

LVI. *A Discourse on the Locus for three and four Lines celebrated among the ancient Geometers, by H. Pemberton, M. D. R. S. Lond. et R. A. Berol. S. In a Letter to the Reverend Thomas Birch, D. D. Secretary to the Royal Society.*

S I R,

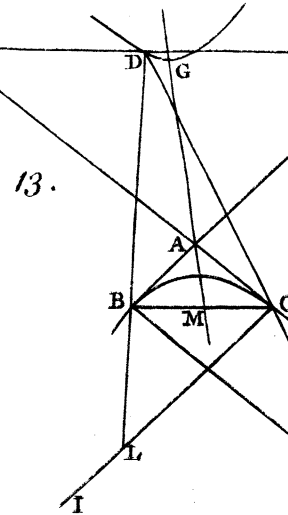
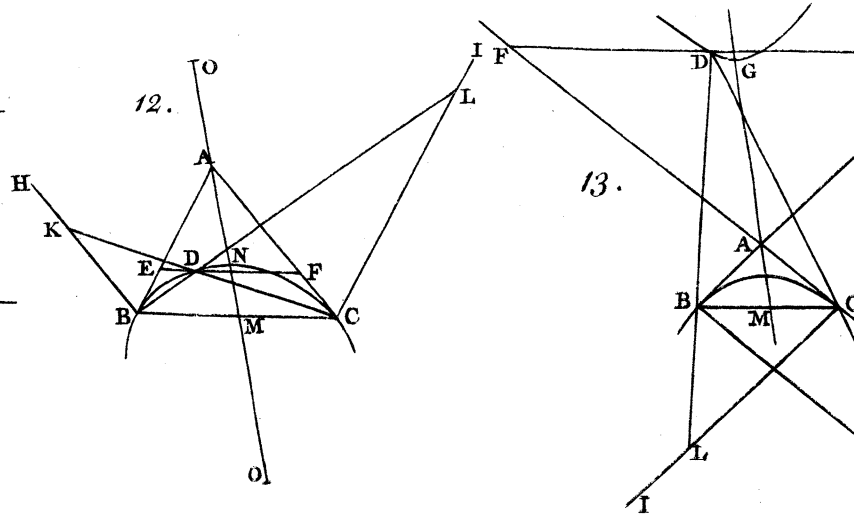
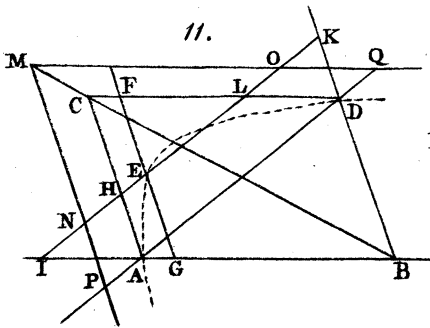
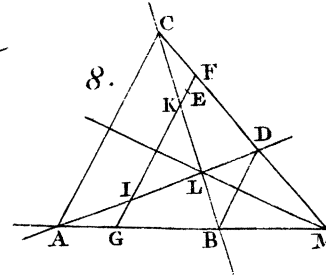
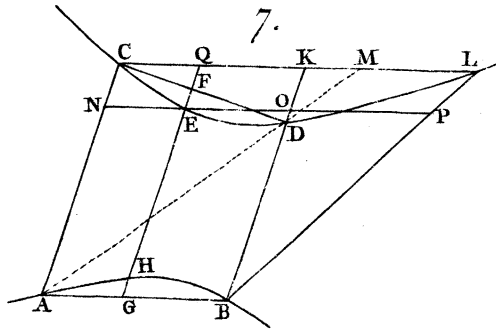
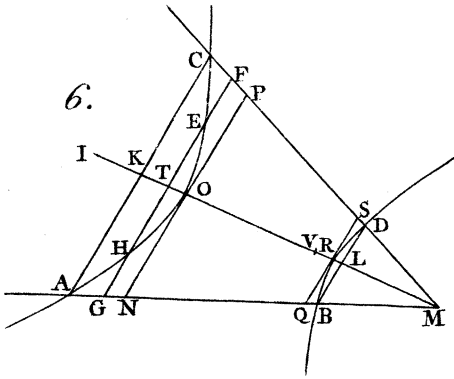
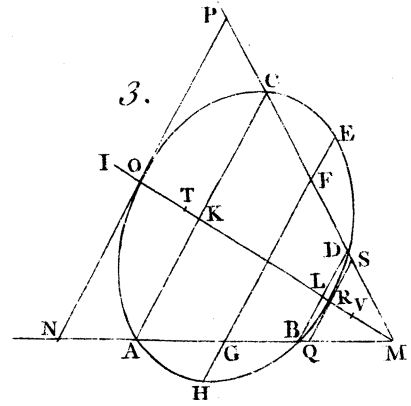
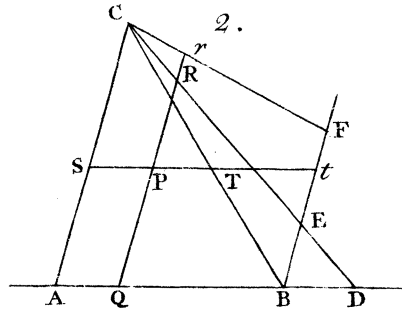
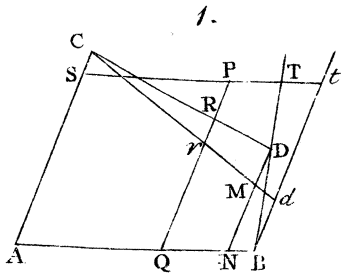
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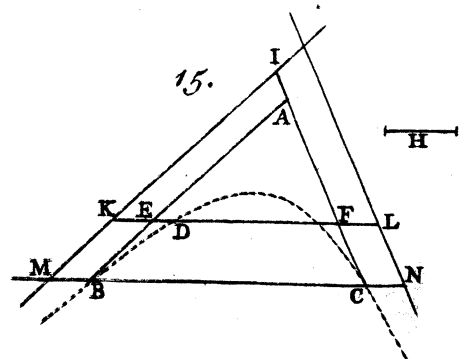
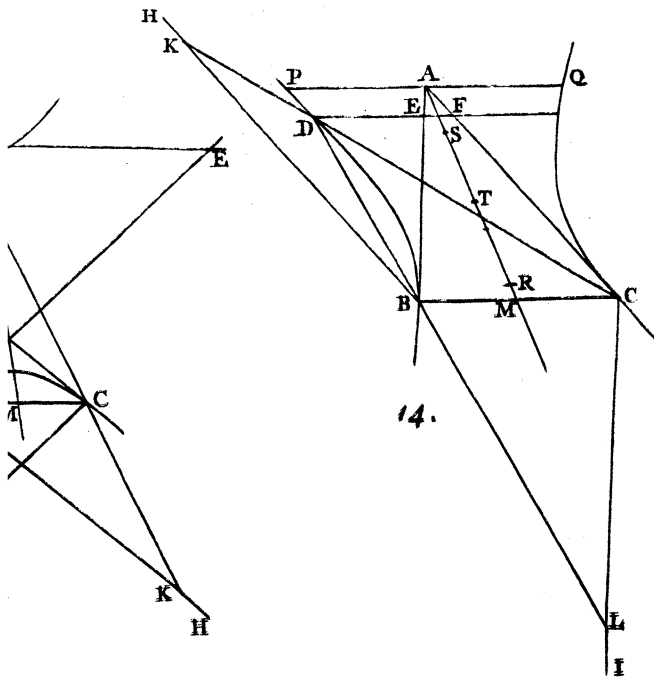
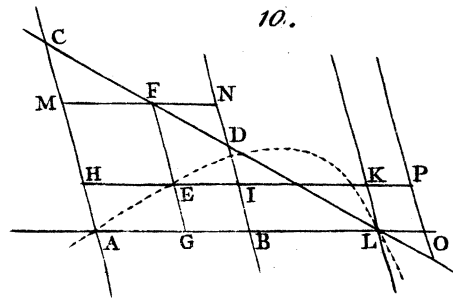
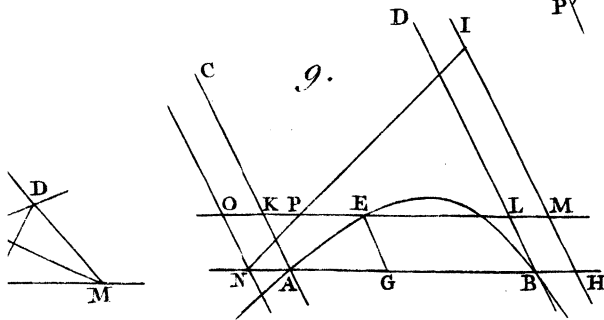
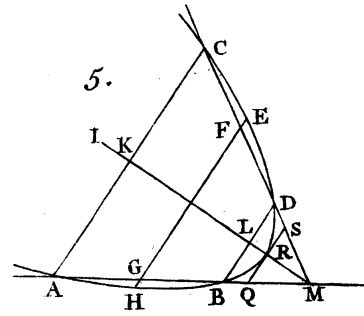
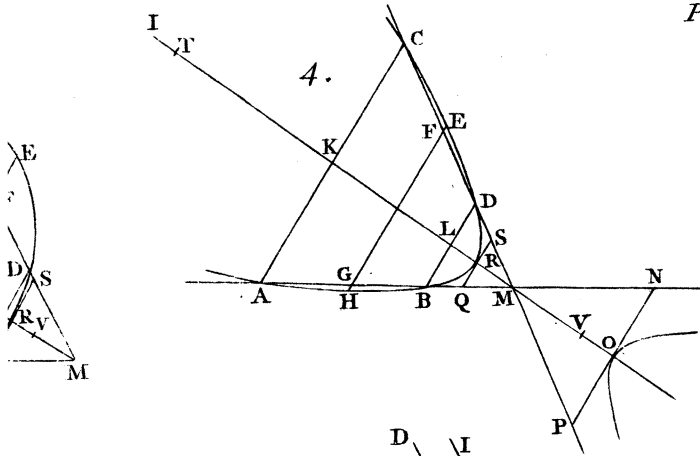
Read at R. S. 15 Dec. 1763. **M**Y worthy friend, and affociate in my early studies, the collector of the late Mr. Robins's mathematical tracts, thought it conducive to a more compleat vindication of the memory of his friend against an insinuation prejudicial to his candour, to make some mention of the course, I took in my early mathematical pursuits, and how soon I became attached to the ancient manner of treating geometrical subjects. This gave occasion to my looking into some of my old papers, amongst which I found a discussion of the problem relating to the *locus ad tres & quatuor lineas* celebrated among the ancients, which I then communicated to a friend or two, whose sentiments of those ancient sages were the same with mine. What I had drawn up on this subject is contained in the papers, I herewith put into your hands, which if you shall think worthy of being laid before our honourable society, they are intirely at your disposal.

