

$$\text{et} \dots \frac{\pi e}{b} = \frac{A B a b + A M a^2 \frac{a}{b} + A B a^2 + A M \frac{a^2 c \mu}{b}}{A a^2 + B b^2 + M d^2 - \left(1 + \frac{a}{b}\right) A a c \mu}$$

$$\text{et preffio integra} = \pi + \frac{\pi a}{b} =$$

$$\frac{A B (a + b)^2 + A M (a^2 + a c \mu) \left(1 + \frac{a}{b}\right)}{A a^2 + B b^2 + M d^2 - \left(1 + \frac{a}{b}\right) A a c \mu}$$

Si jam frictionem et pondus machinæ excludere placuerit, habetur preffio integra  $= \frac{A B (a + b)^2}{A a^2 + B b^2}$ : et si,

ut in troclea evenit, supponatur  $a = b$ , erit preffio integra  $= \frac{A B (a + a)^2}{(A + B) a^2} = \frac{4 A B}{A + B}$

Christianus Hée,

*Professor Mathes. et Phys. Experim. in Statu Navali Hafnienfi, Societatum Scient. Hafnienfis et Berolinensis Membrum.*

## II. *An Investigation of a General Rule for the Resolution of Isoperimetrical Problems of all Orders. By Mr. Thomas Simpson, F. R. S.*

Read Jan. 9, 1755. **T**HE different species of problems comprehended under the name of Isoperimetrical ones, are of much greater extent than the name imports; since, not only the determination of the greatest areas and solids, under equal

equal perimeters or bounds (whence the name is derived), but whatever relates to the Maxima and Minima of quantities depending on a line, space, or body, whereof the figure is unknown, is, by mathematicians, included under that denomination.

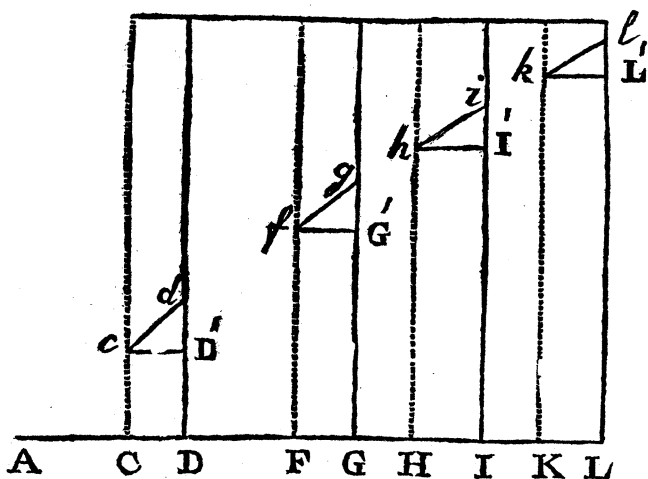
But notwithstanding the usefulness and great extent of this subject, nothing (that I know of) had been done thereon farther than the resolution of certain particular cases (such as finding the line of the swiftest descent, and the solid of the least resistance), 'till the celebrated mathematician M<sup>c</sup> Laurin, in his treatise of fluxions, gave the investigation of an elegant and very easy method, whereby the principal problems belonging to the first order may be resolved.

The paper I have now the honour to lay before the Society contains farther improvements on this subject: as it is by far more general than any thing yet offered, and is drawn up with a view to obviate the difficulties attending the resolution of a very intricate kind of problems, and thereby to open an easy way to some very interesting inquiries in natural philosophy, I cannot doubt of its meeting with a favourable reception.

