XXVI. *An Inveſtigation of a general Theorem for finding the Length of any Arc of any Conic Hyperbola, by Means of Two Elliptic Arcs, with ſome other new and uſeful Theorems deduced therefrom.* *By* John Landen, *F.R.S.*

Redde, Mar. 23, 1775.

IN a paper, which the Society did me the honour to publiſh in the Philoſophical Tranſactions for the year 1771, I announced, that I had diſcovered a general theorem for finding the length of any arc of any conic hyperbola, by means of two elliptic arcs; and I promiſed to communicate the inveſtigation of ſuch theorem. I now purpoſe to perform my promiſe; and, being pleaſed with the diſcovery (by which we are enabled to bring out very elegant concluſions in many intereſting enquiries, as well mechanical as purely geometrical), I cannot but flatter myſelf, that what I am about to communicate will be acceptable to gentlemen who are curious in ſuch inquiries.

1. From the theorem taken notice of in Art. 1. of the paper I have juſt now mentioned, it follows, that in the hyperbola AD (TAB. VII. fig. 1.), if the ſemi-tranſverſe axis AC be $=m-n$; the ſemi-conjugate $=2\times\overline{mn}|^{\frac{1}{2}}$; and the perpendicular CP, from the center C upon the tangent DP, $=\overline{\overline{m-n}|^2-t^2}|^{\frac{1}{2}}$; the difference (DP − AD) between the ſaid tangent

tangent DP and the arc AD will be equal to the fluent of $\left(\frac{(m-n)^2-t^2}{(m+n)^2-t^2}\right)^{\frac{1}{2}} \times \dot{t}$.

2. It is well known, that in any ellipsis whose semi-tranverse axis is m, and semi-conjugate n; if x be the abscissa, measured from the center upon the transverse axis, and z the arc between the conjugate axis and the ordinate corresponding to x, $\left(\frac{m^2-gx^2}{m^2-x^2}\right)^{\frac{1}{2}} \times \dot{x}$ will be $= \dot{z}$, g being $= \frac{m^2-n^2}{m^2}$.

Hence, $\left(\frac{(m+n)^2-t^2}{(m-n)^2-t^2}\right)^{\frac{1}{2}} \times \dot{t}$ being $= \left(\frac{(m+n)^2-t^2}{(m+n)^2-\left(\frac{m+n}{m-n}\right)^2 t^2}\right)^{\frac{1}{2}} \times \frac{m+n}{m-n}\dot{t}$, it appears, that in the ellipsis aed (fig. 2.) whose semi-transverse axis cd is $= m+n$, semi-conjugate ca $= 2 \times (mn)^{\frac{1}{2}}$, and abscissa cb (corresponding to the ordinate be) $= \frac{m+n}{m-n}t$; the arc ae is equal to the fluent of $\left(\frac{(m+n)^2-t^2}{(m-n)^2-t^2}\right)^{\frac{1}{2}} \times \dot{t}$.

3. In the ellipsis aefd (fig. 3.), the semi-transverse axis cd being $= m$; the semi-conjugate ca $= n$; and the abscissa cb (corresponding to the ordinate be) $= x$; if ep, the tangent at e, intercepted by a perpendicular (cp) drawn thereto from the center c, be denoted by t; $gx \times \left(\frac{m^2-x^2}{m^2-gx^2}\right)^{\frac{1}{2}}$ (as is well known) will be $= t$, g being as in the preceding article.

Hence

Hence $x^2 = \frac{m^2g+t^2}{2g} - \frac{\overline{\overline{m^2-n^2}|^2-2\times\overline{m^2+n^2}\times t^2+t^4}|^{\frac{1}{2}}}{2g}$. From which equation, by taking the fluxions, we have,

$$x\dot{x}=\frac{t\dot{t}}{2g}+\frac{\overline{m^2+n^2}\times t\dot{t}-t^3\dot{t}}{2g\times\overline{\overline{m^2-n^2}|^2-2\times\overline{m^2+n^2}\times t^2+t^4}|^{\frac{1}{2}}}$$

