I. *Of the Bases of the Cells wherein the* Bees *deposite their Honey. Part of a Letter from Mr.* Mac Laurin, *Professor of Mathematics at* Edinburgh, *and* F. R. S. *to* Martin Folkes, *Esq; Pr.* R. S.

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THE Sagacity of the *Bees* in making their Cells of an hexagonal Form, has been admired of old; and that Figure has been taken notice of, as the beſt they could have pitched upon for their Purpoſes: But a yet more ſurpriſing Inſtance of the Geometry of theſe little *Inſects* is ſeen in the Form of the Baſes of thoſe Cells, diſcovered in the late accurate Obſervations of Monſieur *Maraldi* and Monſieur *de Reaumur*, who have found thoſe Baſes to be of that Pyramidal Figure, that requires the leaſt Wax for containing the ſame Quantity of Honey, and which has at the ſame time a very remarkable Regularity and Beauty, connected of Neceſſity with its Frugality.

Theſe Baſes are formed from Three equal *Rhombus's*, the obtuſe Angles of which are found to be the Doubles of an Angle that often offers itſelf to Mathematicians in Queſtions relating to *Maxima* and *Minima*; that is, the Angle, whoſe Tangent is to the *Radius*, as the Diagonal is to the Side of the Square. By this Conſtruction, of the Six ſolid Angles at the Baſe that correſpond to the Angles of the

Hexagon, Three are equal as well to each other, as to the ſolid Angle at the *Apex* of the Figure, each of which ſolid Angles is reſpectively formed from Three equal plane obtuſe Angles: And the other Three ſolid Angles are alſo equal to each other, but ſeverally formed each from Four equal plane acute Angles, Supplements to the former obtuſe ones.

By this Form the utmoſt Improvement is made of their Wax, of which they are on all Occaſions very ſaving, the greateſt Regularity is obtained in the Conſtruction, and with a particular Facility in the Execution; as there is one ſort of Angle only with its Supplement, that is required in the Structure of the whole Figure.

Monſieur *Maraldi* * had found by Menſuration, that the obtuſe Angles of the *Rhombus's* were of 110 Degrees nearly; upon which he obſerved, that if the Three obtuſe Angles which formed the ſolid Angles above-mentioned, were ſuppoſed equal to each other, they muſt each be of 109°. 28′; from whence it has been inferred, that this laſt was really the true and juſt Meaſure of them: And lately Monſieur *de Reaumur* † has informed us, that Mr. *Koënig* having, at his Deſire, ſought what ſhould be the Quantity to be given to this Angle, in order to employ the leaſt Wax poſſible in a Cell of the ſame Capacity, that Gentleman had found, by a higher *Geometry* than was known to the Antients, by the Method of *Infiniteſimals*, that the Angle in queſtion ought in this Caſe to be of 109°. 26′. And we ſhall now make

* *Memoires de l'Acad. Royale des Sciences*, 1712.
† *Memoires ſur les Inſectes*, Tom. V.

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it appear from the Principles of common *Geometry*, that the moſt advantageous Angle for theſe *Rhombus's* is indeed, on that Account alſo, the ſame which reſults from the ſuppoſed Equality of the Three plane Angles that form the above-mentioned ſolid ones.

Let GN and NM repreſent any Two adjoining Sides of the Hexagon, that is, the Section of the Cell perpendicular to its Length. The Sides of the Cell are not complete Parallelograms as $CGNK$, $BMNK$, but *Trapezia* $CGNE$, $BMNE$, to which a *Rhombus* $CEBe$, is fitted at E, and that has the oppoſite Point e in the *Apex* of the Figure, ſo that Three *Rhombus's* of this kind, with Six *Trapezia*, may complete the Figure of the Cell. Let O be the Centre of the Hexagon, of which CK and KB are adjoining Sides; join CB and KO, interſecting it in A; and, becauſe COB is equal to CKB, and KE equal to Oe, the Solid $EBCK$ is equal to the Solid $eBCO$; from which it is obvious, that the Solid Content of the Cell will be the ſame, where-ever the Point E is taken in the Right Line KN, the Points C, K, B, G, N, and M, being given. We are therefore to inquire where the Point E is to be taken in KN, ſo that the *Area* of the *Rhombus* $CEBe$, together with that of the Two *Trapezia* $CGNE$, $ENMB$, may form the Leaſt Superficies. Becauſe Ee is perpendicular to BC in A, the *Area* of the *Rhombus* is $AE \times BC$, that of the *Trapezia* $CGNE$, $ENMB$, is $\overline{CG+EN} \times KC$; theſe, added to the *Rhombus*, amount to $AE \times BC + 2KN \times KC - KE \times KC$; and becauſe $2KN \times KC$ is invariable, we are to

TAB. I. Fig 1. and 2.