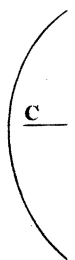
I am your most obedient servant,

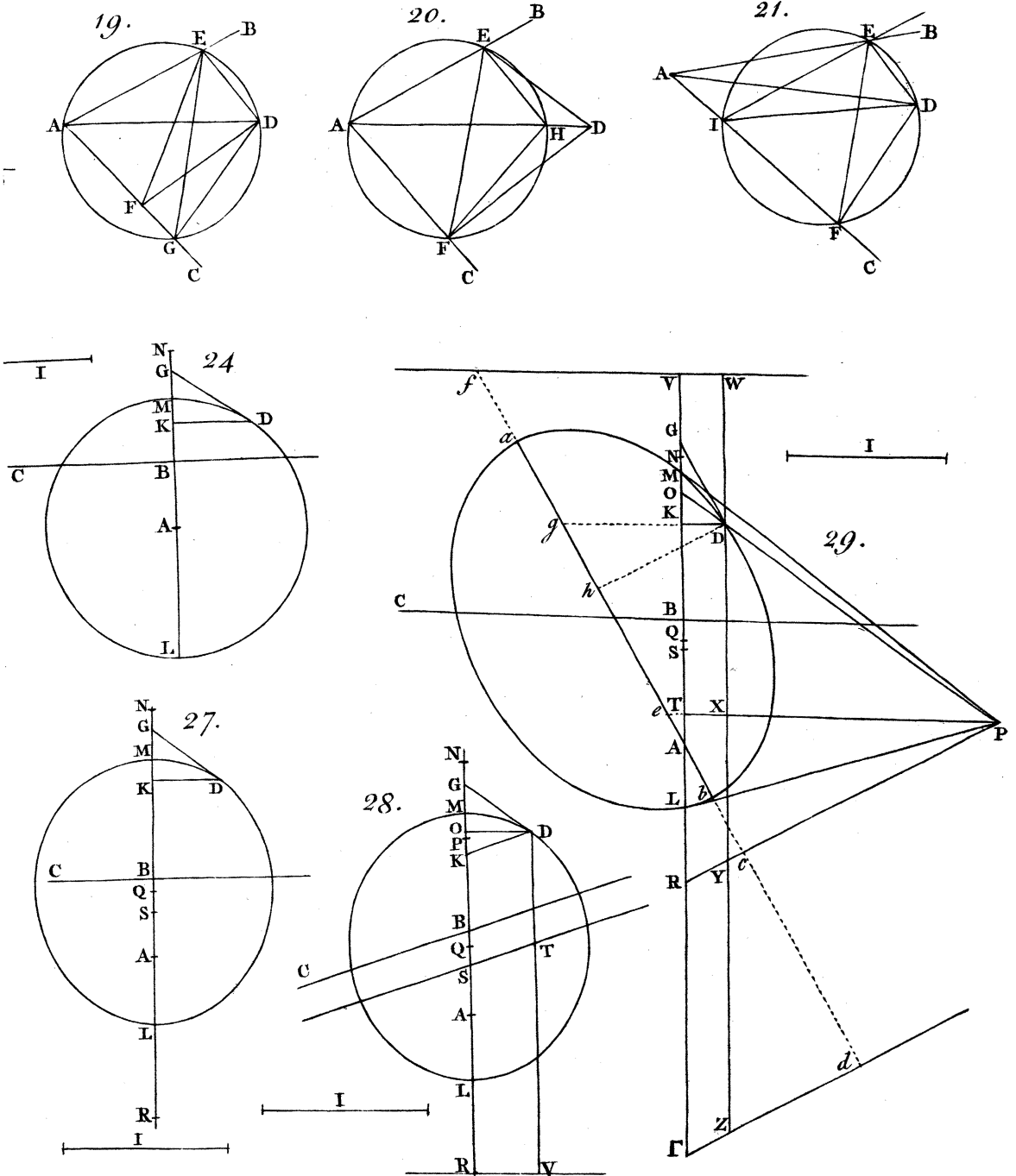
H. Pemberton.

