

XXXV. *A Memoir concerning the most advantageous Construction of Water-Wheels, etc. by Mr. Mallet of Geneva. Communicated by M. Maty, M. D. Sec. R. S. Translated from the French, by J. Bevis, M. D. R. S. S. Read March 26, 1767.*

§ I **T**HE stream of rivers is of such importance in moving machines of all kinds, that any attempt towards perfecting this part of mechanics may be considered as of very great utility.

The first wheel on which the stream acts is one of the most essential members of the machine, and it is easy to discern that the greater or less effect thereof must depend, in a great measure, on the manner of constructing this wheel, and on the dimensions given to it. I shall not at present inquire whether wheels of different constructions from those which have been long in use might be advantageously substituted in their stead; but confine myself, in this essay, to an examination of the most common ones, and to discover the means by which they may be made to produce the greatest possible effects. Their construction is very simple; they consist of several planes inserted into the same axle placed horizontally above the surface of the water, and in a position perpendicular to the stream. These planes, called
float-

Fig. 1.

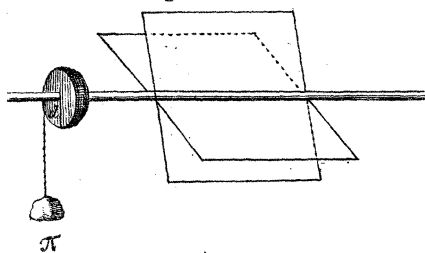


Fig. 2.

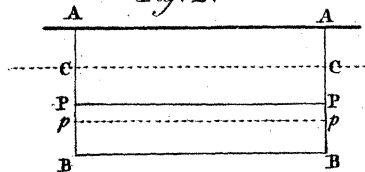


Fig. 3.

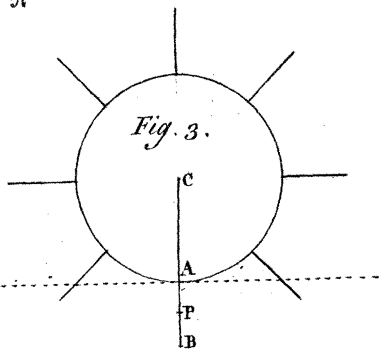


Fig. 4.

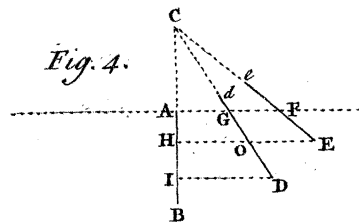


Fig. 5.

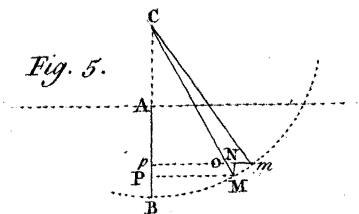


Fig. 6.

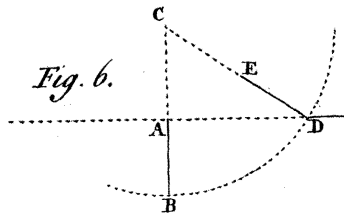
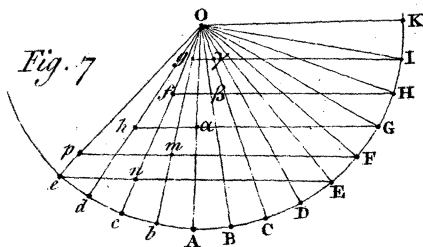


Fig. 7.



float-boards, by yielding to the action of the stream, cause the axle on which they are fixed to turn round, by means of several wheels, which take into each other, and give motion to the part destined to produce some purposed effect, as the mill-stone in a corn-mill.

The size of the float-boards, the velocity with which the wheel is to turn, and the number of the float-boards to produce the greatest possible effect, are three main things I propose to examine in the following inquiry.