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inquire, when $AE\times BC - KE\times KC$ is a *Minimum?*

Suppose the Point L to be so taken upon KN, that KL may be to AL as KC is to BC. From the Centre A describe in the Plane AKE with the *Radius* AE, an Arc of a Circle ER meeting AL, produced, if necessary, in R; let EV be perpendicular to AR in V, and KH be perpendicular to the same in H; then the Triangles LEV, LKH, LAK, being similar, we have $LV:LE::LH:LK::LK:LA::$ (by the Supposition last made) $KC:BC$. Hence, when E is between L and N, we have $LH+LV\ (=VH):LK+LE\ (=KE)::KC:BC$; and when E is between K and L, we have $LH-LV(=VH):LK-LE\ (=KE)::KC:BC$; that is, in both Cases we have $KE\times KC=VH\times BC$; and consequently $AE\times BC - KE\times KC = AE\times BC - VH\times BC = \overline{AE-VH}\times BC = \overline{AR-VH}\times BC = \overline{AH+VR}\times BC$; which, because AH and BC do not vary, is evidently Least when VR vanishes, that is, when E is upon L. Therefore $CLBl$ is the *Rhombus* of the most advantageous Form in respect of Frugality, when KL is to AL as KC is to BC. This is the same Method by which we have elsewhere determined the *Maxima* and *Minima*, in the Resolution of several Problems that have usually been treated in a more abstruse Manner. *See Treatise of Fluxions, Art.* 572, &c.

Now because OK is bisected in A, $KC^2=OK^2=4AK^2$; and $AC^2=3AK^2$, or $BC=2AC=2\sqrt{3}\times AK$; consequently $KC:BC::2AK:2\sqrt{3}\times AK::1:\sqrt{3}$, and $KL:AL::(KC:BC)::$

$1:\sqrt{3}$,

$1:\sqrt{3}$, or $AL:AK::\sqrt{3}:\sqrt{2}$; and (becauſe $AK:AC::1:\sqrt{3}$) $AL:AC::1:\sqrt{2}$; that is, the Angle CLA is that, whoſe Tangent is to the *Radius* as $\sqrt{2}$ is to 1, or as 14142135 to 10000000; and therefore is of $54^{\circ}.\ 44'.\ 08''$, and conſequently the Angle of the *Rhombus* of the Beſt Form is that of $109^{\circ}.\ 28'.\ 16''$.

By this Solution it is further eaſy to eſtimate what their Savings may amount to upon this Article, in conſequence of this Conſtruction. Had they made the Baſe flat, and not of the pyramidal Form deſcribed above, then, beſides completing the Parallelograms $CGNK$ and $BMNK$, the Surface of the Baſe had been $3\,CB\times AK$; what they really do form amounts in Surface to the ſame Parallelograms, and $3\,CB\times AH$: the Savings therefore amount to $3\,CB\times\overline{AK-AH}$ $=3\,CB\times AH\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$, which is almoſt a Fourth-part of the Pains and Expence of Wax, they beſtow above what was neceſſary for completing the parallelogram Sides of the Cells: And at the ſame time they ſeem alſo to have other Advantages from this Form, beſides the ſaving of their Wax; ſuch as a greater Strength of the Work, and more Convenience for moving in theſe larger ſolid Angles.

It remains that we ſhould ſhew, that the plane Angles CLB, CLN, and BLN, are equal to each other. We before found, that $KL:AL::KC:BC::KA:(=\frac{1}{2}KC)\ AC$; conſequently $KL:KA::AL:AC$, and the Triangles LKA, LAC, are ſimilar: Therefore $LK:AL::AL:LC::KC:BC::1:\sqrt{3}$, and $LC=3\,LK$. With the Centre L and *Radius* LC, deſcribe in the Plane $CGNK$ the Semicircle DCP, meeting the Line KN, in D and P; join

Tab. I. Fig. 3.

join CP and CD, and let LQ be perpendicular to CP in Q, then will the Angle CDK be equal to QLP, and we ſhall have $PQ : LQ :: PC : DC :: \sqrt{PK} : \sqrt{DK} :: \sqrt{LC+LK} : \sqrt{LC-LK} :: \sqrt{4} : \sqrt{2} :: \sqrt{2} : 1 :: AC : AL$. Conſequently the Angle $QLP = ALC$, and $CLP = CLB$, or the obtuſe Angle of the *Rhombus* $CLBl$ is equal to CLP, the obtuſe Angle of the *Trapezium*; and conſequently, the Three plane Angles that form the ſolid Angle at L, or the *Apex* at l, are equal to each other: From which it is obvious, that the Four acute plane Angles, which form the ſolid Angle at C or B, are likewiſe equal among themſelves.