*Lemma I.*

*Fig. 1.* At any given points  $D, G, I, L$ , in a right-line  $AL$ , suppose perpendiculars to be erected; and from any other given points  $c, f, b, k$ , at equal distances ( $cD, fG, bI, kL$ ), from the said perpendiculars, respectively, conceive right-lines  $cd, fg, bi, kl$ , to be drawn, to terminate somewhere in the said perpendiculars; let  $Q, R, S, T$ , denote

denote any quantities expressed in terms of  $AC$ ,  $cD'$ , and  $D'd$ , (independent of  $Cc$ ) and  $\mathcal{Q}$ ,  $R$ ,  $S$ ,  $T$ , as many other quantities affected in the very same manner with  $\Delta F$ ,  $fG'$ , and  $G'g$ ; and let  $\mathcal{Q}'$ ,  $R'$ ,  $S'$ ,  $T'$ , and  $\mathcal{Q}''$ ,  $R''$ ,  $S''$ ,  $T''$ , be quantities, still, expressed in the same manner, in terms of  $AH$ ,  $bI$ ,  $Ii$ , and  $AK$ ,  $kL'$ ,  $L'l$ , respectively: 'tis proposed to find an equation expressing the relation of the indeterminate perpendiculars  $D'd$ ,  $G'g$ ,  $Ii$ ,  $L'l$ , so that the quantity  $\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}'' + \mathcal{Q}'''$  may be a Maximum or Minimum, at the same time that the values of the other quantities  $R + R' + R'' + R'''$ ,  $S + S' + S'' + S'''$ , and  $T + T' + T'' + T'''$ , are given, or continue invariable.



Put  $D'd = \alpha$ ,  $G'g = \beta$ ,  $Ii = \gamma$ ,  $L'l = \delta$ ; and let the fluxion of  $\mathcal{Q}$  (supposing  $\alpha$  variable) be denoted by  $q\dot{\alpha}$ , that of  $R$ , by  $r\dot{\alpha}$ , &c. &c. then, since (by the nature of the proposition) the fluxion of  $\mathcal{Q} +$

$\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}'' + \mathcal{Q}'''$ , as well as those of  $R + R' + R'' + R'''$ ,  $S + S' + S'' + S'''$ , &c. must be equal to nothing, we therefore

$$\text{have } \begin{cases} q\dot{\alpha} + q'\dot{\beta} + q''\dot{\gamma} + q'''\dot{\delta} = 0 \\ r\dot{\alpha} + r'\dot{\beta} + r''\dot{\gamma} + r'''\dot{\delta} = 0 \\ s\dot{\alpha} + s'\dot{\beta} + s''\dot{\gamma} + s'''\dot{\delta} = 0 \\ t\dot{\alpha} + t'\dot{\beta} + t''\dot{\gamma} + t'''\dot{\delta} = 0 \end{cases}$$

In order, now, to exterminate the fluxions  $\dot{\alpha}$ ,  $\dot{\beta}$ ,  $\dot{\gamma}$ ,  $\dot{\delta}$ , let these equations be respectively multiplied by 1,  $e$ ,  $f$ ,  $g$ , (yet unknown), and let all the products thence arising be added together, whence will be had

$$\frac{q + er + fs + gt \times \dot{\alpha} + q' + er' + fs' + gt' \times \dot{\beta} + q'' + er'' + fs'' + gt'' \times \dot{\gamma} + q''' + er''' + fs''' + gt''' \times \dot{\delta}}{\times \dot{\delta}} = 0.$$

$$\text{Make, now, } q' + er' + fs' + gt' = 0$$

$$q'' + er'' + fs'' + gt'' = 0$$

$$q''' + er''' + fs''' + gt''' = 0$$

From whence (there being as many equations as quantities,  $e$ ,  $f$ ,  $g$ , to be determined), the values of these quantities will be always given in terms of  $q'$ ,  $r'$ ,  $s'$ , &c. that is,  $e$ ,  $f$ ,  $g$ , will always be represented by quantities depending on  $q'$ ,  $r'$ ,  $s'$ , &c. (or on  $AF$ ,  $G'g$ , &c.) exclusive of  $q$ ,  $r$ ,  $s$ ,  $t$ , (or of  $AC$  and  $D'd$ ), which have nothing to do in these last equations.