In the first place I will suppose the total resistance which this wheel has to encounter, on the part of the machine, and which hinders it from moving so swift as the stream, to be expressed by a weight π , suspended to the extremity of a cord fixed to the circumference of a wheel whose radius is d , and which has the same axle as the float-board wheel, so that the effect of the stream is to raise the said weight π , as expressed in TAB. XV. *fig.* 1. I will likewise suppose, that the stream, by its velocity, moves through v feet in one second of time, and that this velocity is the same, though at different depths.

§ II. After these suppositions, the first thing that presents, is to determine what should be the size of the float-boards for the stream to be capable of raising the weight π with a certain determinate velocity.

Let AA BB, *fig.* 2. be one of the float-bords let into the axle AA, and placed vertically in the water, so as to receive the perpendicular impulse of the stream. Its horizontal length BB= β feet, its vertical height AB= α feet, the velocity of the wheel

wheel at the point B, such that it shall run through z feet in a second; n pounds the weight of a cubic foot of water; and I will suppose the impulse of the stream on a plane perpendicular to it is (as Dr. *Daniel Bernoulli* has stated in his *Hydrodynamica*) equal to the weight of a prism of water, whose base is the plane, and its altitude the generating height of the velocity with which the plane is impelled. This being supposed; let $AP = x$, Pp , its differential, $= dx$, which will give the velocity of the float-board at the point $p = \frac{x}{a}z$, and the relative velocity of the stream with which the plane is impelled at the same

point $= v - \frac{x}{a}z$ whose generating height is $\frac{(v - \frac{x}{a}z)^2}{60}$

feet, whence we have the weight of the parallelo-piped of that height, and of the base $PP pp$ equal to $\frac{n\beta dx}{60} (v - \frac{x}{a}z)^2$ pounds, which weight multiplied by the length AP (x) of the lever which tends to turn the plane, will give $\frac{n\beta x dx}{60} (v - \frac{x}{a}z)^2$ for the total effect of the stream on the little rectangle $PP pp$, whose integral is $\frac{n\beta}{60} \int (v v x dx - \frac{2vz}{a} x x dx + \frac{z^2}{a^2} x^3 dx) =$
 $\frac{n\beta}{60} \left(\frac{1}{2} v v x x - \frac{2}{3} v z \frac{x^3}{a} + \frac{1}{4} z z \frac{x^4}{a^2} - \frac{1}{2} v v f f + \frac{2}{3} v z \frac{f^3}{a} - \frac{1}{4} \frac{z z f^4}{a^2} \right)$ (putting $AC = f$ for the distance between the axle and the surface of the water when the float-board has only its part CB plunged in the water) which (putting $x = a$) will become
 $\frac{n\beta}{60} \left(\frac{1}{2} v v - \frac{2}{3} v z + \frac{1}{4} z z \frac{a a}{a^2} - \frac{1}{2} v v f f + \frac{2}{3} \frac{v z f^3}{a} - \frac{1}{4} \frac{z z f^4}{a^2} \right)$

$-\frac{1}{x} \frac{zzf^4}{\alpha\alpha}$) and will express the effect on the whole plane CC BB, equal πd , the product of the weight π by the length d of the lever on which it acts in opposing the motion of the wheel.

§ III. If the wheel be plunged as deep as its axle, that is, $f = 0$, the equation is changed into this $\frac{n\beta\alpha\alpha}{60} \left(\frac{1}{x} vv - \frac{2}{3} vx + \frac{1}{x} zz \right) = d\pi$, where it appears

1°. That the quantities d , π , v and x remaining the same, we have β inversely proportional to the square of α , whence it follows, that if the length β is to be diminished without altering the effect of the float-board, the height α must be encreased proportionally to the square root of β ; for example, if β is to be made four times less, it will be sufficient to double the height α . 2°. That likewise the velocity of the float-board remaining the same, the weight π will be in the compound ratio of the length β , and of the square of the height $\alpha\alpha$. 3°. Without meddling with the dimensions of the float-board, the more the quantity x is increased, the more must the weight π be diminished. If x be made $= 0$, we have

$$\pi = \frac{n\beta\alpha\alpha}{60d} \cdot \frac{1}{x} vv, \text{ and if } x = v \text{ we have } \pi = \frac{n\beta\alpha\alpha}{60d} \cdot \frac{1}{v} vv,$$

that is six times greater than in the first case; which is very conformable to the nature of things, for when the wheel is in motion, the stream then not acting upon it but with the excess of its velocity above that of the wheel, it follows, that the greater such velocity is, the more will the effect of the stream be diminished.