T H E









THE describing a conic section through the angles of a quadrilateral with two parallel sides is so ready a means of assigning *loci* for the solution of solid problems, that it cannot be doubted, but this gave rise to the general problem concerning three and four lines mentioned by Apollonius, and described by Pappus; and it may be learnt from Sir Isaac Newton, who has considered the problem, how easily the most extensive form of it is reducible to the case, which I have supposed to give rise to it.

Sir Isaac Newton refers the general problem to this: Any quadrilateral $ABCD$ being proposed, to find the *locus* of the point P , whereby PRQ being drawn parallel to AC and SPT parallel to AB , the ratio of the rectangle contained under QP , PR to that under SP , PT shall be given; and this by pursuing the steps, whereby he proves, that the point P will in every quadrilateral be in a conic section, may be readily reduced to the case of a quadrilateral with two sides parallel, after this manner. Draw Bt and DN parallel to AC , then find the point M in ND , that the rectangle under MDN be to that under ANB in the ratio given, and draw Cr Md .

Here Rr will be to AQ , or SP , as MD to AN , and Bt , or QP , to Tt as ND to NB whence the rectangle under Rr , QP will be to that under SP , Tt as that under MDN to that under ANB , that is, in the ratio given of the rectangle under RPQ to that under SPT . Therefore, by taking the sum of the antecedents and of the consequents,

consequents, the rectangle under $r P Q$ will be to that under $S P t$, that is, to the rectangle under $A Q B$, in the quadrilateral $A B C d$, whose two sides $A C$, $B d$, are parallel, in the given ratio.

In like manner, if three of the given lines passed through one point, as the lines $C A$, $C B$, $C D$, and the rectangle under $Q P R$ be to that under $S P T$ in a given ratio, this case is with the same facility reduced to the like quadrilateral thus.

Draw $B E$ parallel to $A C$, that shall cut $S T$ produced in t , and let the point F be taken, that the rectangle under $C A$, $E F$ be to the square of $A B$ in the ratio given; then $C r F$ being drawn, $B t$, or $Q P$, will be to $T t$ as $A C$ to $A B$, and $R r$ to $A Q$, or $S P$, as $E F$ to $A B$; whence the rectangle under $Q P$, $R r$ will be to that under $T t$, $S P$, as that under $A C$, $E F$ to the square of $A B$, that is, in the given ratio of the rectangle under $Q P R$ to that under $S P T$, and the rectangle under $Q P r$ will be to that under $S P t$ or $A Q B$ in the quadrilateral $A B C F$, whose two sides $A C$, $B F$ are parallel, in the same given ratio.

Now let $A B C D$ be a quadrilateral having the two sides $A C$, $B D$ parallel, with any conic section passing through the four points A , B , C , D ; also, the point E being taken in the section, and $E F G$ being drawn parallel to $A C$ or $B D$, let the ratio of the rectangle under $A G B$ to the rectangle under $F E G$ be given: then the conic section will be given.

Let the sides $A B$, $C D$ meet in M , and draw $M I$ bisecting $A C$ and $B D$ in K and L . Then the diameter

diameter of the section, to which AC and BD are lines ordinately applied, will be in the line MI ; and if NP , QS are tangents to the section, and parallel to AC and BD , the points O , R , in which they intersect MI , will be the points of their contact, and the vertexes of that diameter. But the square of NO is to the rectangle under ANB , and the square of QR to the rectangle under AQB , as the rectangle under EGH or FEG to that under AGB , therefore in a given ratio; but the ratio of NM to NO , the same as that of QM to QR , is also given; whence the ratio of the square of NM to the rectangle under ANB , or of the square of OM to the rectangle under KOL , is given, as likewise the ratio of the square of RM to the rectangle under KRL .

Now in the ellipsis the square of MO , the distance of the remoter vertex of the diameter OR from M , is greater than the rectangle under KOL ; that is, the ratio given of the rectangle under FEG to that under AGB must be greater than the ratio of the square of half the difference between AC and BD to the square of AB . But in the hyperbola the square of MO is less than the rectangle under KOL ; whereby the ratio of the rectangle under FEG to that under AGB shall be less than that of the square of half the difference between AC and BD to the square of AB [a].

[a] As the square of OM shall be greater or less than the rectangle under KOL , the square of NM will be respectively greater or less than the rectangle under ANB ; therefore the ratio of the square of NO to the rectangle under ANB , that is, of the rectangle under FEG to that under AGB , will be accordingly greater

In both cases, if the point T be such, that the rectangle under MOT be equal to that under LOK , whereby MO shall be to OT in the given ratio of the square of MO to the rectangle under LOK , the given rectangle under KML will be to the rectangle under LTK (by Prop. 35. L. 7. Papp. [b]) in this given ratio, and therefore given; consequently the points T and O will be given.

In like manner, if the rectangle under MRV be equal to that under LRK , so that MR be to RV in the given ratio of the square of RM to the rectangle under LRK , the given rectangle under KML (by Prop. 22. L. 7. Papp.) will be to the rectangle under LVK in the same given proportion, whence the points V and R will be given.

Thus in both cases the points T and V will be found by applying to the given line KL a rectangle exceeding by a square, to which the given rectangle under KML shall be in the given ratio of the square of MO to the rectangle under KOL , or of the square of MR to the rectangle under KRL ; MO being to OT , and MR to RV , in that given ratio.

But in the last place, if this given ratio be that of equality, so that the square of RM be equal to the rectangle under KRL , by adding to both the rectangle under MRL , that under RML will be equal to that under KM , LR , and MR to RL as KM to ML , and the vertex R of the diameter

greater or less than the ratio of the square of NO to the square of NM , which is the same with that of the square of the difference between AK , BL to the square of AB .

[b] See pag. 511.

R I will be given, the conic section being here a parabola, this diameter having thus but one vertex.

Hitherto the point E, when the line E F G falls between A C and B D, is without the quadrilateral, and within the lines A B, C D, when E F G is without the quadrilateral.

But when E is within the lines A C, B D in the first case, and without in the second, the *locus* of the point E will be opposite sections, each passing through two angles of the quadrilateral.

When one section passes through A and C, and the other through B and D; then if the diameter M I be drawn, as before, and to K L be applied a rectangle deficient by a square, to which the given rectangle under K M L shall be in Fig. 6. the given ratio of the square of M O to the rectangle under K O L, or of the square of M R to the rectangle under K R L, the points T and V, constituting the rectangles under K T L and under K V L, being thus found, M O will be to O T, and M R to R V, in this given ratio (by prop. 30. L. 7. Papp.) O and T being the vertexes of the diameter M I.

But the rectangles under K T L, K V L cannot be assigned, as here required, unless the ratio given for that of the square of O M to the rectangle under K O L, or that of the square of R M to the rectangle under K R L, be not less than that of the rectangle under K M L to the square of half K L; that is, when the ratio of the square of O N to the rectangle under A N B, and that of the square of R Q to the rectangle under A Q B, or that of the given ratio of the rectangle under F E G to that under A G B is not less than that of the rectangle

under AK , BL to the square of half AB , or of the rectangle under AC , BD to the square of AB .

