XXIX. *De Polygonis Areâ vel Perimetro maximis et minimis, inſcriptis Circulo, vel Circulum circumſcribentibus. Auctore* S. Horſley, *LL. D. R. S. Sec.*

Redde, May 19, 1775.

THEOREMA I. (TAB. IX.)

Si linea recta arcum circularem contingentibus duabus interceptum contingat, ſegmentum ejus, contingentibus primo poſitis interceptum, in contactûs ſui puncto vel æqualiter vel inæqualiter diviſum eſt, prout arcus ipſe circularis æqualiter vel inæqualiter in eodem puncto diviſus eſt. Segmentaque arcûs (inæqualiter ſcilicet diviſi) et rectæ contingentis majora et minora ab iiſdem ſunt partibus mutui contactûs.

ARCUM circularem AEC, contingentibus duabus, AB, CD, interceptum, recta BD in puncto E contingat. Recta vero BD contingentibus primo poſitis, AB, CD, in punctis B, D, occurrat. Dico rectam BD in puncto E vel æqualiter vel inæqualiter diviſam, prout arcus AEC æqualiter vel inæqualiter in eodem puncto E diviſus eſt. Primo puta arcum AC in puncto E æqualiter diviſum. Dico igitur et rectam BD, in puncto E, bifariam ſecari. Circuli AEC centrum eſto F. Jungantur FE, FB, FD, quarum

rum FB, FD, circuli peripheriæ in punctis G, H, occurrant. Rectæ FB, FD arcus AE, CE bifariam dividunt. Arcus igitur GE, HE, arcuum AE, CE semisses. Æquales igitur GE, EH, propter AE, EC, ex hypothesi, æquales. Quare anguli BFE, DFE æquales. Æquales autem anguli BEF, DEF: rectus enim uterque. In triangulis igitur BFE, DFE, quæ latus commune habent EF, duo anguli BEF, BFE, duobus DEF, DFE, singuli singulis æquales. Reliqua igitur reliquis æqualia (per El. I. 26.). Quare BE = ED. *Q. E. D.*

Jam vero puta arcum AC inæqualiter in E divisum, et segmentorum AE, CE, majus esse AE, CE minus. (fig. 2.) Dico rectam BD inæqualiter in puncto E divisam, segmentumque BE majus esse, DE minus. Jungantur enim ut prius FB, FE, FD, quarum FB, FD peripheriæ in punctis G, H, occurrant. Rectæ FB, FD, arcus AE, CE, bifariam dividunt. Arcus igitur GE, HE, arcuum AE, CE, semisses. Cum igitur arcus AE, arcu CE major sit, arcus GE arcu HE major erit. Angulus igitur BFE angulo DFE major. Fiat angulus EFK angulo DFE æqualis. Quoniam angulus KFE angulo DFE æqualis est, nec non angulus rectus KEF, recto DEF æqualis; in triangulis EFK, EFD, quæ latus EF commune habent, anguli duo KFE, KEF, duobus DFE, DEF, singuli singulis æquales. Reliqua igitur reliquis æqualia. Latera igitur EK, ED, æqualia. Propter angulum vero EFK angulo EFB minorem, punctum K punctis, B, E, necessario interjacet. Recta igitur BE, rectâ KE major. Major itaque quam ED. *Q. E. D.*

THEOREMA.

THEOREMA II.

Linea recta quæ arcum circularem contingentibus duabus interceptum in puncto medio contingit, et contingentibus primo positis hinc inde occurrit, minima est omnium quæ, eundem arcum contingentes, contingentibus primo positis intercipiuntur.