It follows from our last remark, that the greatest weight with which the stream can constitute an equilibrium, will be $= \frac{n\beta\alpha\alpha vv}{120d}$, but then the wheel will not have any motion, nor consequently the weight π : If the float-board be increased, or the weight diminished, from that instant the wheel will begin to turn, and the swifter as the float-board is greater, or the weight less; but in most machines, it is required that the weight may be the greatest possible, as also the velocity with which it is raised. A question therefore here offers itself, whose solution is of much importance. What must be the velocity of the float-board whose dimensions are given, that the product of the weight by its velocity shall be the greatest possible?

§ IV. The velocity of the weight π is $\frac{d}{\alpha}z$ feet in a second, which being multiplied by the value of $\pi = \frac{n\beta\alpha\alpha}{60d}(\frac{1}{2}vv - \frac{2}{3}vz + \frac{1}{4}z^2)$ will give the product $\frac{n\beta\alpha\alpha}{60}(\frac{1}{2}v^2z - \frac{2}{3}vz^2 + \frac{1}{4}z^3)$ which must be a *maximum*; for which purpose make $\frac{1}{2}vvdz - \frac{4}{3}vzdz + \frac{3}{4}z^2dz = 0$; whence we have $z = \frac{8 - \sqrt{10}}{9} = 0,53752v$: this value of z being substituted, make the equation $\frac{1}{2}vv - \frac{2}{3}vz + \frac{1}{4}z^2 = \frac{11 + 2\sqrt{10}}{81}vv$, so that we have the equation $\beta\alpha\alpha = \frac{11 + 2\sqrt{10}}{4860} + \frac{d\pi}{nvv} = 280,529 + \frac{d\pi}{nvv}$, which expresses the dimensions of the float-board where the effect will be the greatest possible. If the float-board be plunged no deeper than to CC, as we have at first supposed, the most advantageous value of

of z may be determined in the same manner, which will be found

$$z = \frac{8a^4 - 8af^3 - a\sqrt{10a^6 + 54a^4f^2 - 128a^3f^3 + 54a^2f^4 + 10f^6}}{9(a^4 - f^4)} v$$

If $f = 0$, this value of $z = 0.537v$

$$f = 0.25 a = \frac{1}{4} a. \quad = 0.498v$$

$$f = 0.3 a \quad = 0.486v$$

$$f = 0.5 a = \frac{1}{2} a. \quad = 0.436v$$

$$f = 0.7 a \quad = 0.390v$$

$$f = 0.9 a \quad = 0.353v$$

$$f = a \quad = 0.333v$$

By the inspection of these different values it appears that this value of z diminishes as the plunged part is greater, and that this velocity can never exceed the quantity $0.537v$, nor be less than $\frac{1}{3}v$ *.

This value of z and of its square zz being substituted in the general formula (§ II.) we shall obtain from it the following equation :

$$\frac{60d\pi}{v\beta vv} = \frac{11a^6 - 27a^4f^2 + 32af^3 - 27a^2f^4 + 11f^6 + 2(a^3 - aff)\sqrt{10a^6 + 54a^4f^2 - 128a^3f^3 + 54a^2f^4 + 10f^6}}{81(a^4 - f^4)}$$

which for a given relation between f and a will shew the breadth β for producing the greatest effect.

