Telephium repens folio non deciduo. C. B. P. 287.

1646 Thalictrum minus alterum Parisiensium foliis crassioribus et lucidis. H. R. Par.

1647 Tithymalus maritimus. C. B. P. 291.
Tithimalus paralius. J. B. 3. 674.

1648 Trifolium Bitumen redolens. C. B. 327.

1649 Trifolium Bitumen redolens angustifolium. Boer. Ind. Alt. 2. p. 32.
Trifolium bituminosum arboreum angustifolium ac sempervirens. Hort. Cath.

1650 Virga aurea Canadensis foliis carnosis non serratis latioribus. Hist. Oxon.

XIX. *A Letter to the Right Honourable* George *Earl of* Macclesfield, *President of the* Royal Society, *on the* Advantage *of taking the Mean of a Number of Observations, in practical Astronomy: By* T. Simpson, *F. R. S.*

My Lord,

Read April 10, 1755.

IT is well known to your Lordship, that the method practised by astronomers, in order to diminish the errors arising from the imperfections of instruments, and of the organs of sense, by taking the Mean of several observations, has not been so generally received, but that some persons, of considerable note, have been of opinion, and even publickly maintained, that one single observation, taken

taken with due care, was as much to be relied on as the Mean of a great number.

As this appeared to me to be a matter of much importance, I had a ſtrong inclination to try whether, by the application of mathematical principles, it might not receive ſome new light; from whence the utility and advantage of the method in practice might appear with a greater degree of evidence. In the proſecution of this deſign (the reſult of which I have now the honour to tranſmit to your Lordſhip) I have, indeed, been obliged to make uſe of an hypotheſis, or to aſſume a ſeries of numbers, to expreſs the reſpective chances for the different errors to which any ſingle obſervation is ſubject; which ſeries, to me, ſeems not ill-adapted: but this I ſhall ſubmit intirely to the judgment of your Lordſhip, who have made ſo great a number of obſervations, at your ſeat at Shirburn; where, to the beſt collection of mathematical books, your Lordſhip has added a more complete ſet of aſtronomical inſtruments than (perhaps) are to be found in the poſſeſſion of any nobleman in Europe.

Should not the aſſumption, which I have made uſe of, appear to your Lordſhip ſo well choſen as ſome others might be, it will, however, be ſufficient to anſwer the intended purpoſe: and your Lordſhip will find, on calculation, that, whatever ſeries is aſſumed for the chances of the happening of the different errors, the reſult will turn out greatly in favour of the method now practiſed, by taking a mean value. But I ſhall no longer detain your Lordſhip with general obſervations, but proceed to the

matter propofed; which I fhall confider in the following propofitions.

PROPOSITION I.

Suppofing that the feveral chances for the different errors that any fingle obfervation can admit of, are expreffed by the terms of the progreffion r^{-v} - - - r^{-3}, r^{-2}, r^{-1}, r^{0}, r^{1}, r^{2}, r^{3} - - - - r^{v} (where the exponents denote the quantities and qualities of the particular errors, and the terms themfelves the refpective chances for their happening): 'tis propofed to determine the probability, or odds, that the error, by taking the Mean of a given number (n) of obfervations, exceeds not a given quantity $\left(\frac{m}{n}\right)$.

It is evident, from the laws of chance, that, if the given feries, $r^{-v} - - - + r^{-3} + r^{-2} + r^{-1} + r^{0} + r^{1} + r^{2} + r^{3} - - - + r^{v}$, expreffing all the chances in one obfervation, be raifed to the nth power, the terms of the feries thence arifing will truly exhibit all the different chances in all the propofed (n) obfervations. In order to raife this power, with the greateft facility, our given expreffion may be reduced to $r^{-v} \times \frac{1 - r^{2v+1}}{1 - r}$: whereof the nth power (making $w = 2v + 1$), will be $r^{-nv} \times \overline{1 - r^{w}}|^{n} \times \overline{1 - r}|^{-n}$; which, expanded, becomes

$$r^{-nv} - nr^{w-nv} + \frac{n}{1}\cdot\frac{n-1}{2}r^{2w-nv} - \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}r^{3w-nv} +, \&c.$$

multiplied into

$$1 + nr + \frac{n}{1}\cdot\frac{n+1}{2}r^{2} + \frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}r^{3} +, \&c.$$

Now,

Now, to find from hence the ſum of all the chances whereby the exceſs of the poſitive errors above the negative ones can amount, preciſely, to a given number m, it will be ſufficient (inſtead of multiplying the former ſeries by the whole of the latter) to multiply by ſuch terms of the latter, only, as are neceſſary to the production of the given exponent m, in queſtion. Thus, the firſt term (r^{-nv}) of the former ſeries is to be multiplied by that term of the ſecond, whoſe exponent is $nv+m$, in order that the power of r, in the product, may be r^m. But it is plain, from the law of the ſeries, that the coefficient of this term (putting $nv+m=q$) will be $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q)$, q being the number of factors; and conſequently, that the product under conſideration will be $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q)\times r^m$. Again, the ſecond term of the former ſeries being $-nr^{w-nv}$, the exponent of the correſponding term of the latter will be $-w+nv+m$ $(=q-w)$, and therefore the term itſelf equal to $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-w)\times r^{q-w}$: which, drawn into $-nr^{w-nv}$, gives $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-w)\times -nr^m$, for the ſecond term required.