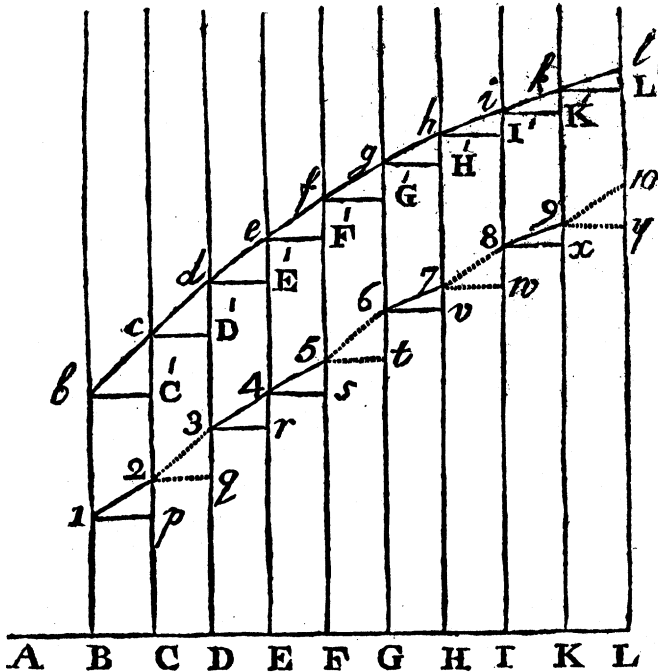
But, because all the terms of the equation  $\frac{q + er + fs + gt \times \dot{\alpha} + q' + er' + fs' + gt' \times \dot{\beta} + q'' + er'' + fs'' + gt'' \times \dot{\gamma} + q''' + er''' + fs''' + gt''' \times \dot{\delta}}{\times \dot{\delta}} = 0$ , after the first ( $q + er + fs + gt \times \dot{\alpha}$ ) do vanish (by their coefficients being made equal to nothing), it is evident that  $q + er + fs + gt$  must also be  $= 0$ : which is an equation expressing the general relation of  $AC$ ,  $cD'$ , and  $D'd$ , with regard to

to the other proposed quantities  $AF, fG', G'g, \&c.$  whereon the coefficients  $e, f, g,$  depend: and this relation will, evidently, continue the same, at whatever distances from the line  $AL$ , the points  $c, f, b, k,$  are taken, as these distances have nothing to do in the consideration, all the proposed quantities (as well the  $Q$ 's as  $R$ 's,  $\&c.$ ) being (by hypothesis) expressed in terms intirely independent thereof.

*Lemma II.*

*Fig. 2.* Upon a given right-line  $BL$ , suppose perpendiculars  $Bb, Cc, Dd, \&c.$  to be erected at equal distances; and upon the same line  $BL$ , as a base, suppose a polygon  $BbcdefghiklL$  to be constituted, having its angular points  $b, c, d, \&c.$  posited in the said perpendiculars; let  $y$  denote the distance of any of these perpendiculars ( $Cc, Dd, \&c.$ ) from any given point  $A$ , in  $LB$  produced; and, supposing  $bC', cD', dE', \&c.$  to be drawn parallel to  $AB$ , let the base of any of the little triangles  $bCc, cD'd, \&c.$  be represented by  $y'$ , and the perpendicular corresponding by  $x'$  ( $y'$  being given, or the same, in every triangle, and  $x'$  indeterminate): then, supposing  $Q, R, S, T,$  to denote any quantities expressed in terms of  $y, y',$  and  $x'$ , it is proposed to find an equation exhibiting the general relation of the quantities  $y, y',$  and  $x',$  so that the sum of all the  $y' Q$ 's (resulting from the several triangles) may be a Maximum or Minimum, at the same time that the sums of all the  $y' R$ 's,  $y' S$ 's,  $\&c.$  are given quantities.