* If in the value of z we make $f = a$, we have $z = \frac{0}{0}$

which obliges us to take, according to the common method, the differentials of the numerator and of the denominator, considering f as variable, and the relation of these differentials will give the value of z : but on account of the radical quantity, the calculus being somewhat tedious, and, again bringing out $z = \frac{0}{0}$

and that after several similar operations, it is better to have recourse to the equation from which the value of z was deduced;

this equation is $\frac{3}{4}avz \cdot \frac{a^3 - f^3}{a^4 - f^4} - \frac{1}{2}avv \cdot \frac{a^2 - f^2}{a^4 + f^4}$, which

by the above operation will be $\frac{3}{4}zz = vz - \frac{1}{4}vv$, and $z = \frac{1}{3}v$.

As the extremity of the float-board must have a certain velocity depending on the relation of the height a to the plunged part, and as the velocity of the weight $\pi = \frac{d}{a} z$, it follows that if we would increase the velocity of π , we must diminish the height a and increase the breadth β , so that the product $\beta a a$ and the relation of f to a may be the same as before ; for example, if the wheel be plunged as deep as the axle, to double the velocity of the weight, the height of the float-board must be reduced one half, and its breadth be quadrupled.

§ V. It may so happen that the channel on which the wheel is placed shall be so shallow and narrow, as not to allow the float-boards the necessary dimensions, for raising the weight with a convenient velocity. In this case we are obliged to raise the axle of the wheel above the surface of the water, so much that the lever on which the stream acts may be long enough to recompense the smallness of the float-boards. Herein it is necessary to solve the following problem :

The breadth b , and the height a , of the float-board AB being given ; to find the radius CA (r) of the wheel which shall cause the weight π to ascend with the velocity $\frac{d}{r+a} z$.

The exact solution of this problem might be deduced from the formula (§ II.) which would render the operation tedious, the equation being of the fourth degree ; but it may be rendered far more simple by a supposition which is but little wide of the truth when AB is but small in comparison of CA, and this is to consider

consider all the points of the float-board AB as affected with the same velocity z .

Let CP fig. 3. = x , we shall have $\frac{n\beta}{60} (v-z)^2 \int x dx$ for the effect of the portion AP, and $\frac{n\beta}{60} (v-z)^2 \times \frac{2ar+aa}{2}$ for the effect of the whole float-board AB.

This quantity must be made equal to $d\pi$, and then, just as in the foregoing cases, such a value of z be sought, that the weight π and its velocity may be the greatest possible; that is, the differential of $z (v-z)^2$ must be made = 0, which gives $z = \frac{1}{3} v$. Therefore $d\pi = \frac{nabvv}{270} (2r+a)$, and $r = \frac{270 d\pi - naabvv}{2 nabvv}$, and the velocity of the weight π will be = $\frac{2ndabvv}{810d\pi + 3nbaav} \times v$.

§ VI. We have seen that the calculus was much simplified by supposing one of the velocities constant for all points of the float-board. For this velocity being c , the effect of the whole float-board will be simply $\frac{n\beta}{60} (vc)^2 \frac{aa-ff}{2}$. It will therefore not be unuseful to inquire what this velocity c must be, that the effect of the float-board may be the same, as supposing, as we have hitherto done, a variable velocity, and proportional to the distances from the axle, we have only to make $\frac{n\beta}{60} \frac{(v-c)^2}{(v-c)^2} \left(\frac{aa-ff}{2} \right)$
 $= \frac{n\beta}{60} \left(\frac{1}{2} vv (aa-ff) - \frac{2}{3} vz \frac{a^3-f^3}{a} + \frac{1}{4} zz \frac{a^4-f^4}{aa} \right)$
 (§ IV.) but the equation whence we got the value of z

C c c 2
(§ IV.)

(§ IV.) $\frac{1}{2} v v - \frac{4}{3} \frac{v z}{a} \cdot \frac{a^3 - f^3}{a^2 - f^2} + \frac{1}{4} z z \cdot \frac{a a + f f}{a a} = 0$, we shall have $c = v - \sqrt{\frac{1}{2} v v - \frac{1}{4} z z \cdot \frac{a a + f f}{a a}}$.

