Received March 8, 1770.

XXIV. *Obſervations on the proper Method of calculating the Values of Reverſions depending on Survivorſhips: By* Richard Price, *D. D. F. R. S.*

Read May 10, 1770.

ALL queſtions relating to the values of lives and reverſions, are at preſent of particular importance in this kingdom. Much buſineſs is continually tranſacted in this way; and any conſiderable errors in the method of ſolving ſuch queſtions muſt in time produce very bad conſequences. The deſign of the following obſervations is, to point out a particular error, into which there is danger of falling, in finding the values of ſuch reverſions as depend on ſurvivorſhips. In doing this, I ſhall, in order to be as plain as poſſible, take the following caſe.

" A, aged 40, expects to come to the poſſeſſion " of an eſtate, ſhould he ſurvive B, aged likewiſe " 40. In theſe circumſtances, he offers, in order to " raiſe a preſent ſum, to give ſecurity for 40 *l. per ann.* " out of the eſtate *at his death*, provided he ſhould " get into poſſeſſion; that is, provided he ſhould ſur- " vive B. What is the ſum that ought now to be

" advanced

"advanced to him in consideration of such security, "reckoning compound interest at four *per cent.*?"

M. de Moivre's directions, in his treatise on annuities, prob. XVII. and XX. lead us to seek the required sum in this case, by the following process.

Find first, the present sum A should receive for the reversion of 40 *l. per ann.* for ever after his death, supposing it not dependent on his surviving B.—The present value of such a reversion, is the value of the life subtracted from the perpetuity *. The value of the life, taken from Mr. de Moivre's tables, is 13.2 years purchase. This subtracted from 25, the perpetuity, leaves 11.8, the value of the supposed estate after the life of A; which value therefore, is in money 472 *l.* But (as M. de Moivre observes), the lender having a chance to lose his money, a compensation ought to be made to him for the risk he runs, which is founded on the possibility that a man of the age of 40 may not survive another person of the same age. This chance is an equal chance; and therefore *half* the preceding sum, or 236 *l.* is the sum which should be advanced now on the expectation mentioned.

This solution carries a plausible appearance; and most persons will, probably, be ready to pronounce it right; nor will this be any wonder, as so great a master of these subjects as M. de Moivre, appears to have been misled by it. Nothing more is necessary to prove it to be fallacious, than proceeding in the same way to solve the following similar question.

"A, aged 40, offers to give security for 40 *l. per* "*ann.* to be entered upon at his death, provided it

* By Prob. XXVI. p. 293, of Mr. Simpson's Select Exercises.

"should

"should happen before the death of B, aged like-
"wise 40. What sum should now be advanced to
"him for such a reversion, interest being reckoned
"at 4 *per cent.*?"

In solving this problem, agreeably to the method just described; we are to find the value of 40*l. per ann.* to be entered upon *certainly* at the death of A, and then to multiply this value by the chance that A shall not survive B, or by $\frac{1}{2}$; and in this way the answer comes out the same as that already given. Now, it may be easily seen that this must be wrong. The value of a reversion to be received when a person of a given age dies, cannot be the same whether the condition of obtaining it is, that he shall die *before*, or that he shall die *after* another person; that is, whether it is provided that a purchaser, if he succeeds, shall get into possession sooner or later. The reversion in the latter case must, without doubt, be of less value than in the former.

The first question here proposed resolves itself into the following general question. "What is the pre-
"sent value of a given reversionary sum or estate, to
"be received after the failure of two lives, provided
"one in particular of them should be the longest
"life?"

Now, the present value of an estate to be enjoyed for ever after the failure of the longest of two lives, is the value of the longest of the two lives subtracted from the perpetuity. The value of the longest of two lives is, it is well known, the value of the two joint lives subtracted from the sum of the values of the two single lives. In the present case, therefore, it is 9.82 (the value of two joint

joint lives of 40 *l.* *) ſubtracted from twice 13.2, (the value of a ſingle life of 40) that is, 16.58 years purchaſe. And this ſubtracted from 25 (the perpetuity) gives 8.42, which multiplied by 40 gives 336.8 *l.* the value of the given eſtate, were it certainly to be enjoyed after the extinction of the longeſt of two lives both 40; that is, whether one or other of them failed laſt. But that A's life in particular ſhould fail laſt rather than B's, is an even chance. The true value of the reverſion, therefore, is $\frac{1}{2}$ the laſt value, or 168.4 *l.*

In like manner. The ſecond queſtion is the ſame with the queſtion. "What is the preſent value of "40 *l.* *per ann.* for ever, to be entered upon after the "extinction of two joint lives both 40; that is, "whenever *either* of them ſhall fail, provided the "firſt that fails ſhould happen to be A's life in par-"ticular?" And the anſwer is found by ſubtracting the preſent value of the two joint lives from the perpetuity, and multiplying the remainder by $\frac{1}{2}$, or the chance that A in particular ſhall die firſt; and this will give the required value, 303 *l.*