Though Monſieur *Maraldi* had found, by his Menſuration, theſe obtuſe Angles to be of about 110 Degrees; the ſmall Difference between this and the $109°. 28'. 16''$, juſt found by Calculation, ſeems to have been either accidental, or owing to the Difficulty of meaſuring ſuch Angles with Exactneſs: Beſides that he ſeems to admit the real Equality of the ſeveral plane Angles, that form as well the *Apex*, as the other ſolid ones we have been treating of. And, as to the ſmall Difference between our Angle and that determined by Mr. *Koënig*, who firſt conſidered this Problem, but has not yet publiſhed his Demonſtration of it, that can only be owing to his not carrying on his Computation ſo far, and would ſcarcely have been worth the mentioning, were it not yet in Favour of the Practice of theſe induſtrious little Inſects; and did it not therefore give us ground to conclude, that in general, and when the particular Form and Circumſtance of the Honey-comb does not require a Variation from their Rule,

the *Bees* do truly construct their Cells of the best Figure, and that not only nearly, but with Exactness; and that their Proceeding could not have been more perfect from the greatest Knowledge in *Geometry*. How they arrive at this, and how the wonderful Instinct in Animals is to be accounted for, is a Question of an higher Nature; but this is surely a remarkable Example of this Instinct, as it has suggested a Problem that had been overlooked by Mathematicians, though they have treated largely on the *Maxima* and *Minima*; and such an one, as has been thought to exceed the Compass of the common *Geometry*.

It may be worth while to add here, that if the Cells had been of any other Form than hexagonal, and the Bases had still been pyramidal, these must have been terminated by *Trapezia*, and not by *Rhombus's*, and therefore had been less regular, because OA and AK would have been unequal: Nor could there have been room for such an advantageous or frugal a Construction as that we have described, because the solid Content of the Cell would have increased with the Right Line KE. The Cells, by being hexagonal, are the most capacious, in proportion to their Surface, of any regular Figures that leave no Interstices between them, and at the same time admit of the most perfect Bases. Thus, by following what is best in one respect, unforeseen Advantages are often obtained; and what is most beautiful and regular, is also found to be most useful and excellent.

Fig. 1.