Because



Because the values of the quantities  $\dot{y}Q$ ,  $\dot{y}R$ ,  $\dot{y}S$ ,  $\dot{y}T$ , depending on the different triangles  $bC'c$ ,  $cD'd$ , &c. are supposed to be no-ways affected by the distances ( $Bb$ ,  $Cc$ , &c.) of the bases of those triangles, from the base  $BL$  of the polygon, it is evident, that those values may be changed, by altering the species of one, or more, of the said triangles at pleasure, without any-ways affecting the values depending on the other triangles: for another polygon  $LB12345$ , &c. may be so described as to have all its sides, respectively, parallel to those of the former, excepting only those (23, 56, 78, 910) you would have to be different: so that the whole variation in the

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several sums (whether of the  $y'Q$ 's,  $y'R$ 's, or  $y'S$ 's, &c.) will depend intirely upon the difference of the particular triangles  $2q3$ ,  $cD'd$ ;  $5t6$ ,  $fG'g$ , &c. assigned.

Since, therefore, the values of the  $y'Q$ 's,  $y'R$ 's,  $y'S$ 's, &c. may be varied, at pleasure, by altering the species of any number of corresponding triangles ( $2q3$ ,  $cD'd$ ;  $5t6$ ,  $fG'g$ ;  $7w8$ ,  $bI'i$ ;  $9y10$ ,  $kL'l$ ), while the other triangles, and the values depending on them, remain the same, it is manifest, that, when the sum of the  $y'Q$ 's, answering to all the triangles, is a Maximum or Minimum, the sum of any number of them, taken at pleasure (other things remaining the same), will likewise be a Maximum or Minimum; and, consequently, that the sum of as many  $Q$ 's will, at the same time, be a Maximum or Minimum, because  $y'$  is every-where the same, or a constant quantity.

Hence, if the construction of the preceding Lemma be retained (supposing all the  $Q$ 's,  $R$ 's,  $S$ 's, &c. to be here expressed as before, in terms of  $AC$ ,  $cD'$ , and  $D'd$ , &c.) it is plain that the sum of all the  $Q$ 's, (or of the  $y'Q$ 's), depending on the said particular triangles (and consequently of all the  $y'Q$ 's in general), will be a Maximum or Minimum, when the general relation of  $y$ ,  $y'$ ,  $x'$ , (or of  $AC$ ,  $cD'$ ,  $D'd$ ) is expressed by the same equation  $q + er + fs + gt = 0$ , there determined: in which  $q$ ,  $r$ ,  $s$ ,  $t$ , represent the fluxions of  $Q$ ,  $R$ ,  $S$ ,  $T$ , divided by that of  $x'$  ( $= \alpha = D'd$ ), and wherein the coefficients  $e$ ,  $f$ ,  $g$ , will be constant quantities; because it is proved that their values depend intirely on the triangles  $fG'g$ ,  $bI'i$ ,  $kL'l$ , which remain the same,

# [ 11 ]

same, let the perpendicular (or ordinate)  $Cc$  be taken at what distance you will from the given point  $A$ ; that is, let  $y$  stand for which you will of the distances  $AB, AC, AD, \&c.$  Q. E. I.

## Corollary.

If the sides of the polygon  $bcdefgh, \&c.$  be diminished, and their number increased *in infinitum*, the sum of all the  $yQ$ 's will (it is well known) be expressed by the fluent of  $yQ$ ; the sum of all the  $yR$ 's, by the fluent of  $yR, \&c.$  whence it follows, that, to have the fluent of  $yQ$  (answering to a given value of  $y$ ) a Maximum, or Minimum, and the fluents of  $yR, yS, \&c.$  at the same time, given quantities, the relation of  $y, j$ , and  $\dot{x}$ , must be defined by the equation  $q + er + fs + gt = 0$ , above exhibited;  $q, r, s, \&c.$  being the respective fluxions of  $Q, R, S, \&c.$  divided by that of  $\dot{x}$  (or  $\dot{x}$ ); this quantity  $\dot{x}$ , or  $\dot{x}$ , (in finding the said fluxions) being, alone, considered as variable. Hence we have the following