ARCUM circularem BED, contingentibus duabus AB, CD, interceptum recta AC in puncto E contingat, et contingentibus primo positis AB, CD, in punctis A, C, occurrat. Punctum E arcûs BED medium esto. Dico rectam AC omnium minimam, quæ, arcum BED contingentes, contingentibus AB, CD, intercipiuntur. Sumatur enim in arcu BED punctum quodlibet F, et ducatur recta GH quæ circulum in F contingat. Recta vero GH contingentibus AB, CD, in punctis G, H, occurrat. Dico rectam AC rectâ GH minorem. Si parallelæ sint contingentes AB, CD (fig. 1.) res manifesta est: parallelas enim AB, CD, recta GH oblique secat, recta autem AC normaliter. Rectæ vero AB, CD, non sint parallelæ. (fig. 2. et 3.) Si recta AC non sit minor quam GH, aut æqualis erit, aut major. Sit primo æqualis. Arcûs autem BD, inæqualiter in F divisi, segmentum majus esto BF. Rectæ igitur GH, inæqualiter in F divisæ, segmentum GF majus erit (per 1. hujus). Rectæ BA, DC, productæ occurrent; occursus esto K: rectarum vero GH, AC, occursus esto L. Junctâ BD, per H ducatur HM rectis BD, AC, parallela: et per G ducatur GN rectæ DK parallela, quæ rectæ AC in N occurrat.

 Rectæ

Rectæ GF, GB, quæ circulum in punctis B, F, contingentes in puncto G conveniunt, æquales sunt. Pari ratione AE, AB, æquales. Recta igitur GF duabus AE, AG, simul sumptis æqualis est. Rursum rectæ HF, HD, quæ circulum in punctis F, D, contingentes in puncto H conveniunt, æquales sunt. Pari ratione CD, CE, æquales. Recta igitur CD, vel CE, duabus FH, HC, simul sumptis æqualis. Tres igitur GF, FH, HC, simul sumptæ, tribus AG, AE, EC, simul sumptis æquales; id est, duæ GH, HC, simul sumptæ duabus AG, AC, simul sumptis. Rectæ autem GH, AC, ex hypothesi, æquales. Ablatis igitur GH, AC, æqualibus, relinquuntur HC, AG, æquales. Propter rectas autem AC, MH, BD, parallelas, et triangulum BKD isosceles, æquales sunt CH, AM: quare AG, AM, æquales. Propter parallelas, autem AL, MH, rectæ GM, GH, in punctis A, L, similiter divisæ sunt: bifariam autem GM in A: quare et GH bifariam in L: et propter GN, CD, parallelas, CN bifariam in L. Cum CL igitur semissis sit rectæ CN, et CE semissis rectæ CA, erit EL rectæ AN semissis, sive AN rectæ EL dupla. Rectâ autem GH æqualiter in L divisâ, cum ejusdem rectæ, inæqualiter in F divisæ, segmentum GF majus est (per I. hujus), recta GF, rectis HF et duplæ FL simul sumptis æqualis est. Rectæ autem GF recta GB æqualis. Quare GB, sive duæ GM, MB, simul sumptæ, duabus, HF et duplæ FL, simul sumptis, sive duabus, HD et duplæ FL, simul sumptis, æquales. Et ablatis hinc inde MB, HD, æqualibus, relinquetur GM æqualis duplæ FL, id est duplæ EL, id est, ex prius ostensis, rectæ AN. Propter æquales autem GA, AM, recta GM rectæ GA dupla est: et propter parallelas GN,