In like manner, the third term, of the product, whoſe exponent is m, will be found $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-2w)$ $\times\frac{n}{1}.\frac{n-1}{2}r^m$: and the ſum of all the terms having the ſame given exponent (m) will conſequently be

$$+\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}(q)\times r^m$$
$$-\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}(q-w)\times n r^m$$
$$+\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}(q-2w)\times\frac{n}{1}\cdot\frac{n-1}{2} r^m$$
$$-\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}(q-3w)\times\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} r^m$$
&c. &c.

From which general expreſſion, by expounding m by 0, $+1$, -1, $+2$, -2, &c. ſucceſſively, the ſum of all the chances, whereby the difference of the poſitive and negative errors can fall within the propoſed limits, will be found; which, divided by $r^{-wn}\times\overline{1-r^w}|^n\times\overline{1-r}|^{-n}$, will give the true meaſure of the probability required: from whence the advantage of taking the Mean of ſeveral obſervations might be ſhewn: but this I ſhall exemplify in the next propoſition; which is better adapted to the purpoſe, and to which this is premiſed, as a Lemma.

Remark.

If r be taken $=1$, or the chances for the errors in exceſs and defect be ſuppoſed exactly the ſame; then our expreſſion, by expunging the powers of r, will become the very ſame with that ſhewing the chances for throwing $n+q$ points with n dice; each die having as many faces (w), as the reſult of any one ſingle obſervation, can come out different ways. Which may be otherwiſe made to appear, independent of any kind of calculation, from the bare conſider-

fideration, that the chances for throwing precifely the number m, with n dice, whereof the faces, of each, are numbered $-v - - - - -3, -2, -1, 0, +1, +2, +3, - - - - +v$, muft be the very fame as the chances whereby the pofitive errors can exceed the negative ones by that precife number; which laft are, evidently, the fame as the chances for throwing precifely the number $\overline{v+1}.n+m$ (or $n+q$) with the fame n dice, when they are numbered in the common way, with the terms of the natural progreffion 1, 2, 3, 4, 5, *&c.*: becaufe the number upon each face being, here, increafed by $v+1$, the whole increafe upon all the n faces will be expreffed by $\overline{v+1}.n$; fo that there will be now the very fame chance for the number $\overline{v+1}.n+m$, as there was before for the number m; fince the chances for throwing any faces affigned will continue the fame, however thofe faces are numbered.

PROPOSITION II.

Suppofing the refpective chances, for the different errors which any fingle obfervation can admit of, to be expreffed by the terms of the feries $r^{-v} + 2r^{1-v} + 3r^{2-v} - - - - - - + \overline{v+1}.r^{0} - - - - - + 3r^{v-2} + 2r^{v-1} + r^{v}$ (whereof the coefficients, from the middle one ($v+1$), decreafe, both ways, according to the terms of an arithmetical progreffion): 'tis propofed to determine the probability, or odds, that the error, by taking the Mean of a given number (t) of obfervations, exceeds not a given quantity $\left(\frac{m}{t}\right)$.

Purfuing

Pursuing the method laid down in the preceding problem, the sum of the series here given will appear to be $r^{-v} \times \dfrac{\overline{1-r^{v+1}}\big|^2}{\overline{1-r}\big|^2}$ (being the same with the square of the geometrical progression $r^{-\frac{1}{2}v} \times \overline{1+r+r^2+r^3 \cdots +r^v}$), and the t^{th} power thereof, by making $n=2t$, and $w=v+1$. will therefore be given $= r^{-tv} \times \overline{1-r^w}\big|^n \times \overline{1-r}\big|^{-n} =$

$$r^{-tv} - nr^{w-tv} + \frac{n}{1}.\frac{n-1}{2}r^{2w-tv} + \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}r^{3w-tv} +, \&c.$$

multiplied into

$$1+nr+\frac{n}{1}.\frac{n+1}{2}r^2+\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}r^3+, \&c.$$