But if one of the opposite sections pass through A and B , and the other through C and D , the ratio of the rectangle under FEG to that under AGB will be less than that of the rectangle under

Fig. 7. AC , BD to the square of AB . For CL being drawn parallel to AB , and AD joined and continued to M , the line DM falls wholly within the section passing through C and D : therefore KM is less than KL , and the ratio of KD to KL less than that of KD to KM , that is, of BD to AB ; whence BK being equal to AC , and CK to AB , the ratio of the rectangle under BKD to that under CKL , being the ratio of the rectangle under EGH , or FEG , to that under AGB , will be less than the ratio of the rectangle under AC , BD to the square of AB .

And here the point L is given; for the given rectangle under BKD is to that under CKL in the given ratio of the rectangle under HGE , or that under FEG , to the rectangle under AGB ; hence CK , equal to AB , being given, KL is given, and consequently the point L .

Again, BL being joined, and $NEOP$ drawn parallel to AB , also GEF continued to Q , as AG , equal to CQ , to FQ so will CK be to DK , and OP to EG , equal to OB , as KL to BK , consequently the rectangle under OP , AG will be to that under EG , FQ as that under KL , CK to that under KB , DK , that is, as the rectangle under AGB to that under FEG ; and by combining the antecedents and consequents the rectangle under PEN will be to that under QEG in the same given ratio. More-

Moreover DK being to AC as KM to CM , the ratio of DK to AC , that is, the ratio of the rectangle under BKD to the square of AC , will be less than the ratio of KL to CL , or the ratio of the rectangle under CKL to that under AB, CL ; therefore, by permutation and inversion, the ratio of the rectangle under CKL to the rectangle under BKD , that is, the given ratio of the rectangle under NEP to that under ANC , equal to that under GEQ , is greater than the ratio of that under AB, CL to the square of AC . And hence the opposite sections passing through the angles of the quadrilateral $ABCL$, whose sides AB, CL are parallel, will be given as before.

When the given ratio of the square of OM to the rectangle under LOK shall be that of the rectangle under KML to the square of half KL , whereby the given ratio of the rectangle under FEG to that under AGB shall be that of the rectangle under AC, BD to the square of AB , the points T and V shall unite in one, bisecting KL , and the points O and R shall also unite in one, dividing the line KLM harmonically; and then the *locus* of the point E will be each of the diagonals of the quadrilateral. Fig. 6.

In the last place, if the diagonals AD, BC of the quadrilateral were drawn, cutting GE in I and K , and the ratio of the rectangle under KEI to that under AID were given, and not that of the rectangle under GEF to that under AGB ; then the intersection of these diagonals, as L , will be in the line drawn from M bisecting AC , and BD , and the point L will fall within the quadrilateral, whereby the *locus*, when an Fig. 8.

ellipsis or single hyperbola, will be assigned by the 36th proposition of the foresaid book of Pappus: and when opposite sections, by the 30th proposition, or be reduced to the preceding cases thus.

Since KG will be to GB as CA to AB , and IG to GA as BD to AB , the rectangle under KGI will be to that under AGB , in the given ratio of the rectangle under AC , BD to the square of AB . Therefore when the ratio of the rectangle under KEI to that under AID is given, the rectangle under AID also bearing a given ratio to that under AGB , the ratio of the rectangle under KEI to that under AGB will be given, and in the last place the ratio of the rectangle under GEF to that under AGB will be given, this rectangle under GEF being the excess of that under KGI above that under KEI [c]. And thereby the sections will be determined, as before.

AND thus may the *locus* of the point sought be assigned in all the cases of this ancient problem, which Sir Isaac Newton has distinctly explained. The other cases, he has alluded to, may be treated, as follows.

When three of the given lines shall be parallel, as AC , BD , and HI , the fourth line being AB , and $KELM$ being parallel to AB , the
Fig. 9. ratio of the rectangle under KEL to the rectangle under EG and EM shall be given; that is, three points A , B , and H being given in the line AB , with the line GE insisting on AB in a given angle, that the rectangle under AGB shall be to that under GH and GE in a given ratio: then

[c] By Prop. 193. Lib. 7. Papp.

take AN equal to BH , and draw NO parallel to AC , BD , and HI .

Then if NP be drawn, that PO be to ON in the given ratio, NP will be given in position, and PO will be to ON , that is, EG , as the rectangle under KEL to that under EM , EG ; so that the rectangle under KEL will be equal to that under PO , EM . But the rectangle under OKM is equal to the excess of that under OEM above that under KEL [d]; therefore the rectangle under OKM , or that under NAH , or under NBH , is equal to that under EM and the excess of OE above OP , that is, to the rectangle under PEM ; the point E therefore is in an hyperbola described to the given asymptotes PN , MH , and passing through A and B .

Again if two of the given lines only are parallel, but the rectangles otherwise related to them, than as above. Suppose the ratio of the rectangle under AG , EF to that under BG , GE is given. Let CD meet AB in L , and let HEI , MFN be drawn parallel to AB , and LK parallel to AC and BD . Then the parallelogram EM will be to the parallelogram EB in the given ^{Fig. 10.} ratio. Take AO to OB in that ratio, and draw OP parallel to AC and BD . Here the point O will be given, and the parallelogram PA will be in the given ratio to the parallelogram PB ; whence AB will be to BO as the parallelogram BH to the parallelogram BP , and as the difference between the parallelograms EM and EB to the parallelogram EB , consequently as the parallelogram GM to the parallelogram PG ; therefore the ratio of the rectangle under AG , FG to the rectangle under

[d] By Prop. 194. Lib. 7. Papp.

EG ,

EG, EP or OG will be given; and in the last place the ratio of FG to GL being given, the ratio of the rectangle under AG and GL to that under EG, OG will be given. And thus three points A, L, O, will be given with GE insisting on AB in a given angle, as in the preceding case.

Moreover, AC and BD being parallel, AB and CD may be also parallel. And then, when the ratio of the rectangle under AGB to that under GEF Fig. 11. is given, the determination of the *locus* is so obvious as not to have required a distinct explanation. But when the rectangle under AG, EF bears a given ratio to that under BG, GE; let the diagonals AD, BC be drawn, and HELK drawn parallel to AD. Then the rectangle under HEL will be to that under KEI in the same given ratio; and if CM be taken to MB in the same ratio, the lines MNP, MOQ drawn, the first parallel to AC, BD, and the other parallel to AB, CD, will be given in position, and the diagonal BM will bisect both IK, NO, and HL; therefore the rectangle under HEL being to that under KEI as MC to MB, that is, as NH to NK, here by division the rectangle under HEL will be to that under IHK [e] as NH to HK; therefore equal to that under NH and IH or KL. But the rectangle under NEO is equal to the sum of the rectangles under HNL and under HEL [f]; therefore the rectangle under NEO is equal to that under NH, NK, equal to that under APD, that is, equal to that under PAQ, or that under PDQ, the diagonal BM bisecting both PQ and

[e] By the prop. of Papp. before cited. [f] By the same.

A D.

A D. But thus the point E is in an hyperbola described to the asymptotes MN, MO, and passing through A and D.

THE determination of this *locus* for three lines is solved almost explicitly by Apollonius in the three last propositions of his third book of Conics. For if the three lines proposed were AB, AC, BC, and the point sought D, that the ratio of the rectangle under EDF (the line EF being drawn parallel to BC) should be in a given ratio to the square of a line drawn from D to BC in a given angle, the square of which line will be in a given ratio to the rectangle under BE, CF; then if BH, CI are drawn parallel to AC and AB respectively, also BDL, CDK drawn through D, the square of BC will be to the rectangle under BK, CL as the rectangle under DF, DE, to that under CF, BE. Fig. 12,
13, 14.