KC, triangula AKC, AGN ſimilia: latera autem KA, KC æqualia: æqualia igitur GA, GN. Duæ igitur GA, GN, ſimul ſumptæ, duplæ GA æquales ſunt; id eſt, rectæ GM. Rectæ autem GM, AN oſtenſæ ſunt æquales. Duæ igitur GA, GN, ſimul ſumptæ, rectæ AN æquales ſunt: duo nempe trianguli latera ſimul ſumpta æqualia reliquo. Quod eſt abſurdum. Non ſunt igitur AC, GH, æquales. Dico neque majorem eſſe AC quam GH. Eſto enim major AC, ſiquidem eſſe poteſt. Duæ GH, HC ſimul ſumptæ duabus AG, AC ſimul ſumptis, ut prius, æquales ſunt. Cum igitur AC major ſit quam GH, erit AG minor quam HC. Æquales autem HC, AM, ut prius. Ergo AG minor erit quam AM; ac proinde, propter rectarum GM, GH, NC, ſimilem in punctis A, L, diviſionem, GL minor quam LH, et NL minor quam LC. Cum igitur CE ſemiſſis eſt rectæ CA, et CL major quam ſemiſſis rectæ CN, erit EL minor quam ſemiſſis rectæ AN: ſive AN duplâ EL major. Porro cum rectæ GH inæqualiter in L diviſæ, ſegmentum GL, ex oſtenſis, minus eſt, ejuſdem autem rectæ inæqualiter in F diviſæ ſegmentum GF majus, (per I. hujus) duæ, FH et dupla FL, ſimul ſumptæ rectâ GF majores ſunt, ſive æquali GB, ſive duabus GM, MB ſimul ſumptis. Et æqualibus FH, MB hinc inde ablatis, relinquetur dupla FL rectâ GM major. Æquales autem LF, LE. Quare dupla EL rectâ GM major: et AN, quæ duplâ EL jam oſtenſa eſt major, rectâ GM multo major erit. Propter GA minorem vero quam AM, recta GM duplâ AG major. Æquales autem ut prius GA, GN. Dupla igitur GA, duabus GA, GN ſimul ſumptis æqualis eſt. Recta igitur GM,

 duabus

duabus GA, GN ſimul ſumptis major. Et proinde recta AN, quæ oſtenſa eſt major quem GM, duabus GA, GN, ſimul ſumptis multo major. Trianguli igitur AGN latus AN duobus reliquis ſimul ſumptis majus. Quod eſt abſurdum. Recta igitur AC rectâ GH major non eſt. Sed nec æqualis. Minor igitur. Simili ratione et aliâ omni minor, quæ arcum BD contingens contingentibus primo poſitis AB, CD intercepta eſt. Omnium igitur minima. *Q. E. D.*

THEOREMA III.

Polygonorum omnium, lateribus numero datis, datum circulum circumſcribentium, æquiangulum perimetro minimum eſt.

CIRCULUM ABCD circumſcriptum puta polygono, quot volueris laterum, FGHKE, quod omnium quæ, æquali laterum numero, circa eundem circulum circumſcribi poſſunt, perimetrum minimam habeat. Dico polygonum FGHKE æquiangulum eſſe. Nam ſi æquiangulum non ſit, neceſſe eſt ut duos aliquos angulos proximos inæquales habeat: nam ſi nullos proximos, omnino nullos; ſed æquiangulum erit. Sunto igitur inæquales proximi duo anguli GFE, KEF. Latera vero GF, KE, quæ cum intermedio FE angulos illos complexa ſunt, circulum in A, D punctis contingant: et latus intermedium FE eundem in L contingat. Circuli centrum eſto O. Jungantur OA, OL, OD. Anguli AFL, AOL ſimul ſumpti duobus rectis æquales ſunt, propter angulos ad A, L rectos. Similiter anguli