$=\frac{t\dot{t}}{2g}+\frac{\overline{m^2+n^2}\times t\dot{t}-t^3\dot{t}}{2g\times\overline{\overline{m-n}|^2-t^2\times\overline{m+n}|^2-t^2}|^{\frac{1}{2}}}$. But $\dot{z}$ being $=\overline{\frac{m^2-gx^2}{m^2-x^2}}|^{\frac{1}{2}}\times\dot{x}$, as observed in the preceding article, it appears that $\frac{g}{t}\times x\dot{x}$ is $=\dot{z}$. It is obvious, therefore, that

$$\dot{z}\text{ is }=\tfrac{1}{2}\dot{t}+\tfrac{1}{2}\times\frac{\overline{m^2+n^2}\times\dot{t}-t^2\dot{t}}{\overline{\overline{m-n}|^2-t^2\times\overline{m+n}|^2-t^2}|^{\frac{1}{2}}}$$

$$=\tfrac{1}{2}\dot{t}+\tfrac{1}{4}\times\frac{\overline{m-n}|^2\times\dot{t}-t^2\dot{t}}{\overline{\overline{m-n}|^2-t^2\times\overline{m+n}|^2-t^2}|^{\frac{1}{2}}}+\tfrac{1}{4}\times\frac{\overline{m+n}|^2\times\dot{t}-t^2\dot{t}}{\overline{\overline{m-n}|^2-t^2\times\overline{m+n}|^2-t^2}|^{\frac{1}{2}}}$$

$=\tfrac{1}{2}\dot{t}+\tfrac{1}{4}\times\overline{\frac{\overline{m-n}|^2-t^2}{\overline{m+n}|^2-t^2}}|^{\frac{1}{2}}\times\dot{t}+\tfrac{1}{4}\times\overline{\frac{\overline{m+n}|^2-t^2}{\overline{m-n}|^2-t^2}}|^{\frac{1}{2}}\times\dot{t}$. From whence taking the fluents by the theorems in art. 1. and 2. we have $z=$ ae (fig. 3.) $=\tfrac{1}{2}t+\frac{\text{DP}-\text{AD}}{4}$ (fig. 1.) $+\frac{\text{ae}}{4}$ (fig. 2.) consequently the hyperbolic arc AD is $=$ DP + ae + $2t-4$ae. Thus, beyond my expectation, I find, that the hyperbola may in general be rectified by means of two ellipses.

Writing E and F for the quadrantal arcs ad, ad, (fig. 2. and 3.) respectively, and L for the limit of the difference DP − AD, whilst the point of contact (D) is supposed to be carried to an infinite distance from the vertex A of the hyperbola (fig. 1.), we find

$$2\text{F}-\text{E}=\text{L},$$

the value of ae being $=\tfrac{1}{2}\text{F}+\tfrac{1}{2}m-\tfrac{1}{2}n$ when t is $=m-n$; that

that is, when e coincides with d (fig. 2.), and P with C (fig. 1.), by what I have proved in the before mentioned paper, art. 10.

4. From what is done above, the following uſeful theorems are deduced.

THEOREM I.

The fluent of $\frac{1}{2}a^{\frac{1}{2}}z^{-\frac{1}{2}}\dot{z} \times \left(\frac{\frac{b^2}{a}+z}{a-z}\right)^{\frac{1}{2}}$ is $=$ de.

THEOREM II.

The fluent of $\frac{1}{2}a^{\frac{1}{2}}z^{-\frac{1}{2}}\dot{z} \times \left(\frac{a-z}{\frac{b^2}{a}+z}\right)^{\frac{1}{2}} = \overline{\frac{2a^2}{b^2}+1}.\mathrm{de} - \overline{\frac{2a^2}{b^2}+2}.\mathrm{ef}$.

THEOREM III.

The fluent of $\frac{\frac{1}{2}a^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\left(b^2+2kz-z^2\right)^{\frac{1}{2}}} = 2\,\mathrm{ef} - \mathrm{de} = 2\,\mathrm{F} - \mathrm{E} + \mathrm{AD} - \mathrm{DP}$.

THEOREM IV.

The fluent of $\frac{\frac{1}{2}a^{-\frac{1}{2}}b^2 z^{-\frac{1}{2}}\dot{z}}{\left(b^2+2kz-z^2\right)^{\frac{1}{2}}} = 2 \times \overline{\mathrm{de}-\mathrm{ef}}$. N.B. $k = \frac{a^2-b^2}{2a}$.