Which series being the same with those in the preceding problem (excepting only that the exponents of the former of them are expressed in terms of t, instead of n), it is plain, therefore, that if q be made $=tv+m$ (instead of $nv+m$), the conclusion, there brought out, will answer equally here: so that the sum of all the chances whereby the excess of the positive errors above the negative ones, can amount to a given number m, precisely, will be truly represented by

$$\begin{array}{l}
+\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q) \times r^m \\
-\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-w) \times n\,r^m \\
+\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-2w) \times \frac{n}{1}.\frac{n-1}{2}r^m \\
-\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-3w) \times \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}r^m \\
\qquad \&c. \qquad\qquad\qquad \&c.
\end{array}$$

But

But this general expression, as several of the factors destroy each other, may be transformed to another, which is more commodious. Thus $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q)$, in the first line, will, by breaking the numerator and denominator into two parts, become

$$\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}.\text{----}\,q\times\overline{q+1}.\overline{q+2}.\overline{q+3}.\text{----}\,\overline{q+n-1}}{1.\ 2.\ 3.\ 4.\text{----}\,n.\ \overline{n+1}.\overline{n+2}.\overline{n+3}.\text{----}\,q}:$$

which, by equal division, is reduced to

$$\frac{q.\overline{q+1}.\overline{q+2}\text{----}\,\overline{q+n-1}}{1.\ 2.\ 3.\text{----}\,\overline{n-1}} = \frac{\overline{q+n-1}.\ \overline{q+n-2}\text{----}\,q}{1.\ \ 2\text{----}\,\overline{n-1}} =$$

$\frac{p-1}{1}.\frac{p-2}{2}.\frac{p-1}{3}(n-1)$; supposing $p\ (=q+n)=tv+m+n$.

In the very same manner, making $q'=q-w$, and $p'\ (=q'+n)=p-w$, it appears, that $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-w)=\frac{p'-1}{1}.\frac{p'-2}{2}.\frac{p'-3}{3}(n-1)$, &c. &c.

and consequently, that our whole given expression (supposing $p''=p-2w$, $p'''=p-3w$, &c.) will be transformed to

$$\begin{aligned}
&+\frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}(n-1)\times r^m\\
&-\frac{p'-1}{1}.\frac{p'-2}{2}.\frac{p'-3}{3}(n-1)\times n r^m\\
&+\frac{p''-1}{1}.\frac{p''-2}{2}.\frac{p''-3}{3}(n-1)\times\frac{n}{1}.\frac{n-1}{2}r^m\\
&-\frac{p'''-1}{1}.\frac{p'''-2}{2}.\frac{p'''-3}{3}(n-1)\times\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}r^m\\
&\quad\text{\&c.}\qquad\qquad\qquad\qquad\text{\&c.}
\end{aligned}$$

Which expression is to be continued till the factors become nothing, or negative; and which, when

 $r=1$,

$r=1$, will be the very ſame with that exhibiting the number of chances for p points, preciſely, with n dice, having each w faces: and in this caſe, where the chances for the errors in exceſs and defect are the ſame, the ſolution is the moſt ſimple it can be; ſince, from the chances above determined, anſwering to the number p, preciſely, the ſum of the chances for all the inferior numbers (incluſive) may be readily obtained; being given (from the Method of Increments) equal to

$$\begin{array}{l}
+\frac{p}{1}\cdot\frac{p-1}{2}\cdot\frac{p-2}{3}\cdot\frac{p-3}{4}(n) \\
-\frac{p'}{1}\cdot\frac{p'-1}{2}\cdot\frac{p'-2}{3}\cdot\frac{p'-3}{4}(n)\times n \\
+\frac{p''}{1}\cdot\frac{p''-1}{2}\cdot\frac{p''-2}{3}\cdot\frac{p''-3}{4}(n)\times\frac{n}{1}\cdot\frac{n-1}{2} \\
-\frac{p'''}{1}\cdot\frac{p'''-1}{2}\cdot\frac{p'''-2}{3}\cdot\frac{p'''-3}{4}(n)\times\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} \\
\quad\&c. \qquad\qquad\qquad \&c.
\end{array}$$

The difference between which and half (w^n), the ſum of all the chances, (which difference I ſhall denote by D), will conſequently be the number of the chances whereby the errors in exceſs (or in defect) can fall within the given limit m: ſo that $\frac{D}{\frac{1}{2}w^n}$ will be the true meaſure of the required probability, that the error, by taking the Mean of t obſervations, exceeds not the quantity $\frac{m}{t}$, propoſed.