anguli DEL, DOL ſimul ſumpti duobus rectis æquales; propter angulos ad D et L rectos. Duo igitur AFL, AOL, ſimul ſumpti, duobus DEL, DOL ſimul ſumptis æquales. Inæqualibus igitur AFL, DEL hinc inde ablatis, relinquuntur AOL, DOL inæquales. Arcus igitur AL, LD inæquales. Bifariam igitur ſecetur arcus ALD in M puncto, quod neceſſario à puncto L diverſum erit. Per M ducatur recta quæ arcum AD contingat. Contingens per M contingentibus AF, DE in punctis N, P, occurrat. Recta NP rectâ FE minor erit (per præc). Quare et dupla rectæ NP duplâ rectæ FE minor. Sed propter NM=NA et PM=PD, tres rectæ AN, DP, PN, ſimul ſumptæ, duplæ rectæ NP æquales ſunt. Et propter FA=FL, et DE=EL, tres AF, DE, FE, ſimul ſumptæ, duplæ rectæ FE æquales ſunt. Tres igitur AN, DP, PN ſimul ſumptæ tribus AF, DE, FE, ſimul ſumptis minores. Polygonum autem NGHKP, circulum ABCD circumſcribit, et latera numero totidem habet ac polygonum FGHKE; quod omnium quæ æquali laterum numero, circulum ABCD circumſcribunt, perimetro, ex hypotheſi, minimum eſt. Perimeter igitur FGHKEF perimetro NGHKPN minor. Utrique auferatur pars communis AGHKD: relinquentur AF, DE, FE, reliquis AN, DP, PN, ſimul ſumptæ ſimul ſumptis, minores. Aſt majores jam oſtenſæ ſunt. Simul igitur majores et minores. Quod eſt abſurdum. Non ſunt igitur anguli GFE, KEF, inæquales, exiſtente perimetro FGHKEF minimâ. Similiter oſtendetur, nec alios duos quoſvis angulos proximos polygonum FGHKE inæquales habere.

tur

Nullos igitur proximos inæquales habet. Omnino igitur nullos. Omnes igitur æquales. Æquiangulum igitur. *Q. E. D.*

THEOREMA IV.

Polygonorum omnium, lateribus numero datis, datum circulum circumſcribentium, æquiangulum areâ minimum eſt.

PATET ex præcedente, cum area æqualis eſt recangulo ſub ſemiperimetro et circuli inſcripti ſemidiametro.

THEOREMA V.

Polygonorum omnium, lateribus numero datis, dato circulo inſcriptorum, æquilaterum perimetro maximum eſt.

CIRCULO ABCD inſcriptum puta polygonum ABCDE, quot volueris laterum, quod omnium, quæ, æquali laterum numero, eidem circulo inſcribi poſſunt, perimetrum maximam habeat. Dico polygonum ABCDE æquilaterum. Non enim. Duo igitur proxima quædam latera inæqualia habet: nam ſi nulla proxima, omnino nulla; ſed æquilaterum erit; quod negaſti. Inæqualia ſunto latera proxima AB, AE. Propter rectas AB, AE inæquales, arcus AB, AE inæquales erunt. Bifariam igitur ſecetur arcus

BAE in puncto G, quod à puncto A diverſum erit. Jungantur BG, EG. Circuli centrum eſto F: juncta GF peripheriæ iterum in H occurrat. Denique jungantur BE, HA. Arcubus GB, GE æqualibus, ſemicirculis GBH, GEH, ablatis, relinquuntur æquales BH, EH. Quare et anguli BGH, EGH, nec non BAH, EAH, æquales. Angulos igitur BGE, BAE, qui in eodem ſunt ſegmento circulari, rectæ GH, AH, bifariam dividunt. Duæ igitur GB, GE, ſimul ſumptæ ad rectam GH eandem proportionem habent, ac duæ BA, AE, ſimul ſumptæ, ad rectam AH. (*Vide* Demonſtrationem prop. 94. *Datorum* EUCLIDIS) In circulo autem ABCD, diameter GH, rectâ AH, major eſt. Duæ igitur GB, GE, ſimul ſumptæ, duabus AB, AE, ſimul ſumptis, majores. (per xiv. 5. Elem.). Polygonum autem GBCDE circulo ABCD inſcriptum eſt, et latera numero totidem habet ac polygonum ABCDE; quod omnium quæ, æquali laterum numero, circulo ABCDE inſcribi poſſunt, perimetro, ex hypotheſi, maximum eſt. Perimeter igitur ABCDEA perimetro GBCDEG major. Utrique auferatur pars communis BCDE. Relinquuntur AB, AE, reliquis GB, GE, ſimul ſumptæ ſimul ſumptis, majores. Duæ autem GB, GE, duabus AB, AE, oſtenſæ ſunt majores. Simul igitur majores et minores. Quod eſt abſurdum. Non ſunt igitur latera AB, AE, inæqualia. Simili modo oſtendetur, de aliis quibuſlibet polygoni lateribus proximis, inæqualia non eſſe, exiſtente perimetro maximâ. Nulla igitur proxima inæqualia. Omnino igitur nulla. Omnia igitur æqualia. *Q. E. D.*

