

L. *A Supplement to a former Paper, concerning Difficulties in the Newtonian Theory of Light: by the Rev. S. Horley, L L. B. F. R. S.*

PROBLEM I.

Read Dec. 19, 1771. *A PARCEL of equal Circles being disposed upon a plane surface, of any figure whatsoever, in the closest arrangement possible, to determine the ultimate proportion of the space covered by all the Circles, to the space occupied by all their Interstices, when each circle is infinitely small, and the space, over which they are disposed, is of a finite magnitude.*

The closest manner, in which a parcel of equal circles can be disposed upon a plane, is when the centers of every three contiguous circles are situated at the angles of an equilateral triangle, which hath each of its sides equal to a diameter of any one of the circles.

A number of circles, thus disposed, may be divided, as TAB. XVII. shews, into several rows of circles, having their centers ranged upon parallel right lines, A G, H P, Q X, R Z, &c. Every
4 A 2 circle,

circle, which is not in an outermost row, or at the extremity of any other row, touches six others, namely two in its own row, and two in the row on either side of its own: and each adjacent pair of these six do also touch each other. In the outer rows, every circle, which is not at one extremity of its row, touches four others, two in its own row, and two in the row next beside it; which last two do likewise touch each other. A circle at either extremity of an outer row, touches only a single circle in its own row, but either one or two in the row next beside it. The bare inspection of the figure (TAB. XVII.) will make these assertions manifest.

Now, imagine the equal circles, exhibited in the figure, to be each infinitely small, the number of them being infinitely great, and the whole space over which they are disposed being of a finite magnitude. The ultimate proportion of the space covered by all the Circles, to the space occupied by all their Interstices, is that of $\frac{3}{4}$ the area of one of the circles to the whole of one interstitial area, *i. e.* the proportion of 39 to 4 very nearly.

DEMONSTRATION.

The circles ranged along the parallel right lines AB, HP, form two rows of interstices; the row marked *a, b, c, d, &c.* and the row marked $\alpha, \beta, \gamma, \delta, \&c.$ and, in like manner, two rows of interstices are formed by every two contiguous rows of circles.

Now, the numbers of the circles ranged along the several parallel right lines, AG, HP, QX, &c. are either equal or unequal, according to the figure of the space over which they are disposed.

Case 1. First suppose, that an equal number of circles is ranged along each of the parallel lines; in which case, the figure, in which they are included, must be a parallelogram. The number of circles, ranged along the parallel right lines AG, HP, being equal, the number of interstices in each of the rows, $a, b, c, d, \&c.$ $\alpha, \beta, \gamma, \delta, \&c.$ is less by unity than the number of circles upon either line, AG, or HP, be that number what it will. Thus the two circles A, B, upon the line AG, with the two circles H, K, upon the line HP, have the single interstice a , in the row $a, b, c, d, \&c.$ and the single interstice α , in the row $\alpha, \beta, \gamma, \delta, \&c.$ Again, the three circles A, B, C, upon the line AG, with the three, H, K, L, upon the line HP, have the two interstices a, b , in the row $a, b, c, d, \&c.$ and the two α, β , in the row $\alpha, \beta, \gamma, \delta, \&c.$ And universally, if the number of circles in each row be m , the number of interstices, in each of the two rows of interstices, will be $m - 1$. Consequently, the whole number of interstices formed by these two rows of circles is $2m - 2$. In like manner, the two rows of circles HP, QX, form two more rows of interstices. And the number of circles upon each line, HP, QX, being m , the number of interstices in each row is $m - 1$, and the whole number in both rows $2m - 2$. Therefore, the whole number of interstices formed by the three rows of circles, AG, HP, QX, is $2m - 2$ twice taken, or $2m - 2 \times 2$. By the same reasoning if a fourth row of m circles, $\Gamma \Sigma$ be added, the number of interstices formed by the four rows is $2m - 2 \times 3$. And universally, if there be n rows of