Theſe theorems ſtill refer to fig. 1. 2. 3.; but now the values of the ſeveral lines therein (being not as before) are as here ſpecified; *videlicet*,

Fig. 1. In the hyperbola AD, the ſemi-tranſverſe axis AC is now $=a$; the ſemi-conjugate $=b$; the perpendicular CP, from the center C upon the tangent DP, is $=(az)^{\frac{1}{2}}$ the ſaid tangent DP $=\left(\frac{a}{z}\right)^{\frac{1}{2}} \times \left(b^2+2kz-z^2\right)^{\frac{1}{2}}$; and the abſciſſa CB (correſponding to the ordinate BD) is $=\frac{a^{\frac{1}{2}}}{z^{\frac{1}{2}}} \times \left(\frac{az+b^2}{a^2+b^2}\right)^{\frac{1}{2}}$

Fig.

Fig. 2. In the ellipſis aed, the ſemi-tranſverſe axis cd is $= \overline{a^2+b^2}|^{\frac{1}{2}}$; the ſemi-conjugate ca$=b$; the abſciſſa cb$= \overline{\frac{a^2+b^2}{a}}|^{\frac{1}{2}} \times \overline{a-z}|^{\frac{1}{2}}$; and the ordinate be$=b \times \overline{\frac{z}{a}}|^{\frac{1}{2}}$.

Fig. 3. In the ellipſis aefd, the ſemi-tranſverſe axis cd is$=\frac{1}{2}.\overline{a^2+b^2}|^{\frac{1}{2}}+\frac{1}{2}a$; the ſemi-conjugate ca$=\frac{1}{2}.\overline{a^2+b^2}|^{\frac{1}{2}}-\frac{1}{2}a$; the tangents ep, fq, intercepted by perpendiculars (cp, cq) drawn thereto from the center c, each$= a^{\frac{1}{2}} \times \overline{a-z}|^{\frac{1}{2}}$; and the abſciſſa (c b′ or c b″) on cd, correſponding to the point e or f, of the curve is determined by the expreſſion

$$\frac{\overline{\overline{a^2+b^2}|^{\frac{1}{2}}+a-z\mp\overline{z^2+\frac{b^2}{a}z}|^{\frac{1}{2}}}|^{\frac{1}{2}}}{2^{\frac{1}{2}}+\overline{a^2-b^2}|^{\frac{1}{4}}}\times \text{cd}.$$

The quadrantal arc ad (fig. 2.) is denoted by E; and the quadrantal arc ad (fig. 3.) is denoted by F. L the limit of DP$-$AD (fig. 1.) is $=$ 2F$-$E.

From what is now done, I might proceed to deduce many other new theorems, for the computation of fluents; but I ſhall, at preſent, decline that buſineſs: and, after giving a remarkable example of the uſe of theorem 4. in computing the deſcent of a heavy body in a circular arc, conclude this paper with a few obſervations relative to the contents of the preceding articles.

5. Let lpqn (fig. 4.) be a ſemi-circle perpendicular to the horizon, whoſe higheſt point is l, loweſt n, and center m. Let ps, qt, parallel to the horizon, meet the diameter lmn in s and t; and let the radius lm (or mn) be denoted by r; the height ns by d; and the diſtance

st by x. Then, putting b for ($16\frac{1}{12}$ feet) the ſpace a heavy body, deſcending freely from reſt, falls through in one ſecond of time; and ſuppoſing a pendulum, or other heavy body, deſcending by its gravity from p, along the arc pqn, to have arrived at q; the fluxion of the time of deſcent will be $= \frac{\frac{1}{2} r h^{-\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\overline{2rd - d^2 - 2.\overline{r-d}.x - x^2}|^{\frac{1}{2}}}$. The fluent whereof, or the time of deſcent from p to q is (by theor. 4. of the preceding article) $= \frac{2r}{h^{\frac{1}{2}} \times \overline{2r-d}} \times$ de−ef. a (in that theorem) being taken $= d^{\frac{1}{2}}$, $b = \overline{2r-d}|^{\frac{1}{2}}$, cb (fig. 2.) $= \overline{\frac{2r}{d}}|^{\frac{1}{2}} \times \overline{d-x}|^{\frac{1}{2}}$, and ep, fq, (fig. 3.) each $= \overline{d-x}|^{\frac{1}{2}}$. Hence it appears, that the whole time of deſcent from p to n is $= \frac{2r}{h^{\frac{1}{2}} \times \overline{2r-d}} \times \overline{\mathrm{E}-\mathrm{F}}$; when, in fig. 2. and 3. the ſemi-axes are taken according to the values of a and b juſt now ſpecified.