If $f = 0$, we have $z z = 0,288 v v$ and $c = 0,345 v$

$f = \frac{1}{2} a$ $z z = 0,190 v v$ $c = 0,336 v$

$f = a$ $z z = \frac{1}{9} v v$ $c = \frac{1}{3} v$.

so that whatever be the relation of f to a , the velocity c is ever nearly $= \frac{1}{3} v$, and the more exactly so, as f is greater. Wherefore we may always assume $\frac{1}{27} n \beta v v \cdot (a a - f f) = d\pi$ for the effect of the stream upon a float-board whose plunged part is $a - f$; this effect will be increased in the ratio of 4 to 9, when the wheel has no motion, for making $c = 0$, we find it $= \frac{1}{12} n \beta v v \cdot (a a - f f)$.

§ VII. Hitherto we have all along supposed, that the float-board did through its whole plunged part receive the perpendicular impulse of the stream; but it is easily understood, that the wheel coming to turn, presents to the stream the plane of the float-board under an angle which is continually varying, which diminishes its effect every instant, as it removes from the vertical: This inconvenience may be remedied by multiplying the number of the float-boards, so that when the first is removed from the vertical as far as a certain point, the next may occupy that advantageous place, to be in its turn replaced some time after by a third, and so on. Now our third inquiry is, to assign the angle contained between two float-boards, or, which comes to the same, the number of float-boards the wheel should consist of, that its effect may be the greatest possible, being of no less importance than the preceding ones. To begin then with the most simple

simple case ; we will suppose the wheel immoveable, or that $c = 0$, and proceed to investigate, whether, supposing the number of float-boards to be greater, the sum of the effects will come out greater or less than what results from one single float-board placed vertically.

In order to a general solution of this question, we will suppose two float-boards CD and CE fig. 4. making any angles with the vertical, and let us compare the effect of the single float-board GD with the effect resulting from the float-boards FE and GD taken together, which will be reduced to FE and OD, because the part OG becomes useless, as the stream is intercepted by FE. Let $CB = CD = CE = a$, $CA = f$, $\cosin. BCD = m$, $\cosin. BCE = \mu$, which gives $CG = \frac{f}{m}$ $CF = \frac{f}{\mu}$ and $CO = \frac{\mu}{m} a$. Then we shall find, by § VI, the effect of GD $= \frac{n\beta vv}{120} (mmaa - ff)$ that of OD $= \frac{n\beta vv}{120} (mmaa - \mu\mu aa)$. and that of FE $= \frac{n\beta vv}{120} (\mu\mu aa - ff)$; whence it appears, that the sum of the two last is exactly equal to the first, which will ever hold good whatever be the value of f .

Whence arises the following theorem :

Whether the wheel be plunged quite up to the axle, or only in part so, provided it be immoveable, and that one of its float-boards be placed vertically, its effect will be constantly the same, whatever be the number of float-boards opposed to the stream, even though it were infinite.

The

The latter part of this theorem, though flowing from the general demonstration, may be also demonstrated, immediately, in the following manner; Let BP fig. 5. be

$= x$, we have $MO = \frac{adx}{x-x}$, and $CO = \frac{ax - dx - ax}{x - x}$;

$$\overline{CO}^2 = \frac{a^4 - 2a^3x + a^2x^2 - 2a^3dx + 2aaxdx}{aa - 2ax + xx} \text{ neglect-}$$

ing the dx^2) and $aa - \overline{CO}^2 = \frac{2aaxdx}{a-v}$; Therefore

the effect of the stream upon OM, which is $=$

$$\frac{n\beta vv}{120} \cdot (aa - \overline{CO}^2) \frac{\overline{CP}^2}{CP} \text{ will become } = \frac{n\beta vv}{120} (2ax -$$

$2x dx)$ whose integral is $= \frac{n\beta vv}{120} (2ax - xx)$ where-

in putting $x = a - f$, we have $\frac{n\beta vv}{120} (aa - ff)$ for the total effect of the stream upon the wheel, which is the same as that of a single float-board AB in a vertical position.