Hence if the ratio of the rectangle under DF, DE to the square of a line drawn from D on BC in a given angle, is given; the square of this line being in a given ratio to the rectangle under CF, BE, the ratio of the rectangle under BK, CL to the square of BC will be given; whence a conic section passing through D will in all cases be given.

In the first place let the point D be within the angle BAC. Then if BC be bisected by the line AM, this will be a diameter to the conic section, which shall touch BA, AC in the points B, C, and BC will be ordinately applied to that diameter; the vertex of this diameter being N, the given ratio of the rectangle under BK, CL to the Fig. 12.

the square of BC will be compounded of the ratio of the square of MN to the square of NA , and of the ratio of the rectangle under BAC to the fourth part of the square of BC ; and thus the line AM will be divided in N in a given ratio, and the point N , one vertex of the diameter, to which BC is ordinately applied, will be given.

If AN be equal to NM , the point N will be the only vertex of this diameter, and the section will be a parabola.

Otherwise by taking the point O in AM extended, so that the ratio of AO to OM be the same with that of AN to NM , the point O will be the other vertex of the diameter.

And here if the ratio of AN to NM be that of a greater to a less, the point O will fall beyond M from A within the angle BAC , the conic section being an ellipsis.

But if the ratio of AN to NM be that of a less to a greater, the point O will fall on the other side of A , and the section will be an hyperbola.

Fig. 13. And in this case if the opposite section be drawn, that also will be the *locus* of the point D within the angle vertical to the angle BAC .

In the last place, if D be in either of the collateral angles, AM drawn as before will contain a secondary diameter in opposite sections, one of which shall touch BA in B , and the other CA in C .
Fig. 14. Then if one of these sections pass thro' D , the sections will be given. For here PAQ being drawn through A parallel to BC , the given ratio of the rectangle under CL , BK to the square of
of

of BC will be the same with that of the given rectangle under BAC to the square of AP : therefore AP is given, and thence the sections. For let RS be the secondary diameter, to which BC is ordinately applied, and T the center of the opposite sections. Then the square of BM will be to the rectangle under AMT as the square of the transverse diameter conjugate to the secondary diameter RS to the square of this secondary diameter; and if a line were drawn from M to P , this would touch the hyperbola BP in P [g], and the square of AP will be to the rectangle under MAT in the same ratio; therefore the given ratio of the square of MB to the square of AP will be that of the rectangle under AMT to the rectangle under MAT , or the ratio of MT to AT ; consequently the ratio of MT to AT is given, and thence the point T . But also the diameter RS is given in magnitude, the square of RT or of ST being equal to the rectangle under MTA ; whence in the last place the transverse diameter conjugate to this is also given; for the square of this diameter is to the square of RT as the given square of BM to the rectangle under AMT now also given.

But a more simple case may also be proposed in three lines, when the ratio of the rectangle under EDF should be equal to the rectangle under a given line, and that drawn from D to BC in a given angle: Fig. 15.

This line will bear, both to BE and FC , a given ratio, and the rectangle under EDF will be in a given

[g] Apoll. conic. L. II. prop. 40.

ratio to the rectangle under the given line and EB or CF .

Let the line given be H , and take MB and NC , that the rectangle under $MB C$, and that under $BC N$ be to that under BA and H in the given ratio of the rectangle under $ED F$ to that under BE and H , BM and CN being equal. Then draw from M and N lines parallel to BA , CA , which shall intersect EF in K and L , whereby, MK cutting CA in I , the rectangle under $MB C$ will be to that under BA and H as the rectangle under $BM C$ to that under MI and H , and also as the rectangle under EKF to that under KI and H , that is, as the rectangle under $ED F$ to that under H and BE or MK , whence by adding the antecedents and consequents the rectangle under KDL will be to the rectangle under H and MI in the same given ratio, which is also that of the rectangle under $BM C$ to the same rectangle under H and MI : the point D therefore is in an hyperbola passing through B and C having for asymptotes the lines MK and NL given in position, the rectangle under KDL being equal to that under $BM C$, or that under MBN .

If the two lines AB and AC are parallel, the *locus* may be known to be a parabola by the last proposition of the fourth book of Pappus.

But if BC were parallel to one of the other, the *locus* will be an hyperbola, as the preceding, but assigned by a shorter process.

Suppose the given lines to be AE , AF , and BC parallel to AF . And let the rectangle under $ED F$ be equal to that under DG , and the given line H , the line EG making given angles with AE , AF . Here take EI equal to H ,

and deduct from both the rectangles that under $E I$ or H , and $D F$, whereby will be left the rectangle under $I D F$ equal to that under H and $F G$, both whose sides are given. Draw therefore $I K$ parallel to $A E$, and the rectangle under $I D F$ will be equal to this given rectangle, the given lines $K I$, $A F$ being the asymptotes to the hyperbola passing through D .

Coroll. If $L M$ be drawn through B parallel to $E F$, $L B$ shall be equal to $F G$, and $B M$ equal to $E I$ or H , whereby the hyperbola opposite to that passing through D will pass through B .

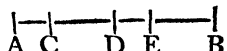
S C H O L I U M.

The propositions of Pappus, which have been referred to in pag. 500, 501, 504, l. 2. are given by him, among others, for Lemmas subservient to the lost treatise of Apollonius *De sectione determinata*, and the four here cited respect and comprehend all the cases of the problem, where three points are given in any line, and a fourth is required such, that the rectangle under the segments of the proposed line intercepted between the point sought, and two of the given points, shall bear a given ratio to the square of the segment terminated by the third point.

The cases indeed of the problem, from the diversity of situation in the points given to the point sought and to one another, are in number six. The given extreme of the segment to constitute the square may either be without the other two given points, or between them. And when it is without, the point sought may be required to be taken without them all, either on the side opposite to the given extreme of the segment to constitute the square, which will be one case, or it may be required to fall on the same side,

which will be a second case. If it be required to fall between this point and the other two, this will be a third case. A fourth case will be, when the point sought shall be required to fall between the other two points. Also when the given extreme of the segment to constitute the square lies between the other two given points, the point sought may be required to fall, either there also, or without, composing the 5th, and 6th cases.

The propositions in Pappus referring to these cases, though but four in number, suffice for them all, each proposition being applicable to the problem two ways. For instance the thirty-fifth proposition, as expressed by Pappus, is this, being the first above cited. Three points C, D, E being taken in the line A B, so that the rectangle under A B E be equal

to that under C B D, A B is to B E  B
as the rectangle under D A C to

that under C E D. Now A B is to B E, both as the square of A B to the rectangle under A B E, and as the rectangle under A B E to the square of B E. Therefore if the ratio of A B to B E be given, the ratio of the square of A B to the rectangle under C B D will be given, which is the first of the cases above described, and also the ratio of the rectangle under C B D to the square of B E given, which is the second case. In both cases the rectangle under D A C will be to that under C E D in the given ratio of A B to B E. But in the first the rectangle under D A C will be given, and the point E in the rectangle under C E D to be found by applying a rectangle, which shall bear a given ratio to the given rectangle under D A C to the given line C D exceeding by a square; and in the second case the
rectangle

rectangle under CED is given, and A in the rectangle under DAC to be found by applying to the given line CD a rectangle exceeding by a square, which shall bear a given ratio to the rectangle under CED now given; whence by the ratio of AB to BE given the point B will be found in both cases.