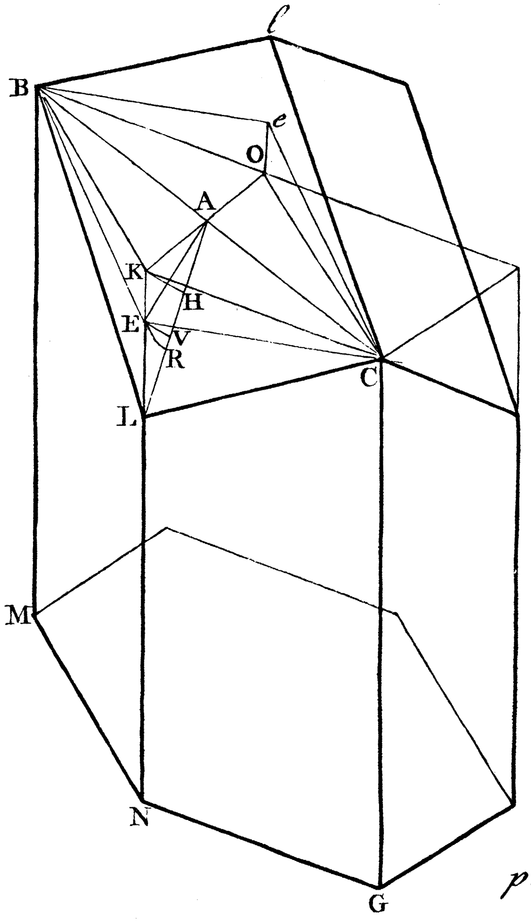

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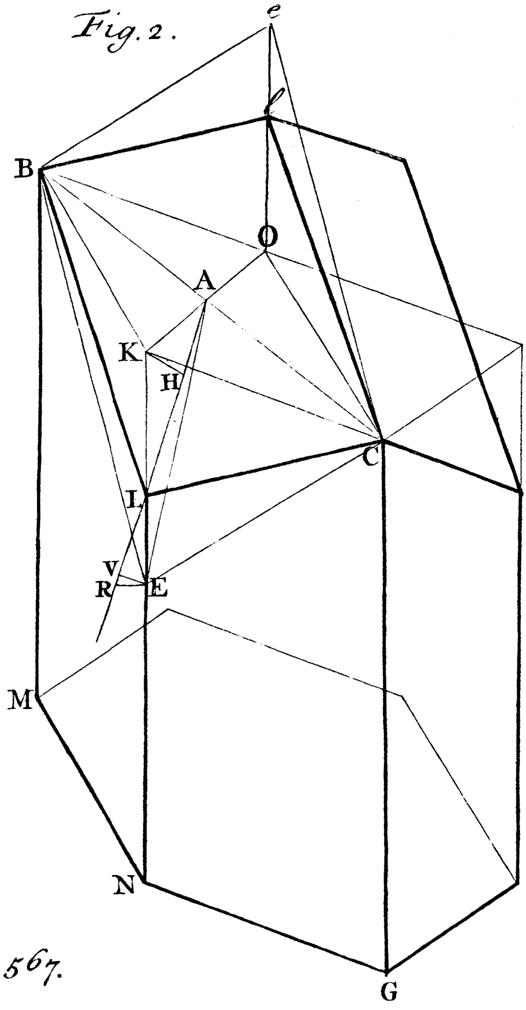
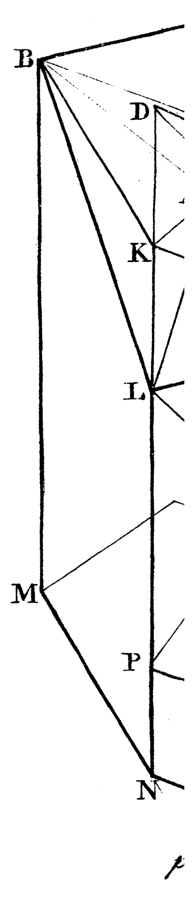
Fig. 2.





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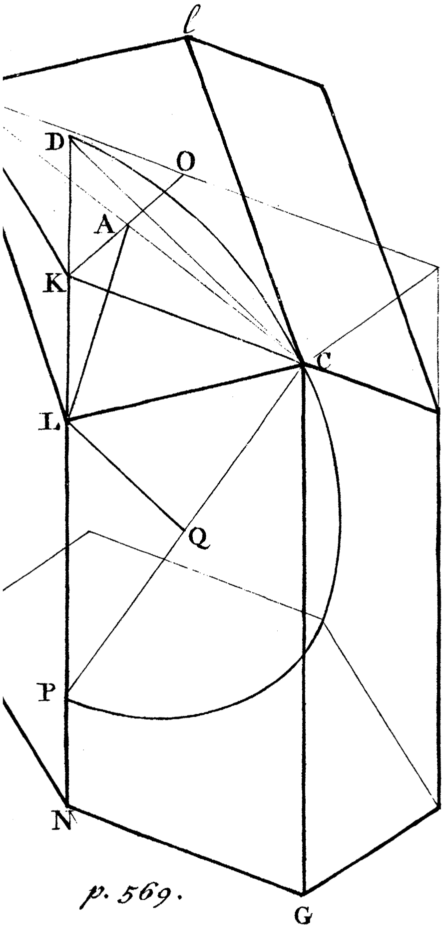

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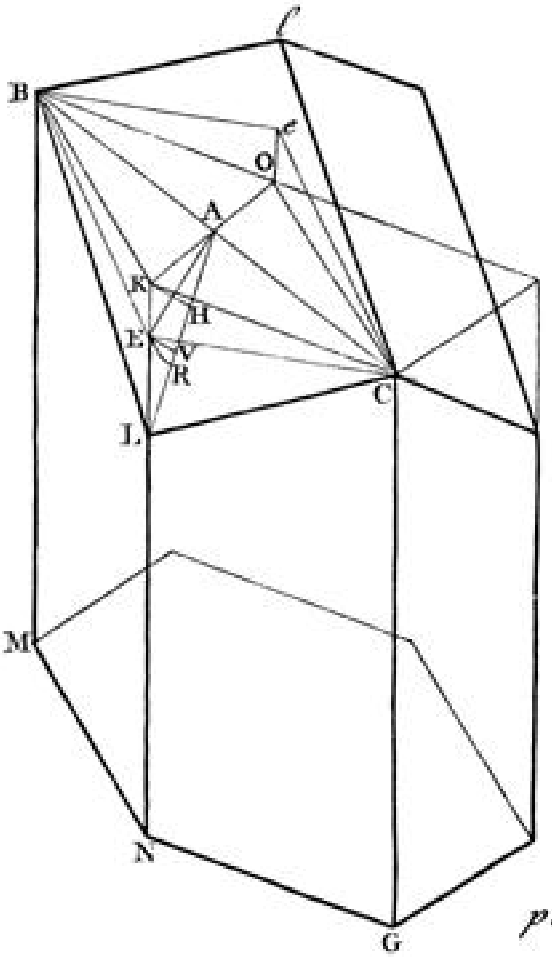