§ VIII. This theorem will also hold true for the case of § V. wherein we have supposed the height of the float-boards very small, in comparison of the radius of the wheel; we have seen that the effect of a single float-board placed vertically was $= nabvv (2r + a)$; the demonstration of the preceding § will be applicable here after the same manner, and will shew that whatever be the number of float-boards, the effect will be ever the same.

It does not however follow that the number of float-boards should be indifferent; for the wheel coming to turn the float-board, its lower part, which received the perpendicular impulse, will no longer

receive

receive it otherwise than obliquely, and the effect will diminish till the angle formed by two neighbouring float-boards be bisected exactly by the vertical, which will render the first entirely useless; after which the effect will increase anew, and will become again greatest, when the second float-board is got to the vertical; so that in order to fix upon the most advantageous number of float-boards, regard must be had to the sum of the different effects for all the situations of the float-boards during one whole turn of the wheel.

Whence it follows that in this case, wherein they are supposed very small, the greater their number is, the greater will be the sum total of the effects, since, if that number were infinite, there would be a float-board in a vertical position every instant.

§ IX. This will no longer hold good, if the height of the float-boards be more considerable, and it be found necessary to take the different velocity of their different points into consideration; by comparing (fig. 4.) the pressure on FE with that on the portion GO, they will be found no longer equal, as in the foregoing case; it is true that the same quantity of fluid acts on these two planes, and the disadvantage which FE has by receiving the impulse more obliquely, is exactly compensated, as before, by the length of the lever, but the difference arises from the different velocity of the corresponding points of FE and GO; those velocities are in the ratio of CF to CG, or as $\cos. ACG$ to $\cos. ACF$, which shews that the effect of FE is always less than that of GO, and consequently the effect must be diminished, by adding a greater number of float-boards;
the

the said effect will be greatest when there is only one float-board placed vertically, and least when their number is infinite: let us enquire what it will be in this latter case. We will suppose the same fig. 5, and the same denominations as in § VII. We had $CO = \frac{aa - ax - xdx}{a - x}$, we shall have (neglect-

ing dx^2 , dx^3 , and dx^4) $aa - \overline{CO}^2 = \frac{2axdx}{a-x}$,

$\frac{a^3 - \overline{CO}^3}{a}$, and $\frac{a^4 - \overline{CO}^4}{aa} = \frac{4axdx}{a-x}$. Now the pref-

ture on OM is, by § II, $= \frac{n\beta}{120} \frac{\overline{CO}^2}{aa} \left(vv (aa - \overline{CO}^2) - \frac{4}{3} vx \cdot \frac{a^3 - \overline{CO}^3}{a} + \frac{1}{2} zx \cdot a^4 - \frac{\overline{CO}^4}{aa} \right)$

which (by putting for CO its value) will become $= \frac{n\beta}{120} (2axdx - 2xdx) (v - x^2)$ whose integral

$\frac{n\beta}{120} (2ax - xx) (v - x^2) = \frac{n\beta (v - x^2)}{120} (aa - ff)$

(making $x = a - f$) will express the effect resulting from an infinite number of float-boards: this least effect will be to the greatest, that is when there is but one float-board, as $(v - x^2) : vv - \frac{4}{3} vx \cdot \frac{a^3 - f^3}{aa - ff} + \frac{1}{2} \frac{zx}{aa} \cdot (aa + ff)$ or as $(v - x^2)$

$: \frac{1}{2} vv - \frac{1}{4} zx \cdot \left(\frac{aa + ff}{aa} \right)$ §. VI.