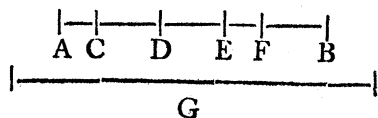
The 22d proposition either way applied refers to the 3d case only, the 30th relates both to the 4th and 5th, and the 36th proposition to the remaining 6th.

The 45th, and other following propositions, are accommodated to the solution of Apollonius's problem, when four points are given, and a fifth required, which with the given points shall form four segments such, that the rectangle under two shall bear a given proportion to the rectangle under the other two. The various cases of this problem appear to have been the subject of the second book of the mentioned treatise of Apollonius; and, according to the character given by Pappus of those propositions, these lemmas serve to reduce them to problems in the first book, not those above mentioned, but those, where three points being given, the rectangle under the segments included by two, and a fourth point shall bear a given proportion to the rectangle under the segment formed by the third point and a given line.

For instance the 46th proposition is this; in the line AB four points

A, C, E, B being given;

and the point F assumed between E and B ;

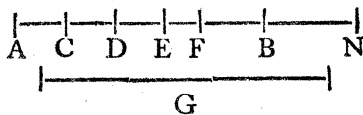


also D taken, according to the 41st proposition, that the rectangle under ADC be equal to that under BDE ; if G be equal to the sum of AE, CB , the rectangle

gle under AFC together with that under EFB will be equal to the rectangle under G and DF .

Here if it were proposed to find the point F , that the ratio of the rectangle under AFC to that under EFB should be given, the ratio of the rectangle under AFC to that under DF and the given line G would be given.

But this analysis may be carried on to a compleat solution of the problem thus. If CN be taken to G in the given ratio of the rectangle under AFC to that under DF and G , the point N will be given, and the rectangle under AF, CN will be to



that under AF, G in this ratio of CN to G ; consequently the excess of the rectangle under AF, CN above that under AFC , that is, the rectangle under AFN , will be to the excess of the rectangle under AF and G above that under DF and G , or the given rectangle under AD, G , in the same given ratio, and in the last place the rectangle under AFN will equal the given rectangle under AD and CN .

Here I have chosen this proposition in particular, because the case of the problem, to which it is subservient, is subject to a determination, when FN shall be equal to AF . And then the rectangle under AFN being equal to that under AD and CN , as CN to FN so is AF to AD , and by division as CF to FN so DF to AD ; therefore when AF is equal to FN , CF will be to AF as FD to AD : consequently CD to FD as FD to AD , and the square of DF equal to the rectangle under ADC , when the problem admits of a single solution only, wherein the rectangle under AFC

AFC will bear to that under EFB a less ratio than in any other situation of the point F between E and B.

Moreover CN is to G as the rectangle under AFC to the sum of the rectangles under AFC and EFB; therefore FN being equal to AF, when the problem is limited to this single solution, the rectangle under AFC shall be to the rectangles under AFC and EFB together as the sum of AF and FC to G, which is equal to the sum of AE and CB; whence by division the ratio of the rectangle under AFC to that under EFB, when the problem is limited to this single solution, will be that of the sum of AF and CF to the excess of FB above EF.

Thus directly do these lemmas correspond with Apollonius's first mode of solution, and lead to the general principle of applying to a given line a rectangle exceeding or deficient by a square, which shall be equal to a space given. This being a simple case of the 28th and 29th propositions of the 6th book of Euclid's elements, admits of a compendious solution. Such a one is exhibited by Snellius in his treatise on these problems (in Apollon. Batav.) and Des Cartes has exhibited another more contracted in it's terms, but not therefore more useful. It may also be performed thus. If upon a given line AB any triangle ACB be erected at pleasure; then if the legs CA, CB, whether equal or unequal, be continued to Fig. 17. D and E, that the rectangles under CAD and CBE be each equal to the given space, and a circle be described through C, D, E cutting AB extended in F and G, the rectangle under BFA and BGA will each be equal to the space given. Also if in the legs CA, CB the rectangles under CAD and CBE be each taken Fig. 18.

taken equal to the space given, and a circle in like manner be described through C, D, E, cutting AB in F and G, the rectangles under AFB and AGB will each be equal to the given space. Here it is evident, that the space given must not exceed the square of half AB, when equal, the circle will touch AB in it's middle point.

P O S T S C R I P T.

AS this application to a given line of a rectangle exceeding or deficient by a square, or the more general problem treated of in the sixth book of the elements, of applying a space to a line so as to exceed or be deficient by a parallelogram given in species, is the most obvious result, to which the analysis of plane problems, not too simple to require this construction, leads; so the descriptions of the conic sections here treated of, stand in the like stead in regard to the higher order of problems styled solid from the use of the conic sections deemed necessary for their genuine solution. And these are the only modes of solution, the modern algebra, which grounds its operations on one or two elementary propositions only, naturally leads to. But as the form of analysis amongst the antients, by expatiating through a larger field, often was found to arrive at conclusions much more concise and elegant, than could offer themselves in a more confined track; the antient sages in geometry, that the solid order of problems might not want this advantage, sought out that copious and judicious collection of properties attending the conic sections,
which

which, with some useful additions from later writers, have been handed down to us.

And as the advantages of this ancient system of analysis cannot be too much inculcated in an age, wherein it has been so little known, and almost totally neglected, permit me, Sir, to close this address to you with an example in each species of problems.

Were it proposed to draw a triangle given in species, that two of its angles might touch each a right line given in position, and the third angle a given point. It is obvious, how difficult it would be to adopt a commodious algebraic calculation to this problem; notwithstanding it admits of more than one very concise solution, as follows.

Let the lines given in position be AB , AC and the given point D , the triangle given in species being EDF . Fig. 19,
20, 21.

In the first place suppose a circle to pass through the three points A , E , D , which shall intersect AC in G . Then EG , DG being joined, the angle DEG will be equal to the given angle DAC , both inscribing on the same arch DG ; also the angle EDG is the complement to two right of the given angle BAC : these angles therefore are given, and the whole figure $EFGD$ given in species. Consequently the angle EGF , and its equal ADE will be given together with the side DE of the triangle in position.

Again, suppose a circle to pass through the three points A , E , F , cutting AD in H , and EH , FH joined. Here the angle EFH will be equal to the given angle EAF , and the an- Fig. 20.

gle FEH equal to the given angle FAH. Therefore the whole figure EHFD is given in species, and consequently the angle ADE, as before.

In the last place suppose a circle to circumscribe the triangle, and intersect one of the lines, as AC, in I. Here DI being drawn, the angle DIF will be equal to the given angle DEF in the triangle; consequently DI is inclined to AC in a given angle, and is given in position, as also the point I given; whence IE being drawn, the angle FIE will be the complement of the angle EDF in the triangle to two right. Therefore IE is given in position, and by its intersection with the line AB gives the point E, with the position of DE, and thence the whole triangle, as before.

Here it may be observed, that the angle D of the triangle EDF given in species touching a given point D, and another of its angles touching AC, the line IE here found is the locus of the third angle E.

Again, in the astronomical lectures of Dr. Keil, it is proposed to find the place of the earth in the ecliptic, whence a planet in any given point of its orbit shall appear stationary in longitude, and a solution is given from the late eminent astronomer, Dr. Halley, upon the assumption, that the orbit of the earth be considered as a circle concentric to the Sun.

But for a compleat solution of this problem let the following lemma be premised.