In ſhort, it appears in both theſe caſes, that, according to the firſt method of ſolution, we are to ſubtract from the perpetuity the value of one of the ſingle lives, when, in the *former* caſe, the value of

* The value of the two joint lives is here given on M. de Moivre's hypotheſis of an equal decrement of life; and it has been calculated by a rule in Mr. Simpſon's treatiſe on the Doctrine of Annuities and Reverſions, p. 16. M. de Moivre's rules, in the ſecond and third problems of his treatiſe on annuities, give the values of joint lives ſo much leſs than the truth, that they ought never to be uſed.

the *longest* of the two lives, and, in the *latter* case, the value of their *joint continuance*, ought in reality to be subtracted. I need not say what prodigious errors may often arise from hence, and how unfit such a method of solution is for practice. The Society in Nicholas-Lane, Lombard-street, for equitable assurances on lives and survivorships, have in constant practice such questions as those now stated; and, had they happened to have adopted this method of solution, they could not have continued long an advantage to the publick.

Mr. Simpson, in p. 322. of his Select Exercises, speaks on this subject in the following manner. "I "have been very particular on these kinds of problems; and the more so, as there has been no "method before published, that I know of, by which "they can be rightly determined. It is true, the "manner of proceeding by first finding the proba-"bility of survivorship (which method is used in my "former work, and which a celebrated author has "largely insisted on in three successive editions) may "be applied to good advantage when the given ages "are nearly equal; but then it is certain, that this is "not a genuine way of going to work, and that the "conclusions hence derived are at best but near ap-"proximations." This accurate and excellent mathematician has here expressed himself much too favourably, of the method of solution on which I have remarked. In both the cases I have specified, the ages are equal; and yet in one of them the error is a good deal above a third of the true value, and in the other a fifth: And it is obvious, that in cases where three equal lives are taken, the errors will be much greater.

greater. Mr. Simpſon's obſervations in this paſſage, are true only, when applied to a different method uſed by himſelf in the 28th and following problems of his Treatiſe on the Doctrine of Annuities and Reverſions. This method is exact when the lives are equal; but it gives reſults that are too far from the truth, when there is any conſiderable inequality between the lives.

It is with reluctance I have made ſome of theſe remarks. M. de Moivre has made very important improvements in this branch of ſcience, and the higheſt reſpect is due to his name and authority. This, however, only renders theſe remarks more neceſſary *.

* The ſtrict demonſtration of the ſolution I have given of the queſtions here ſtated, is as follows. It is plain that the purchaſer of A's right, as ſtated in the firſt queſtion, cannot get into poſſeſſion till the year when A and B ſhall be both dead, nor then, unleſs A happens to die laſt. Now, ſuppoſing the common complement of life n, the probability that A and B ſhall be both dead at the end of the firſt year, and A die laſt, is $\overline{1-\frac{n-1}{n}}\times\overline{1-\frac{n-1}{n}}\times\frac{1}{2}=\frac{1}{2}-\frac{n-1}{2n}-\frac{n-1}{2n}+\frac{\overline{n-1}|^2}{2n^2}$. In like manner, the probability that they ſhall be both dead at the end of the ſecond and third, &c. years, and A ſurvive, is $\frac{1}{2}-\frac{n-2}{2n}-\frac{n-2}{2n}+\frac{\overline{n-2}|^2}{2n^2}$, $\frac{1}{2}-\frac{n-3}{2n}-\frac{n-3}{2n}+\frac{\overline{n-3}|^2}{2n^2}$, &c. The preſent value, therefore, of the 1ſt, 2d, and 3d, &c. rents of the reverſionary eſtate, is $\frac{1}{2r}-\frac{n-1}{2nr}-\frac{n-1}{2nr}+\frac{\overline{n-1}|^2}{2nr^2r}$, $\frac{1}{2r^2}-\frac{n-2}{2nr^2}-\frac{n-2}{2nr^2}+\frac{\overline{n-2}|^2}{2n^2r^2}$, $\frac{1}{2r^3}-\frac{n-3}{2nr^3}-\frac{n-3}{2nr^3}+\frac{\overline{n-3}|^2}{2n^2r^3}$, &c. r ſignifying 1 *l.* increaſed by its intereſt for a year; and the eſtate being 1 *l. per annum*. And the ſum of theſe terms is the

In the practice of aſſurances on the ſurvivorſhip of one life beyond another, there is frequent occaſion for making ſuch aſſurances, for a term of years only. The method of determining the value in ſuch caſes I have not ſeen any where deſcribed. Perhaps, therefore, a brief account of it here may be of ſome uſe.