at

of equal circles, and m circles in each row, the number of interstices formed by all the rows is $2m - 2 \times n - 1$. Now, when the circles are infinitely small, their diameters are infinitely small. Therefore, the space which they cover being of finite magnitude, it is necessary, that both the number of circles in each row, and the number of rows, that is, that each of the numbers, m and n , should be infinitely great. But when m and n are each infinitely great, $2m - 2 \times n - 1$, that is, the number of interstices, becomes ultimately $2mn$; and the interstices being all equal one to another, if the area of one be called P , the sum of their areas will be $2mn \times P$. But the number of circles in n rows, each row consisting of m circles, is mn ; and the circles being equal, if the area of one be called A , the sum of their areas will be $mn \times A$. Hence the space covered by all the circles is to the space covered by all their interstices, when the magnitude of each circle is infinitely diminished, and the number or them so infinitely augmented, as that they shall cover a space of finite magnitude, ultimately, as $mn \times A$ to $2mn \times P$, that is, as A to $2P$, or as $\frac{1}{2} A$ to P , that is, as $\frac{1}{2}$ the area of one circle to the whole area of one interstice.

Case 2. Now, suppose, that unequal numbers of circles are ranged along the several lines AG , HP , QX , &c. which must always be the case, if the figure of the space, in which they are contained, be any other than a parallelogram; and let the number upon AG be the greatest of all, and call that number, as before, m . If from the row HP , the extreme

treme circle P be taken away, all the rest being left, the interstice ζ will be taken away, and all the other interstices, formed by m circles upon HP, with m circles upon AG, will remain. If again the circle O be taken away, besides the interstice ζ already taken away, the two ϵ, f will disappear; and every circle more that is taken away, of those remaining upon HP, from the extremity of the line, two more interstices will disappear. If from the row of circles HP, the extreme circle H be taken away, the two interstices a, α , will disappear. And if the circles K, L, M, be taken away successively, every new circle that is taken away, two more interstices will disappear, of those formed by the two rows AG, HP. Again, if the two circles P and H be taken away, the three interstices ζ, a, α , will disappear; and every circle more that is taken away, from either extremity, two more interstices will disappear. Hence whatever number of circles be taken away out of m circles upon HP, provided they be taken successively, from either or both ends of the row (and when the number of circles upon HP is supposed less than that upon AG, the deficiency must be at the end, not in the middle of the row, otherwise the circles remaining would not be in the closest arrangement), it is evident that the number of interstices which disappear, of those which would be formed by m circles upon HP, with m circles upon AG, must be either double the number of circles taken away, or less than the double of that number by 1. That is, if $m - a$ be the number of circles left upon HP, the number of interstices formed by them, with m circles upon AG, is less than the number which would be

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formed

formed by m circles upon HP, with m circles upon AG, either by $2a$, or by $2a - 1$. The number of interstices formed by m circles upon each row would be, as hath been shewn in the preceding case, $2m - 2$. Therefore, the number formed by m circles upon AG, with $m - a$ circles upon HP, is either $2m - 2 - 2a$, or $2m - 2a - 1$. That is, ultimately (when the number $m - a$ is infinitely increased) $2m - 2a$. Now, suppose the number of circles upon QX to be $m - a - b$. The number of circles upon the two rows AG, HP, is $2m - a$. Upon the three rows AG, HP, QX, the number is $3m - 2a - b$. And if the number of circles upon $\Gamma\Sigma$ be $m - a - b - c$, the number of circles upon the four rows AG, HP, QX, $\Gamma\Sigma$, will be $4m - 3a - 2b - c$. And, universally, the number of rows being n , and the number of circles upon the several rows, m , $m - a$, $m - a - b$, $m - a - b - c$, $m - a - b - c - d$, &c. successively, the whole number upon all the n rows will be

$$nm - a \times n - 1 - b \times n - 2 - c \times n - 3, \text{ \&c.}$$

But, as it hath been shewn that m circles upon AG, with $m - a$ circles upon HP, form $2m - 2a$ interstices, if the number $m - a$ be infinite, in the same manner it may be shewn, that $m - a$ circles upon HP, with $m - a - b$ circles upon QX, form $2m - 2a - 2b$ interstices, when the number $m - a - b$ is infinite. Therefore, the whole number of interstices formed by the three rows upon AG, HP, QX, is $2m - 2a \times 2 - 2b$. And, in like manner, the number of interstices, formed by the circles of four rows, will be