The velocity of a planet in longitude bears to the velocity of the earth the ratio, which is compounded of the subduplicate ratio of the *latus rectum* of the greater axis of the planet's orbit to the *latus rectum* of
of

of the greater axis of the earth's orbit, of the ratio of the cosine of the angle, which the orbit of the planet makes with the plane of the ecliptic, to the radius, and of the ratio of a line drawn in any angle from the center of the sun to the tangent of the orbit of the earth at the point, wherein the earth is, to a line drawn in the same angle from the sun to the tangent of the orbit of the planet projected upon the plane of the ecliptic at the place of the planet in the ecliptic.

Let A be the sun, BC the orbit of any planet, DE the same projected on the plane of the ecliptic, FG being the line of the nodes, B the place of the planet in its orbit, D its projected place: then the plane through B and D, which shall be perpendicular to both the planes BC and DE, intersecting those planes in BH, DH, the lines BH, DH will be both perpendicular to the line of the nodes, and the angle BHD the inclination of the orbit to the plane of the ecliptic. But tangents drawn to BC and DE at the points B and D respectively will meet the line of the nodes, and each other in the same point I, and the velocity of the planet in longitude will be to its velocity in the orbit BC, as DI to BI.

Now from the point A let AK fall perpendicular on BI, and AL be perpendicular to DI: then the ratio of DI to IB will be compounded of the ratio of DI to DH, or of AI to AL, of the ratio of DH to BH, and of that of BH to BI, that is, of AK to AI. But DH is to BH as the cosine of the inclination of the orbit to the radius, and the two ratios, that of AI to AL, and that of AK to AI, compound the ratio of AK to AL: therefore

the velocity of the planet in longitude is to the velocity in its orbit in the ratio compounded of that of the cosine of the inclination of the planet's orbit to the radius, and that of AK to AL.

Moreover the ratio of the velocity of the planet in B to the velocity of the earth in any point of its orbit is compounded of the subduplicate of the ratio of the *latus rectum* of the greater axis of the planet's orbit to the *latus rectum* of the greater axis of the earth's orbit, and of the ratio of the perpendicular let fall from the sun on the tangent of the earth's orbit at the earth to AK, the perpendicular let fall on the tangent of the planet's orbit at B. Therefore the velocity of the planet in longitude, when in B, to the velocity of the earth in any point of its orbit is compounded of the subduplicate ratio of the *latus rectum* of the greater axis of the planet's orbit, to the *latus rectum* of the greater axis of the earth's orbit, of the ratio of the cosine of the inclination of the planet's orbit to the radius, and of the ratio of the fore-said perpendicular on the tangent of the earth's orbit to AL, the perpendicular on DI: these perpendiculars being in the same ratio with any lines drawn in equal angles to the respective tangents.

This being premised, the place of a planet in the ecliptic being given, the place of the earth, whence the planet would appear stationary in longitude, may be assigned thus.

A denoting the sun, let B be a given place of any planet in its orbit projected orthographically on the plane

Fig. 23. of the ecliptic, CB the tangent to the planet's projected orbit at the point B, which will therefore be given in position. Also let DE be

the

the orbit of the earth, and the point *D* the place of the earth, whence the planet would appear stationary in longitude at *B*.

Join *AB*, and draw a tangent to the earth's orbit at the point *D*, which may meet *CB* in *F*, and the line *AB* in *G*; draw also *AH* making with *DF* the angle *AHD* equal to that under *ABC*. Then the point *D* being the place, whence the planet appears stationary in longitude, as *FB* to *FD* so will the velocity of the planet in longitude in *B* be to the velocity of the earth in *D*, this velocity of the planet in *B* being also to the velocity of the earth in *D* in the ratio compounded of the subduplicate of the ratio of the *latus rectum* of the greater axis of the planet's orbit, to the *latus rectum* of the greater axis of the orbit of the earth, of the ratio of the co-sine of the inclination of the planet's orbit to the plane of the ecliptic to the radius, and of the ratio of *AH* to *AB*: therefore the ratio of *FB* to *FD* will be compounded of the same ratios; and if *I* be taken, that the ratio of *AB* to *I* be compounded of the two first of these, *I* will be given in magnitude, and the ratio of *FB* to *FD* will be compounded of the ratio of *AB* to *I*, and of *AH* to *AB*. Whence *FB* will be to *FD* as *AH* to *I*; and the angles *CBA*, or *FBG*, and *AHG* being equal, whereby *FG* will be to *FB* as *AG* to *AH*, by equality *FG* will be to *FD* as *AG* to *I*, and *DK* being drawn parallel to *FB*, *BG* will be to *BK* as *FG* to *FD*, and therefore as *AG* to *I*.

But now, as this problem may be distributed into various cases, in the first place consider the earth as moving in a circle concentric to the sun, and likewise *CB*, the tangent to the planet's orbit, perpendicular to *AB*.

But

But here DK also will be perpendicular to AB , and AB meeting the earth's orbit in L and M , the

rectangle under KAG will be equal to the square of AM . But BG being to BK as AG to I , if BN be taken equal to I , BG will be to BK as AG to BN , and AB to KN also as AG to BN , and the rectangle under NK , AG equal to that under AB and I : therefore the rectangle under KAG being equal to the square of AM , NK will be to KA as the rectangle under AB , I to the square of AM , that is, in a given ratio, and KD with the point D will be given in position:

Again, when CB is not perpendicular to LM , let DO be perpendicular to LM . Then the rectangle under OAG will be equal to the square of AM . But BN being taken equal

to I , as before, the rectangle under NK , AG will be equal to that under AB , I ; whence NK will be to AO in the given ratio of the rectangle under AB , I to the square of AM . Therefore NP being taken to PA in that ratio, the point P will be given, and KP , the excess of NP above NK , will be to PO , the excess of AP above AO , in the same ratio. Hence, as DK is parallel to CB and DO perpendicular to LM , the triangle KOD is given in species, and if PD be drawn, the angle OPD will be given; for the co-tangent of the angle OKD will be to the co-tangent of the angle OPD , as KO to OP , that is, as the rectangle under AB , I together with the square of AM to the square of AM , and hence the point D is given by the line PD drawn from a given point P in a given angle APD ; and if AD be drawn, AD will be to AP as the sine of the angle APD to the sine of the angle PDA ; this angle therefore

therefore is given, and the angles APD , PDA being given, the angle PAD is given.

Coroll. Here, where the orbit of the earth is supposed a circle, the ratio of I to AB , that is, of the rectangle under AB , I to the square of AB , will be compounded of the subduplicate ratio of AM , the femidiameter of the earth's orbit, to half the *latus rectum* to the greater axis of the planet's orbit, and of the ratio of radius to the co-sine of the inclination of the planet's orbit to the plane of the ecliptic; and adding on both sides the ratio of the square of AB to the square of AM , the ratio of the rectangle under AB , I to the square of AM will be compounded of the ratio of the square of AB to the rectangle under AM and the mean proportional between AM and the half of this *latus rectum* of the planet's orbit, and of the ratio of the radius to the co-sine of the inclination of the net's orbit.

In the next place, though the earth's orbit is not a circle concentric to the sun; yet if the projection of the planet falls on the line perpendicular to the axis of the earth's orbit, the point A will still bisect LM .

In this case draw to the points L and M tangents to the ellipsis meeting in P , from whence through D draw PD meeting the ellipsis again in Q , and intersecting LM in O . Here if a tangent be drawn to the ellipsis in Q , it will meet the tangent at D on the line LM in the point G . Fig. 26.