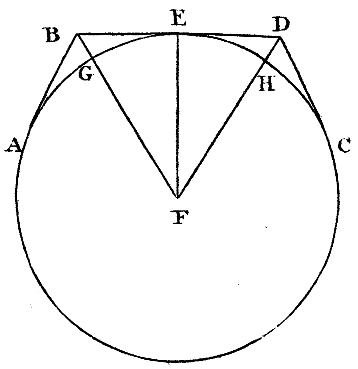
THEOREMA.

THEOREMA VI.

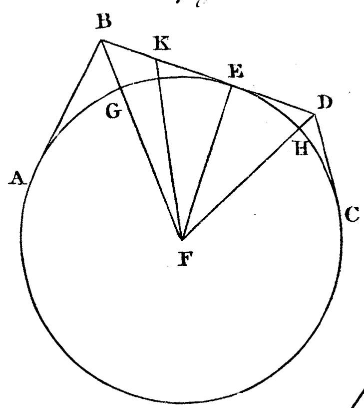
Polygonorum omnium, lateribus numero datis, dato circulo inſcriptorum, æquilaterum areâ maximum eſt.

Demonſtrationem vide apud THOMAM SIMPSON in libello ſuo de *Figuris Geometricis maximis et minimis.*

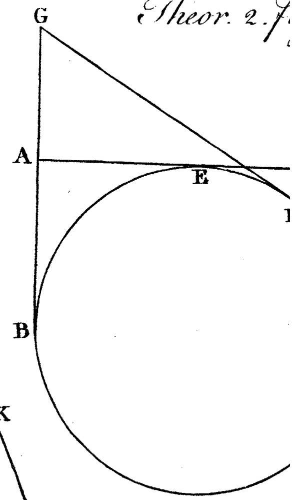
Theor. 1. fig. 1.

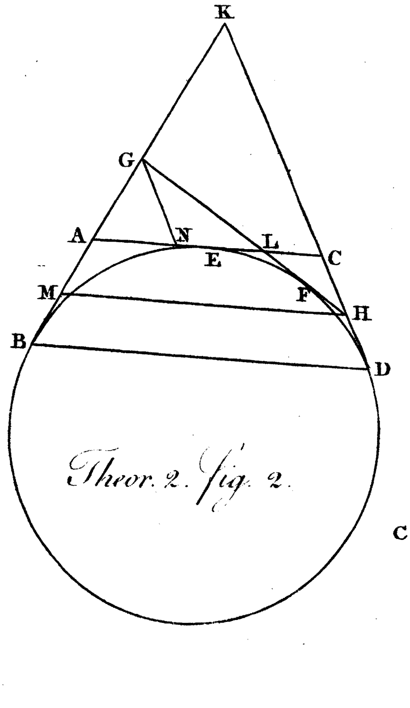


Theor. 1. fig. 2.