This ratio will be that of 1 : 2

1 : 1,485

1 : 1

if $f = 0$

$f = \frac{1}{2} a$

$f = a$

§ X. If

§ X. If we take nothing but the most advantageous position into consideration, and preserve the greatest effect entire, it follows that the angle BCD (fig. 6.) between two float-boards must be such that E should enter the water at the instant when AB quits the vertical, so that the cosine of that angle be $= \frac{f}{a}$; in consequence of which the following table may be constructed, shewing what the number of float-boards should be for a given ratio between f and a .

For 4 float-boards, we have $f = 0$,

5	0,3090a
6	0,5000a
7	0,6236a
8	0,7071a
9	0,7660a
10	0,8090a
12	0,8669a
14	0,9009a
16	0,9239a
18	0,9397a
20	0,9510a
∞.	∞.

§ XI. Certain authors treating of hydraulics, have in this part thereof given the same table, as containing the true number of float-boards the wheel should consist of: but we have seen upon what principle it was formed, and that it was only to preserve entirely the effect of the vertical float-board; from whence it follows not that the number of float-boards which it assigns should be the most advanta-

geous. To which purpose the effect produced from every position of the wheel, and for the different number of the float-boards, should be computed; the number which gives the arithmetical mean between all these effects, the greatest of all, will be that to be chosen, and preferred before what the above table indicates.

It may be sufficiently satisfactory to compute only the effect from 1 to 10 degrees. Thus, for example, for the wheel entirely plunged, we are to find the effect (fig. 7.)

1° on OA, 2° on OI and gb , 3° on OH, and fc , 4° on OG and bd , 5° on OF, and pe , 6° on OE, 7° on OD, 8° on OC, and 9° on OB.

After which the wheel returns into the same position it had at first; and we are to divide the sum of all these effects by 9, to get the arithmetical mean.

We will next suppose the number of six float-boards for the same case of $f=0$, and compute the following effects.

1° on OG + aA , 3° on OE + nc , 5° on OI + γC
2° OF + mb , 4° OD 6° OH + βB .

The sum of all these effects divided by 6 will give the effect of the wheel of 6 float-boards.

The same thing, supposing the angle 40 degrees, or 9 float-boards, and as after a revolution of these 40 degrees, the wheel returns into a similar position, the same must be divided by 4.

Then for an angle of 30 degrees we are to divide by 3, and so on.

I have made this computation to great exactness, for the case of $f = 0$, $f = \frac{1}{2}a$, and $f = a$, $866a = a \cos. 30^\circ$; the result,

1° If $f = 0$.
for 4 float-boards, the arith. mean $= 0.335$
($\frac{1}{3} n \beta a a v v$).

It may be observed in this first case, that there is some advantage in taking 6 float-boards instead of 4 shewn by the table; the effect will be increased in the ratio of 100 to 118, and yet will be more than about $\frac{18}{100}$ of the greatest effect above calculated for a single vertical float-board; so that the found dimensions must be a small matter altered, and the quantity $\beta a a$ increased by $\frac{1}{10}$.

for 6	$= 0,396$ (Gr.
9	$= 0,336$ (Gr.
12	$= 0,323$ (Gr.
18	$= 0,295$ (Gr.
an infinite number	$= 0,214$ (Gr.

2° If $f = \frac{1}{2}a$.

for 6 flo.	$0,277$ (Gr.
7	$0,281$ (Gr.
9	$0,285$ (Gr.
12	$0,284$ (Gr.
18	$0,276$ (Gr.
an infinite number	$0,238$ (Gr.

In this second case 9 float-boards are to be taken instead of 6 shewn by the table, though the difference will be but very small; and we shall have an effect which will be $\frac{86}{100}$ of that of a vertical float-board, and in that ratio that the quantity $\beta a a$ found by the above formulæ, must be increased.

3° If $f = 0,866 a$

for 12 fl.

$= 0,099$ (‰c.

18

$= 0,099$ (‰c.

36

$= 0,104$ (‰c.

an infinite number

$= 0,103$ (‰c.

In this third case the difference is still very small, and the effect resulting from 36 float-boards will be $\frac{0,3}{100}$ of the effect of a single vertical float-board.