpendicular to the Axis, that vary in the ſame Proportion as the Diſtances from the Axis, or in right Lines perpendicular to the Plain of the Equator, that vary as their Diſtances from it; or when any Powers act on the Particles of the Spheroid, that may be reſolved into Forces of this kind; then the

Fluid will be every-where *in Æquilibrio*, if the whole Force that acts at the Pole be to the whole Force that acts at the Circumference of the Equator, as the Semidiameter of the Equator to the Semiaxis of the Spheroid; and that the Forces with which equal Particles at the Surface tend towards the Spheroid, will be in the ſame Proportion as Perpendiculars to its Surface, terminated either by the Plane of the Equator, or by the Axis. Becauſe the centrifugal Force with which any Particle of the Spheroid endeavours to recede from its Axis, in conſequence of the diurnal Rotation, is as the Diſtance from the Axis, it appears that if the Earth, or any other Planet, was fluid, and of an uniform Denſity, the Figure which it would aſſume would be accurately that of an oblate Spheroid generated by an Ellipſis revolving about its Second Axis.

Afterwards the Gravity towards an oblate Spheroid is accurately meaſured by circular Arcs, not only at the Pole, but alſo at the Equator, and in any intermediate Places; and the Gravity towards an oblong Spheroid is meaſured by Logarithms. The Gravity at any Diſtance in the Axis of the Spheroid, or in the Plane of the Equator produced, is likewiſe accurately determined by like Meaſures, without any new Computation or Quadrature, by ſhewing that when Two Spheroids have the ſame Centre and *Focus*, and are of an uniform Denſity, the Gravities towards them at the ſame Point in the Axis or Plane of the Equator produced, are as the Quantities of Matter in the Solids.

This Theory is applied for determining the Figure of the Earth, by comparing the Force of Gravity in any

any given Latitude, derived from the Length of a *Pendulum* that vibrates there in a Second of Time, with the centrifugal Force at the Equator, deduced from the periodic Time of the diurnal Rotation, and the Amplitude of a Degree of the Meridian; or by comparing the Lengths of *Pendulums* that vibrate in equal Times in given unequal Latitudes; or by comparing different Degrees meaſured upon the Meridian. By the beſt Obſervations it would ſeem, that there is a greater Increaſe of Gravitation, and of the Degrees of the Meridian from the Equator towards the Poles, than ought to ariſe from the Suppoſition of an uniform Denſity. Therefore the Author ſuppoſes the Denſity to vary from the Surface towards the Centre; and, in ſeveral Caſes he has conſidered, he finds that a greater Denſity towards the Centre would account for a greater Increaſe of Gravitation towards the Poles, but not for a greater Increaſe of the Degrees of the Meridian; and that the Hypotheſis of a leſs Denſity towards the Centre would account for the latter, but not for the former, ſuppoſing (after Sir *Iſaac Newton*) the Columns of the Fluid to extend from the Surface to the Centre, and there to ſuſtain each other. On this Account he determines the Gravitation towards the Earth, when it is ſuppoſed to be hollow with a *Nucleus* included, according to the Hypotheſis advanced by Dr. *Halley*, with the Difference of the Semidiameters that might ariſe from ſuch a Diſpoſition of the internal Parts. But in this Caſe, and when the Denſity is ſuppoſed variable, the ſpheroidical Figure is only aſſumed as an Hypotheſis. He adds, that by imagining the Denſity to be greater in the Axis than in the Plain of the Equator at equal Di-

Diſtances from the Centre, an Hypotheſis perhaps might be found, that would account for moſt of the *Phænomena*; but that a Series of many exact Obſervations is requiſite, before we can examine with any Certainty the various Suppoſitions that may be imagined concerning the internal Conſtitution of the Earth. This Doctrine is likewiſe applied for determining the Figure of *Jupiter*.

It follows from the ſame Theorem, that if we ſuppoſe the Earth to be fluid, and abſtract from its Motion upon its Axis, and the Inclination of the right Lines in which its Particles gravitate towards the Sun or Moon, the Figure which it would aſſume in conſequence of the unequal Gravitation of its Particles towards either of thoſe Bodies would be accurately that of an oblong Spheroid having its Axis directed towards that Body. The Aſcent of the Water, deduced from this Theorem, agrees nearly with that which Sir *Iſaac Newton* found, by computing it briefly from what he had demonſtrated concerning the Figure of the Earth. Several Obſervations are ſubjoined concerning the Tides, and the Cauſes which may contribute to increaſe or diminiſh them, particularly the Inequality of the Velocities with which Bodies revolve about the Axis of the Earth in different Latitudes.

This Chapter concludes by demonſtrating briefly, that if the Attraction of the Particles decreaſed as the Cube of their Diſtance increaſes, or in any higher Proportion, then any Particle would tend towards the leaſt Portion of Matter in Contact with it, with a greater Force than towards the greateſt Body at any Diſtance, how ſmall ſoever from it. The true

Law

Law of Gravity is better adapted for holding the Parts of each Body in a proper Union, while it perpetuates the Motions in the great Syſtem about the Sun, and preſerves the Revolutions in the leſſer Syſtems nearly regular; and the Author concludes with obſerving, that a remarkable geometrical Simplicity is often found in the Concluſions that are derived from it.

An Account of Book II. *will be given in the next* Tranſaction.

VII. *De* Calculo *prægrandi a* Muliere *cum Urina* excreto *Obſervatio D*[ni] Antonii Leprotti, *R. S. S.* Pont. Max. *Archiat. per Abbatem* Didacum de Revillas, *R. S. S. ad D.* Smart Lethieullier *R. S. S. tranſmiſſa.*

Romæ, pridie Cal. *Januarii An.* 1743.

Read Jan. 27. 1742-3.

QUÆ quindecim abhinc annis Vidua eſt Mulier pauper quinquageſimum agens annum ſumma urinæ difficultate per quadraginta Menſes laborabat; quum nocte ei ſupervenit mictus, imo ſinceri cruoris profluvium ad tres circiter libras; ſimulque Lapis, ejus, quæ in adjecta Figura deſcribitur, formæ & molis, extruſus eſt, cujus pondus jam exſiccati uncias duas & grana novem ac viginti exæquat*. Mulier autem ingenti per eos menſes gravitatis ſenſu, aſſiduoque dolore, ad Veſicæ cervicem afficiebatur ſive cubans ſive erecta; nunc autem ſolido exacto menſe, ea invita, prodere pergunt cum ſanie urinæ. *Vide* Fig. II. *in* TAB. I.

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unc. den. gr.
* i. e. 1 : 17 : 4. TR.