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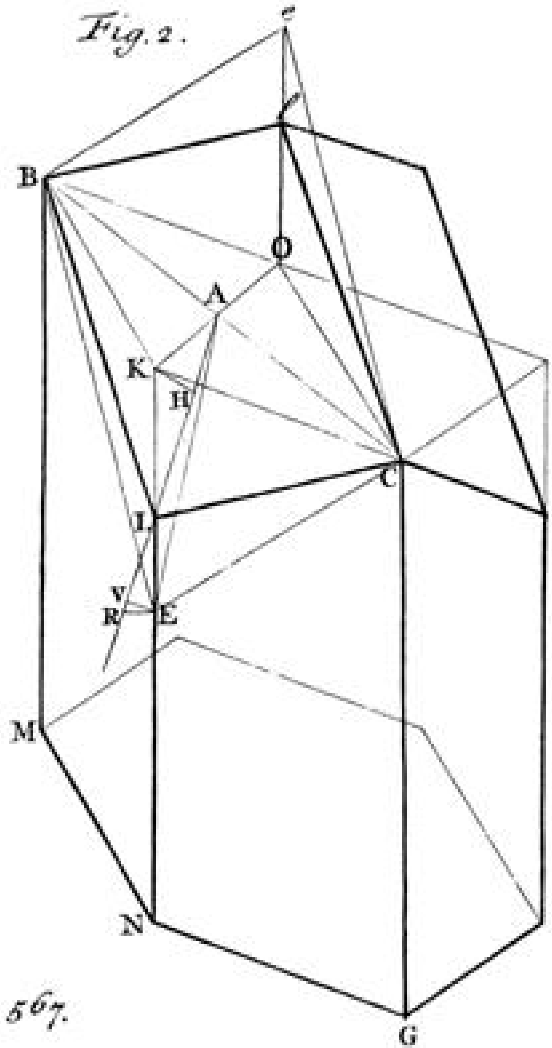
Fig. 2.



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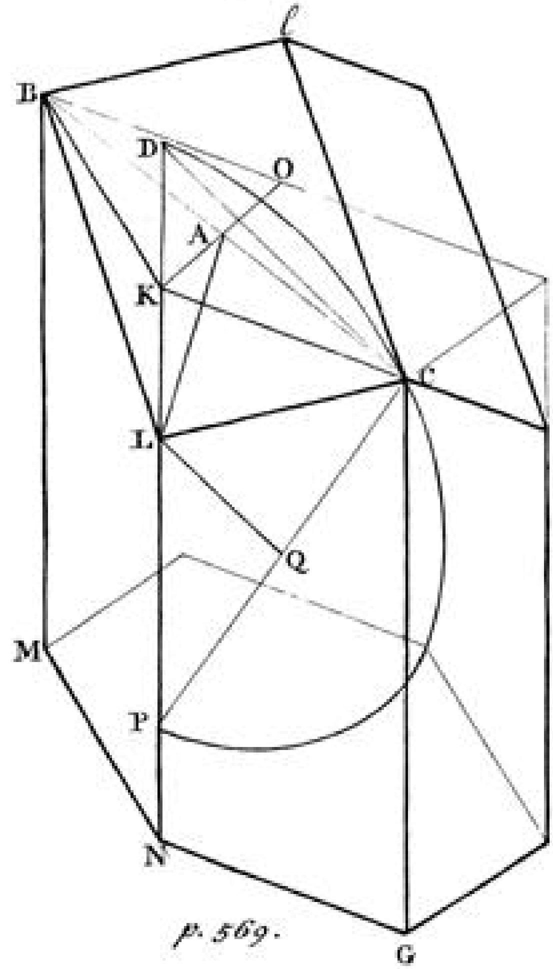

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