## GENERAL RULE.

For the resolution of Isoperimetrical Problems, of all orders, take the fluxions of all the given expressions (as well that respecting the Maximum, or Minimum, as of the others whose fluents are to be given quantities), making that quantity ( $\dot{x}$ ) alone variable, whose fluent ( $x$ ) enters not into the said expressions; and, having divided everywhere by the second fluxion ( $\ddot{x}$ ), let the quantities hence arising, joined to general coefficients,

C 2

1,  $e, f, g,$



1,  $e$ ,  $f$ ,  $g$ , &c. (whose values will depend on the values given, and may be either positive or negative), be united into one sum, and the whole be made equal to nothing; from which equation the true relation of  $\dot{x}$  and  $\dot{y}$ , and of  $x$  and  $y$ , will be given, let the number of restrictions be what it will.

For an example of the general Rule here laid down, let the fluxions given be  $\frac{y \dot{x}^3}{\dot{y} \dot{y}}$ , and  $\dot{x}$ ; the fluent of the former, corresponding to any given value of  $y$ , being to be a Minimum, and that of the latter, at the same time, equal to a given quantity. Here, taking the fluxions of both expressions (making  $\dot{x}$ , alone, variable), and dividing by  $\dot{x}$ , the quantities resulting will be  $\frac{3 y \dot{x} \dot{x}}{\dot{y} \dot{y}}$  and 1; so that, in this case,

we have  $\frac{3 y \dot{x} \dot{x}}{\dot{y} \dot{y}} + e = 0$ , and therefore  $\dot{x} = a^{\frac{1}{2}} y - \frac{1}{2} \dot{y}$  (supposing  $a = -\frac{1}{3} e$ ). From whence, by taking the fluents,  $x = 2 a^{\frac{1}{2}} y^{\frac{1}{2}}$ , or  $x^2 = 4 a y$ , an equation answering to the common parabola.

If the abscisse of a curve be denoted by  $x$ , and the ordinate by  $y$ , it is known, that the several fluxions of the abscisse, curve-line, area, superficies of the generated solid, and of the solid itself, will be represented by  $\dot{x}$ ,  $\sqrt{x x + y y}$ ,  $y \dot{x}$ ,  $2 p y \sqrt{x x + y y}$ , and  $p y^2 \dot{x}$  respectively: if, therefore, the fluxions of these different expressions be taken as before (making  $\dot{x}$ , alone,

$\dot{x}$ , alone, variable), we shall get  $1 + \frac{e \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}}$

$$+ f \dot{y} + \frac{g \dot{y} \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + b \dot{y}^2 = 0, \text{ being a general}$$

equation for determining the relation of  $x$  and  $y$ , when any one of the said five quantities (*viz.* abscissa, curve-line, area, superficies, or solid) is a Maximum or Minimum, and all, or any number of the others, at the same time, equal to given quantities; wherein the coefficients  $e, f, g$ , and  $b$ , may be either positive or negative, or nothing, as the case proposed may require. Thus, for example, if the length of the curve, only, be given, and the area corresponding is required to be a Maximum, our equation will

then become  $\frac{e \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + f \dot{y} = 0$ , or  $a^2 \dot{x}^2 = \dot{y}^2 x$

$\dot{x} \dot{x} + \dot{y} \dot{y} \left( \text{by making } a = -\frac{e}{f} \right)$ ; whence  $\dot{x} =$

$\frac{\dot{y} \dot{y}}{\sqrt{a a - \dot{y} \dot{y}}}$ , and consequently  $x = a - \sqrt{a a - \dot{y} \dot{y}}$ ,

or  $2 a x - x^2 = \dot{y}^2$ , answering to a circle; which figure, therefore, of all others, contains the greatest area, under equal bounds.