By reaſoning in the ſame manner with Mr. Simpſon, p. 322, Select Exerciſes, it will appear, that the value of any given ſum S aſſured, for n years on a given life A, provided in that time another given life B ſhould ſurvive A, is $\frac{S}{a}\times\overline{\frac{b-1}{r}+\frac{b-2}{r^2}+\frac{b-3}{r^3}}$, &c. (continued to n terms) $+\ \frac{S}{a}\times\frac{1}{2br}+\frac{1}{2br^2}+\frac{1}{2br^3}$, &c. (continued likewiſe to n terms) r denoting 1 *l.* increaſed by its intereſt for one year; and a and b, the numbers in the table of obſervations alive at the ages of A and B, divided by the quotient ariſing from dividing the ſum of the differences in the table from theſe

value required. But $\frac{1}{2r}+\frac{1}{2r^2}+\frac{1}{2r^3}$, &c. is half the perpetuity; and $\frac{n-1}{2nr}+\frac{n-1}{2nr}-\frac{\overline{n-1}|^2}{2n^2r}+\frac{n-2}{2nr^2}+\frac{n-2}{2nr^2}-\frac{\overline{n-2}|^2}{2n^2r^2}$ $+\frac{n-3}{2nr^3}+\frac{n-3}{2nr^3}-\frac{\overline{n-3}|}{2n^2r^3}$, &c. is half the value of the joint lives ſubtracted from half the ſum of the values of the two ſingle lives; that is, half the value of the longeſt of the two lives by problem IV. of M. de Moivre's Treatiſe on Life Annuities. A ſimilar demonſtration may be eaſily applied to the ſolution of the other queſtion.

The beſt rules for finding, in all caſes, the values of reverſions depending on ſurvivorſhips, are Mr. Simpſon's, in his Select Exerciſes, p. 297, &c.

ages

ages respectively for n years, by n. Now $\frac{b-1}{br}+\frac{b-2}{br^2}$, &c. (continued to n terms) is the value of an annuity on the given life B for n years. And $\frac{1}{2br}+\frac{1}{2br^2}$, &c. (continued to n terms) is the value of an annuity certain for n years, divided by $2b$.

The general rule, therefore, is this.

"Find (by problem 23. in M. de Moivre's "Treatise on Annuities, fourth edition) the value of "an annuity on the life of B for n years. To this "value add the quotient arising from dividing by $2b$, "the value of an annuity certain for n years, taken "out of M. de Moivre's tables in the treatise just "mentioned, or out of table III. in Mr. Simpson's "Select Exercises: and the sum multiplied by $\frac{S}{a}$ "will be the required value."

Example. Let the rate of interest be 3 *per cent.* or $r=1.03$.

The table of observations, Mr. Simpson's in his Select Exercises, p. 254.

Let the age of A be seven years. B, 30. $n=14$ and $S=100\,l.$

The sum of the decrements in the table for fourteen years from seven years of age, is 73; which divided by 14 is 5.2. The number alive at seven is 430; and this, divided by 5.2, gives 82.6 for the value of a. In like manner the value of b may be found to be 41.7. The value of an annuity for 14 years on a life of 30, is 9.5. The value of an annuity certain for 14 years is 11.296, which divided

 by

by $2b$ or 83.4, gives 0.13; and this added to 9.5, and the ſum multiplied by $\frac{100}{82.6}$ gives 11.66, or 11 *l.* 13 *s.* for the value in preſent payment of 100 *l.* aſſured to a perſon 30 years of age, and payable to him at the death of a child ſeven years of age, provided that ſhould happen before his own death in 14 years.

It deſerves to be particularly remarked, that in this method likewiſe may be determined, what ſums ought to be paid on any ſurvivorſhip, within a given term of years, of one life beyond another, in conſideration of any given ſums now advanced.

The following example of this is a caſe which has offered itſelf in practice, and which I have had occaſion particularly to conſider.

"A perſon aged 30, having in expectation an
" eſtate which is to come to him, provided he ſur-
" vives a minor aged ſeven before he comes of
" age, wants in theſe circumſtances to raiſe
" 1000 *l.* What reverſion, depending on ſuch a
" ſurvivorſhip, is a proper equivalent for this ſum
" now advanced, intereſt being reckoned at 3 *per cent.*
" and the probabilities of life being ſuppoſed the ſame
" with thoſe in the London table of obſervations?"

Anſwer. It appears from what has been juſt determined, that, for 11 *l.* 13 *s.* now advanced, the proper equivalent in theſe circumſtances is 100 *l.* to be paid, in caſe the ſuppoſed ſurvivorſhip ſhould take place. By the rule of proportion, therefore, it will appear, that for 1000 *l.* the proper equivalent is 8576 *l.*

XXV. *De*