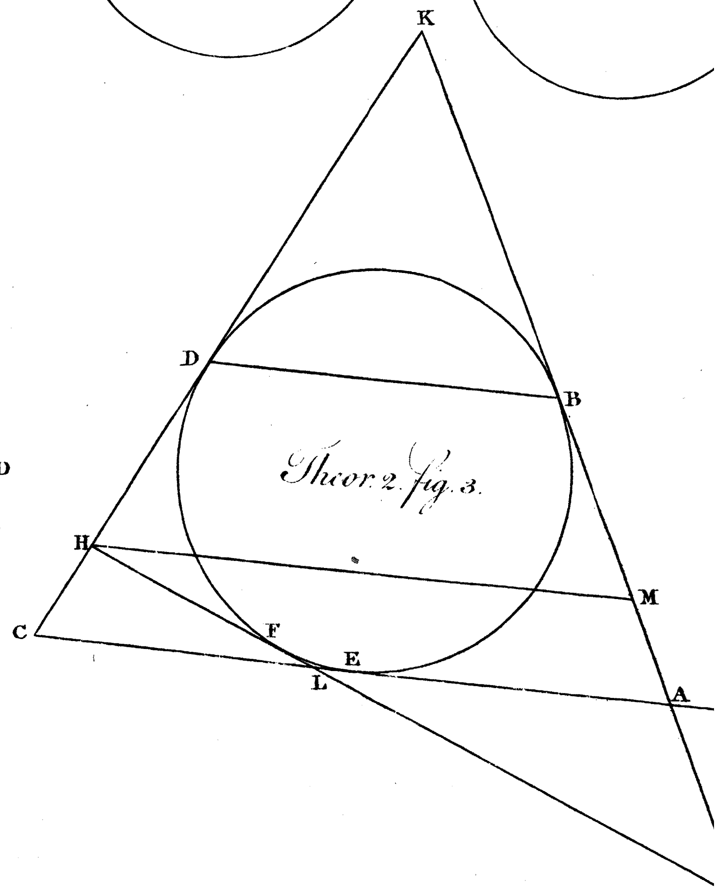


Theor. 2. f



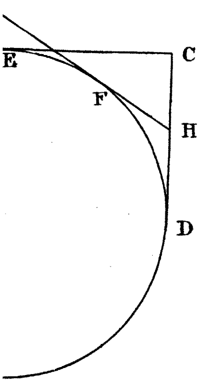


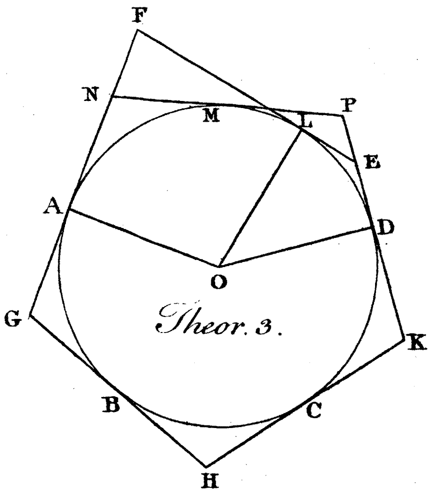
Theor. 2. fig. 2.



Theor. 2. fig. 3.

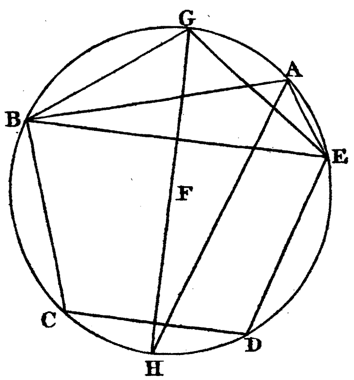
...eor. 2. fig. 1.