If together with the ordinate (which, here, is always supposed given) the abscissa, at the end of the fluent, be given likewise, and the superficies generated by the revolution of the curve about its axis be a Minimum; then, from the same equation, we have

$$1 + \frac{g \dot{y} \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} = 0: \text{ whence } \left( \text{making } a = -\frac{1}{g} \right)$$

$\dot{x}$  is

$\dot{x}$  is found  $= \frac{a \dot{y}}{\sqrt{y y - a a}}$ ; and, from thence,  $x =$   
 $a \times \text{hyp. log. } \frac{y + \sqrt{y y - a a}}{a}$ ; which equation, be-

ing impossible when  $y$  is less than  $a$ , shews that the curve (which is here the Catenaria) cannot possibly meet the axis about which the solid is generated; and, consequently, that the case will not admit of any Minimum, unless the first, or least given value of  $y$  exceeds a certain assignable magnitude.

When any, or all of the above-specified quantities are given, and the contemporary fluent of some other expression, as  $\overline{\dot{x} \dot{x} + \dot{y} \dot{y}}^{\frac{n}{m} \frac{1-2n}{m}}$  is required to be a Maximum, or Minimum; the equation (by taking the fluxion of this last expression, and joining it to the former) will then be  $\overline{\dot{x} \dot{x} + \dot{y} \dot{y}}^{\frac{n-1}{m} \frac{1-2n}{m}} \times 2 n \dot{x} \dot{y} \dot{y}$

$$+ d + \frac{e \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + f y + \frac{g y \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + h y^2 = 0;$$

which, when  $m = 1$ , and  $n = -1$ , will be that defining the solid of the least resistance; and this, when the axis only is supposed to be given (without farther restrictions) will be expressed by

$$\overline{\dot{x} \dot{x} + \dot{y} \dot{y}}^{-2} \times -2 \dot{x} \dot{y} \dot{y}^3 + d = 0, \text{ or } 2 y \dot{y}^3 \dot{x} =$$

$d \times \overline{\dot{x} \dot{x} + \dot{y} \dot{y}}^2$ ; being the case, first, considered by Sir Isaac Newton.— If both the length and the solid content be given, the equation will be

$$-2 \dot{x} \dot{y} \dot{y}^3 \times \overline{\dot{x} \dot{x} + \dot{y} \dot{y}}^{-2} + d + h y^2 = 0; \text{ but if, besides}$$

besides these, the superficies is given likewise, it will then be  $-2 \dot{x} y \dot{y}^3 \times \overline{\dot{x} \dot{x} + \dot{y} \dot{y}}^{-2} + d + \frac{e y \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + b y^2 = 0$ .

Thus, in like manner, by assuming  $m = -\frac{1}{2}$ , and

$n = \frac{1}{2}$ , we have  $\frac{y^{-\frac{1}{2}} \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + d + \frac{e \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + f y + \frac{g y \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + b y^2 = 0$ , for the general equation of the curve of the swiftest descent: which,

when  $e, f, g$ , and  $b$ , are, all of them taken equal to

nothing, will become  $\frac{y^{-\frac{1}{2}} \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + d$ ; which is the case, considered by so many Others, answering to the cycloid. When the length of the arch described in the whole descent (as well as the values of  $x$  and  $y$ )

is given, the equation will then be  $\frac{y^{-\frac{1}{2}} \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} + d + \frac{e \dot{x}}{\sqrt{\dot{x} \dot{x} + \dot{y} \dot{y}}} = 0$ , or  $e - y^{-\frac{1}{2}} \dot{x}^2 = d^2$

$\times \overline{\dot{x} \dot{x} + \dot{y} \dot{y}}$ . And thus may the relation of  $x$  and  $y$  be determined, in other cases, and that under any number of restrictions.