Now LG will be to GM as LO to OM , and the point A bisecting LM , the rectangle under GAO will be equal to the square of AM . But BG is to BK as AG to I . Therefore BN being taken equal to I , AB will be to KN as AG to I , and the rectangle under AB , I equal to that under AG ,

KN: whence AO being to KN as the rectangle under GAO to that under AG and KN, AO will be to KN as the given square of AM to the rectangle under AB and I, also given.

Draw RP parallel to CB, and take PS to AP, also NT to AR in this given ratio inverted. Then will the points T and S be both given, also AO will be to KN, and RO to KT, as AR to NT, that is, as AP to PS. Therefore if TV be drawn parallel to CB, that is, to KD, and VS parallel to LM, these lines will be both given in position; and WDX being also drawn parallel to LM, WD will be equal to KT, and RO being to KT, as AP to PS, DY will be to WD as XP to PS, and by composition YW to WD as XS to PS, and the given rectangle under YW, or SV, and PS equal to that under WD, and XS. Whence SV being parallel to LM, the point D will be in an hyperbola passing thro' P, and having for asymptotes the lines VS, VT given in position.

But if the projection of the planet fall on the axis of the earth's orbit, or the same continued, AB extended to the earth's orbit in L and M will be the axis of that orbit.

If also CB should be perpendicular to AB, KD

would be ordinately applied to LM; and
 Fig. 27. the point R being taken, that Q being the center of the orbit, the rectangle under AQR be equal to the square of QM, the same will be equal also to the rectangle under GQK; whence as GQ to AQ so RQ to QK, and AG to AQ as KR to QK. But, as above, BG being to BK as AG to I, and BN taken equal to I, BG will be to BK as AG to BN, and AB to KN also as AG to BN or I. Therefore if NS be taken to AB as I to AQ, by equality

equality NS will be to NK as AG to AQ , that is, as KR to QK ; and in the last place NS to KS as KR to QR , that is, the rectangle under SKR equal to the given rectangle under NS, QR ; whence the point K , the position of KD , and thence the point D will be given.

But if DK be not ordinately applied to LM , let DO be ordinately applied to LM . Then here the rectangle under AQR , equal to the square of QM , will be equal to that under OQG , and GQ to AQ as QR to OQ , whence by Fig. 28. composition AG to AQ as OR to OQ . But BN being now also taken equal to I , and NS to AB as I to AQ , AB will be here in like manner to KN as AG to I , and NS to KN as AG to AQ ; therefore NS will be to KN as OR to OQ , and by conversion NS to KS as OR to QR . But NS and QR being both given in magnitude, if SP be taken to NS as QR to PR , the point P will be given, and also by equality SP will be to KS as OR to PR ; whence if RV be drawn parallel to DO , and ST to KD , both RV and ST will be given in position, one passing through the given point R , parallel to the ordinates applied to the axis LM , and the other through the point S also given, and parallel to KD or CB : also DTV being drawn parallel to ML , DT will be equal to KS and DV equal to OR , therefore as SP to DT so DV to PR , and the rectangle under SPR equal to that under TDV , consequently the point D in an hyperbola passing thro' P , and having for asymptotes the lines ST, RV given in position.

In the last place when the line LM drawn through the sun in A , and the projected place of the planet in B , is neither the axis of the earth's orbit, nor bi-

sected in A , the tangents to the points L, M being drawn to meet in P , let LM be bisected in Q , and the point R taken, that the rectangle under AQR be equal to the square of QM , whereby PDO being drawn, the rectangle under AQR shall be equal to that under OQG , and QG to AQ as QR to QO , or by composition AG to AQ as OR to QO . Therefore if NB be here also taken equal to I , and NS to AB as I to AQ , AB being as before, to NK as AG to I ; by equality NS will be to NK as AG to AQ , that is, as OR to QO . Whence by conversion NS will be to KS as OR to QR ; and if PT be drawn parallel to CB and SV be here taken to NS as QR to TR , by equality SV will be to KS as OR to TR and also by conversion SV to KV as OR to OT . Moreover SV will be given in magnitude, and the point V given; therefore VW drawn parallel to CB , or KD , will here be given in position. But $WDXY$ being also drawn parallel to RV , SV will be to KV , or DW , as YD to XD , and YZ being taken equal to the given line SV , YZ will be to DW as ZD to XW , equal to TV , and the given rectangle under YZ, TV equal that under WDZ . Therefore ΓZ being drawn parallel to $RP, R\Gamma$, and its equal YZ , being given, the line ΓZ is given in position, and the point D in an hyperbola having for asymptotes $VW, \Gamma Z$, and passing through P .

Thus is this problem in all cases solved either by a right line, or an hyperbola given in position, which shall intersect the projected orbit in the point sought. For though in each case the projection of the planet has here been considered as within the orbit of the earth, the form of argumentation will be altogether similar, were the projection of the planet without. And this is agreeable to the method, I have pursued throughout this discourse, where I have always accommodated the expression to one situation only of the terms given and sought in each article; the variation necessary for the other cases, when one has been duely explained, being sufficiently obvious.

In the 5th volume of the Commentaries of the Royal Academy at Petersbourg is given an algebraical computation for a general solution of this problem in the orbits of any two planets projected on the plane of the ecliptic; but with this oversight of applying to the projected orbits a proposition from Dr. Keil's Astronomical Lectures, which relates to the real orbits (a).

However from the geometrical solution now given a calculation for assigning the point D may be formed without difficulty. L D M being the orbit of the earth, A is the focus, and R P perpendicular to the

(a) The demonstration of Dr. Keil's proposition proceeds on the known property in the planets of having their periodic times in the sesquuplicate ratio of the axes of their orbits, which confines the proposition to the real orbits; for in each planet the periodic time through the projected orbit is the same, as through the real, though the axis in one be not equal to the axis of the other.

axis. Let this axis be ab meeting RP in c , ΓZ in d , PT in e and WV in f . Then the angle aAM is given, being the distance between the heliocentric place of the planet in the ecliptic from the earth's aphelion. Also PT being parallel to CB , the angle ATe , and consequently the angle AeT , will in like manner be given, whence the points Γ , R , T , V being given, as in the solution above, the points d , c , e , and f will be given, the triangles ARc , ATe , being given in species, and similar respectively to the triangles $A\Gamma d$, and AVf . Also the rectangle under WDZ being equal to that under $R\Gamma$, VT , if DK be continued to the axis in g , and Db be drawn parallel to PR , the rectangle under fg , bd is equal to that under fe , dc , and both being deducted from the rectangle under $fb d$ the excess of the rectangle under $fb d$ above that under fe , dc will be equal to that under $gb d$, so that this difference will be a mean proportional between the square of bd and the square of bg , which is in a given ratio to the square of bD , and therefore in a given ratio to the rectangle under $ab b$, $D b$ being ordinately applied to the axis ab .

Thus a biquadratic equation may be formed, whereby the point b shall be found, and thence the point D , whose distance from A is to be as the excentricity of the earth's orbit to half its axis.

Therefore I shall only observe farther, that here occurs an obvious question, what, in so extended a search for principles leading to the solution of any problem, as the ancient analysis admits of, can conduct to the most genuine upon each several occasion.

But

But for this end, where commodious principles do not readily offer themselves, the most general means is to consider first simple cases of the problem in question, and from thence to proceed gradually to the more complex, as has been here done in the present problem, where the several preceding cases lead one after another to the points and lines required for the last case, wherein the problem is stated in its most extensive form.