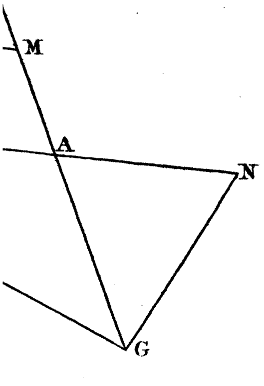


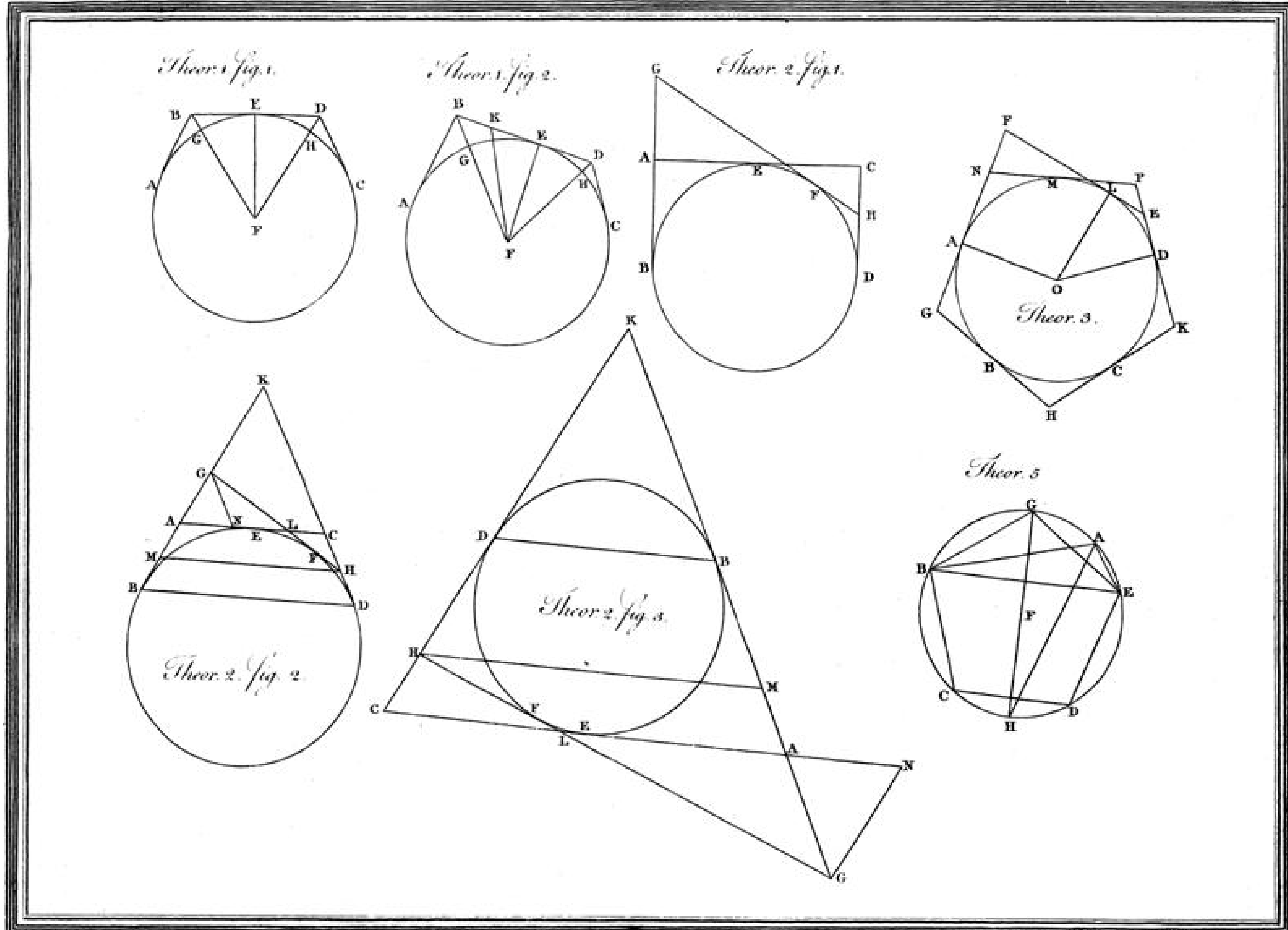


Theor. 3.

Theor. 5






Theor. 1 fig. 1.
Theor. 1 fig. 2.
Theor. 2 fig. 1.
Theor. 3.
Theor. 2 fig. 2.
Theor. 2 fig. 3.
Theor. 5