To illuſtrate this by an example, from whence the utility of the method in practice, may clearly appear, it will be neceſſary, in the firſt place, to aſſign ſome number for v, expreſſing the limits of the

the errors to which any obſervation is ſubject. Theſe limits, indeed, depend on the goodneſs of the inſtrument, and the ſkill of the obſerver; but I ſhall ſuppoſe here, that every obſervation may be relied on to 5 ſeconds; and that the chances for the ſeveral errors, $-5''$, $-4''$, $-3''$, $-2''$, $-1''$, $0''$, $+1''$, $+2''$, $+3''$, $+4''$, $+5''$, included within the limits thus aſſigned, are reſpectively proportional to the terms of the ſeries 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1: which ſeries ſeems much better adapted than if all the terms were to be equal, ſince it is highly reaſonable to ſuppoſe, that the chances for the different errors decreaſe, as the errors themſelves increaſe.

Theſe particulars being premiſed, let it be now required to find, what the probability, or chance, for an error of 1, 2, 3, 4, or 5 ſeconds will be, when (inſtead of relying on one) the Mean of ſix obſervations is taken. Here, then, v being $=5$, and $t=6$, we have $n\ (=2t)=12$, $w\ (=v+1)=6$, and $p\ (=tv+n+m)$ $=42+m$: but the value of m, if we firſt ſeek the chances whereby the error exceeds not 1 ſecond, will be had from the equation $\frac{m}{t}=\pm 1$; where either ſign may be uſed, but the negative one is the moſt commodious: from whence we have $m\ (=-t)=-6$; and therefore $p=36$, $p'=30$, $p''=24$, $p'''=18$, &c. which values being ſubſtituted in the general expreſſion above determined, it will become $\frac{36}{1}\cdot\frac{35}{2}\cdot\frac{34}{3}(12)-\frac{30}{1}\cdot\frac{29}{2}\cdot\frac{28}{3}\cdot(12)\times 12+\frac{24}{1}\cdot\frac{23}{2}\cdot\frac{22}{3}(12)\times 66-\frac{18}{1}\cdot\frac{17}{2}\cdot\frac{16}{3}(12)\times 220=299576368$: and this ſubtracted from 1088391168 $(=\frac{1}{2}\times 6^{12})$, leaves 788814800, for the value of D correſponding.

There-

Therefore the required probability, that the error, by taking the Mean of ſix obſervations, exceeds not a ſingle ſecond, will be truly meaſured by the fraction $\frac{788814800}{1088391168}$; and conſequently the odds will be as 788814800 to 299576368, or as $2\frac{2}{3}$ to 1, nearly. But the proportion, or odds, when one ſingle obſervation is relied on, is only as 16 to 20, or as $\frac{8}{10}$ to 1.— To find, now, the probability, that the reſult comes within 2 ſeconds of the truth, let $\frac{m}{t}$ be made $= -2$; ſo ſhall m $(= -2t) = -12$; and therefore $p=30$, $p'=24$, $p''=18$, &c. And our general expreſſion will here come out $=36079407$; and conſequently $D = 1052311761$: whence $\frac{1052311761}{1088391168}$ will be the true meaſure of the probability here ſought; and the odds, or proportion of the chances, will therefore be as 1052311761 to 36079407, or as 29 to 1, nearly. But the proportion, or odds, when one ſingle obſervation is relied on, is only as 2 to 1: ſo that the chance, for an error exceeding 2 ſeconds, is not $\frac{1}{10}$ part ſo great from the Mean of ſix, as from one ſingle obſervation. And it will be found, in the ſame manner, that the chance for an error exceeding 3 ſeconds, will not be $\frac{1}{1000}$ part ſo great from the Mean of ſix, as from one ſingle obſervation. Upon the whole of which it appears, that the taking of the Mean of a number of obſervations, greatly diminiſhes the chances for all the ſmaller errors, and cuts off almoſt all poſſibility of any great ones: which laſt conſideration, alone, ſeems ſufficient to recommend

uſe of the method, not only to aſtronomers, but to all others concerned in making of experiments of any kind (to which the above reaſoning is equally applicable). And the more obſervations or experiments there are made, the leſs will the concluſion be liable to err, provided they admit of being repeated under the ſame circumſtances.

Other examples, and particulars might be added, in confirmation of what is here determined; but as I would not appear tedious to your Lordſhip, I here conclude, who am,

Woolwich, March 4, 1755.

My Lord,

Your Lordſhip's

moſt obedient humble ſervant,

T. Simpſon.

XX. *An Account of the Succeſs of Agaric, and the* Fungus vinoſus, *in Amputations: By Mr.* James Ford, *Surgeon, of* Briſtol.

Briſtol, March 31, 1755.

Read April 10, 1755.

IN 1753 I had ſome pieces of the agaric of the oak brought me from France, which I have frequently uſed with ſucceſs in hæmorrhages, particularly once after the operation for the ſtone, where a large artery lay ſo deep, that it could not conveniently be taken up with a needle. After the publication of Mr. Warner's obſervations, Mr. Thornhill applied it ſucceſsfully to an