$$2m - 2a \times 3 - 2b \times 2 - 2c. \text{ And, universally,}$$

n being

n being the number of rows, the number of the interstices will be

$$2m - 2a \times n - 1 - 2b \times n - 2 - 2c \times n - 3, \&c. \text{ That}$$

is, $2m \times n - 1 - 2a \times n - 1 - 2b \times n - 2 - 2c \times n - 3, \&c.$ By comparing this expression with the former expression of the number of the circles, it will appear, that when n , the number of the rows of circles, is infinitely augmented, the number of interstices is to the number of circles, ultimately, as 2 to 1. For the two expressions always consist of an equal numbers of terms. The same numerical terms in both are affected with the same signs. The first term of

the latter ($2m \times n - 1$) is ultimately double the first term of the former (mn), when n is infinitely increased, and each succeeding term of the latter is double the corresponding term of the former. Therefore, the whole of the latter expression is ultimately to the whole of the former, as 2 to 1. That is, the number of interstices is ultimately double the number of circles: whence it follows, as in the former case, that the whole space covered by the circles is to the whole space occupied by the interstices, as $\frac{1}{2}$ the area of one circle to the whole area of one interstice.

In this Demonstration I have supposed the number of circles upon the several lines AG, HP, QX, &c. to decrease continually. Had I supposed them to decrease by fits, and in any manner imaginable, still the conclusion would have been the same (a). There-

(a) Suppose the number of circles upon the 1st row to be m , upon the 2d, $m - a$, upon the 3d, $m - a + b$, upon the 4th, $m - a + b - c$, upon the 5th, $m - a + b - c + d$, and so on, and

fore, let the figure of the finite space, including the circles thus closely arranged, with their interstices, be what it will, the proportion of the space covered by all the circles, to the space taken up in interstice, is ultimately that of $\frac{1}{2}$ the area of one circle to the whole area of one interstice.

Now, that this is the proportion of 39 to 4, very nearly, will appear by computing one of the interstitial areas.

The method of computing the interstitial area is obvious. Let A, B, H be the centers of the three circles, which close the interstice $\gamma\phi\psi$. Join AB, AH, BH. The right lines AB, AH, BH, pass through the points of contact γ , ϕ , ψ , respectively.

and each of these numbers to be infinitely increased. Then, n being the number of rows, the whole number of circles will be $nm - a \times n - 1 + b \times n - 2 - c \times n - 3 + d \times n - 4$, &c. Number the interstices formed by every two contiguous rows, and add them all together, and the whole number of interstices will be found to be

$2m \times n - 1 - 2a \times n - 1 + 2b \times n - 3 - 2c \times n - 3 + 2d \times n - 5$, &c. Now, by comparing these two expressions, it appears, that both consist of the same number of terms: That the same numerical terms in order from the first, have the same signs:

That the first term of the latter ($2m \times n - 1$) is ultimately the double of the first term of the former, when n is infinitely increased: That of the terms following the first, the negative terms of the latter are each double the corresponding negative terms of the former: and each positive term of the latter differs from the double of the corresponding positive term of the former, by a number which vanishes with respect to either of those corresponding terms, when n becomes infinite. Therefore, when n becomes infinite, the whole of the latter expression becomes the double of the whole of the former. Hence the conclusion is as before.

The

The area of the triangle, AHB, is equal to the areas of the three sectors A Γ Φ , B Φ Ψ , H Ψ Υ , added to the interstitial area $\Upsilon\Phi\Psi$. But the triangle AHB is equilateral. Therefore each of the sectors A Γ Φ , B Φ Ψ , H Ψ Υ is $\frac{1}{6}$ of the circle to which it belongs: and, the circles being equal, the three sectors are equal to the half of any one of the circles. Therefore, the area of the triangle AHB is equal to $\frac{1}{2}$ the area of one circle (as of A) added to the interstitial area $\Upsilon\Phi\Psi$. Therefore, from the area of the triangle AHB take $\frac{1}{2}$ the area of the circle A, and there will remain the interstitial area $\Upsilon\Phi\Psi$.

Now, if the radius A Φ be put = 1, each side of the triangle AHB will be 2.

Therefore, the area of the triangle AHB	}	= 1,73205
will be		
But the radius being 1, $\frac{1}{2}$ the area of the	}	= 1,5708
circle A is		

The difference is 0,1612

And this is the interstitial area $\Upsilon\Phi\Psi$, the half area of the circle A being 1,5708. Therefore, the semi-circle is to the interstice as 1,5708 to 0,1612, or as 9,74 to 1, or as 39 to 4, very nearly.

C O R O L L A R Y.

If a parcel of equal circles be so disposed upon a plane surface of any figure whatsoever, that the centers of every three adjacent circles are situated at the angles of equal equilateral triangles, having sides greater than the diameters of the circles, but greater in a finite proportion, the ultimate proportion of the space

4 B 2 covered

covered by all the circles to the space occupied by all the interstices, when each circle is infinitely diminished, and the number of them so infinitely increased, that the space over which they spread is of a finite magnitude, is that of $\frac{1}{2}$ the area of one circle to the whole area of one interstice. And the area of any one interstice is equal to the difference of the area of the equilateral triangle, formed by the right lines joining three adjacent centers, and $\frac{1}{2}$ the area of one of the circles.

PROBLEM II.

To determine the greatest possible density of an infinitely thin crust composed of equal spherules, having their centers all in the same plane.

From the number 39 subtract its third part. To the number 4 add the third part of 39. The remainder is to the sum, that is, 26 is to 17, very nearly, as the space occupied by all the matter to the space occupied by all the pore, in an infinitely thin crust, of the greatest possible density, composed of equal spherules, having all their centers in the same plane.

DEMONSTRATION.

Upon a base of innumerable infinitely small circles, arranged in the closest manner possible, (according to Prob. I.) imagine right cylinders to be erected, each cylinder having one of the little circles for its base, and its altitude equal to the diameter of its base.

These

These cylinders are in the closest arrangement possible for equal cylinders; and the spheres, which they circumscribe, are in the closest arrangement possible for equal spheres, which have their centers in the same plane. The solid space occupied by the cylinders, is to the solid space occupied by their interstices, as the surface covered by their circular bases, to the surface covered by the interstices of their bases: That is, as 39 to 4, very nearly, by the first Problem. But the spheres contained within these cylinders are each but $\frac{2}{3}$ of the containing cylinder. The solid content therefore of all the spheres is but $\frac{2}{3}$ of the solid content of all the cylinders; and the remaining third part of the solid content of the cylinders, together with the interstices between the cylinders, makes up the whole of the interstices between the spheres. Therefore, the space occupied by the spheres is to the space occupied by their interstices, as $39 - \frac{29}{3}$ to $4 + \frac{29}{3}$, or as 26 to 17, very nearly.

The spheres being in the closest arrangement possible, if each be a solid atom, or without pore within its own dimensions, then, the infinitely thin crust, which these atoms compose, is plainly the most dense that can be composed of equal spherules, having their centers in one plane. And the space occupied by its matter is to the space occupied by its pore, as 26 to 17, very nearly.

SCHOLIUM.

If the component spherules, instead of being solid, be supposed to be each of the density of gold, in which one half of the bulk may reasonably be supposed to be

be pore, then only $\frac{1}{2}$ of the space, which they occupy, is filled with matter, and the other half is to be added to the pore. Hence spherules of the density of gold, arranged in the closest manner possible, having their centers in one plane, compose a crust, in which, $\frac{1}{4}\frac{3}{3}$ ds, or somewhat more than $\frac{1}{4}\frac{2}{6}$ ths, of its bulk is matter. Therefore, the density of such a crust is somewhat greater than 12 times that of water, since $\frac{1}{4}\frac{1}{6}$ th only of the bulk of water is supposed to be matter, and $\frac{3}{4}\frac{2}{6}$ ths is pore.

S. Horsley.

The first of these two Problems, enabled me to determine the greatest possible number of spherical particles of a given magnitude, that could find room to lie at one time upon the surface of the Sun; and, by the second, I found the density of the crust, which such particles, in the closest arrangement possible, with a given density of each particle separately, would compose.