6. If pqn be a quadrant; that is, if d be $= r$, the whole time of deſcent from p to n will be $= \frac{2}{h^{\frac{1}{2}}} \times \overline{\mathrm{E}-\mathrm{F}}$, by the above theorem. Which time, by what I have ſhewn in the Philoſ. Tranſact. for 1771, is $= \frac{1}{h^{\frac{1}{2}}} \times \overline{\frac{1}{2}\mathrm{E} + \frac{1}{2}\sqrt{\mathrm{E}^2 - 2c}}$, c being $\frac{1}{4}$ of the periphery of the circle whoſe radius is r. Conſequently, $\frac{2}{h^{\frac{1}{2}}} \times \overline{\mathrm{E}-\mathrm{F}}$ being found $= \frac{1}{h^{\frac{1}{2}}} \times \overline{\frac{1}{2}\mathrm{E} + \frac{1}{2}\sqrt{\mathrm{E}^2 - 2c}}$, we find from that equation $\mathrm{F} = \frac{3}{4}\mathrm{E} - \frac{1}{4}\sqrt{\mathrm{E}^2 - 2c}$, where E is the quadrantal arc of the ellipſis, whoſe ſemi-tranſverſe and ſemi-conjugate axes are $\overline{2r}|$ and $r^{\frac{1}{2}}$; and F the qua-

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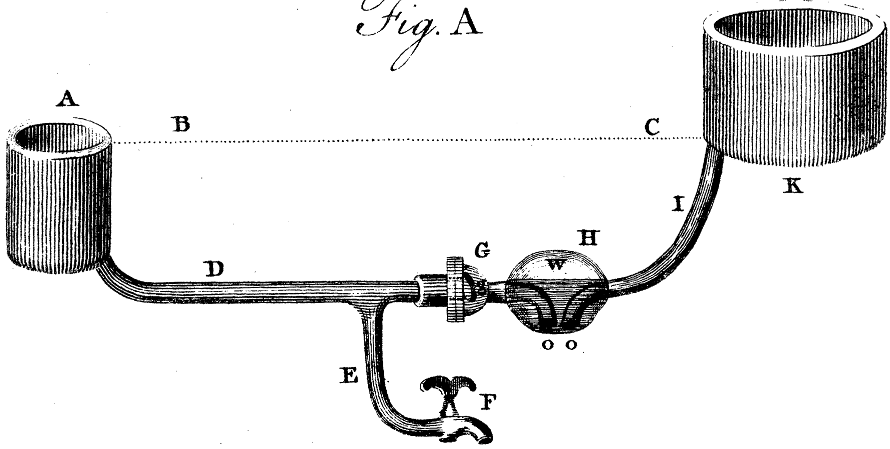
drantal arc of another ellipſis, whoſe ſemi-tranſverſe and ſemi-conjugate axes are $\left(\frac{r}{2}\right)^{\frac{1}{2}}+\frac{1}{2}r^{\frac{1}{2}}$ and $\left(\frac{r}{2}\right)^{\frac{1}{2}}-\frac{1}{2}r^{\frac{1}{2}}$.

Before Mr. MACLAURIN publiſhed his excellent Treatiſe of Fluxions, ſome very eminent mathematicians imagined, that the *elaſtic curve* could not be conſtructed by the quadrature or rectification of the conic ſections. But that gentleman has ſhewn, in that treatiſe, that the ſaid curve may in every caſe be conſtructed by the rectification of the hyperbola and ellipſis; and he has obſerved, that, by the ſame means, we may conſtruct the curve along which, if a heavy body moved, it would recede equally in equal times from a given point. Which laſt mentioned curve Mr. JAMES BERNOULLI conſtructed by the rectification of the elaſtic curve, and Mr. LEIBNITZ and Mr. JOHN BERNOULLI by the rectification of a geometrical curve of a higher kind than the conic ſections. It is obſervable, that Mr. MACLAURIN's method of conſtruction juſt now adverted to, though very elegant, is not without a defect. The difference between the hyperbolic arc and its tangent being neceſſary to be taken, the method always fails when ſome principal point in the figure is to be determined; the ſaid arc and its tangent then both becoming infinite, though their difference be at the ſame time finite. The contents of this paper, properly applied, will evince, that both the *elaſtic curve* and the *curve of equable receſs from a given point* (with many others) may be conſtructed by the rectification of the ellipſis only, without failure in any point

Fig. A



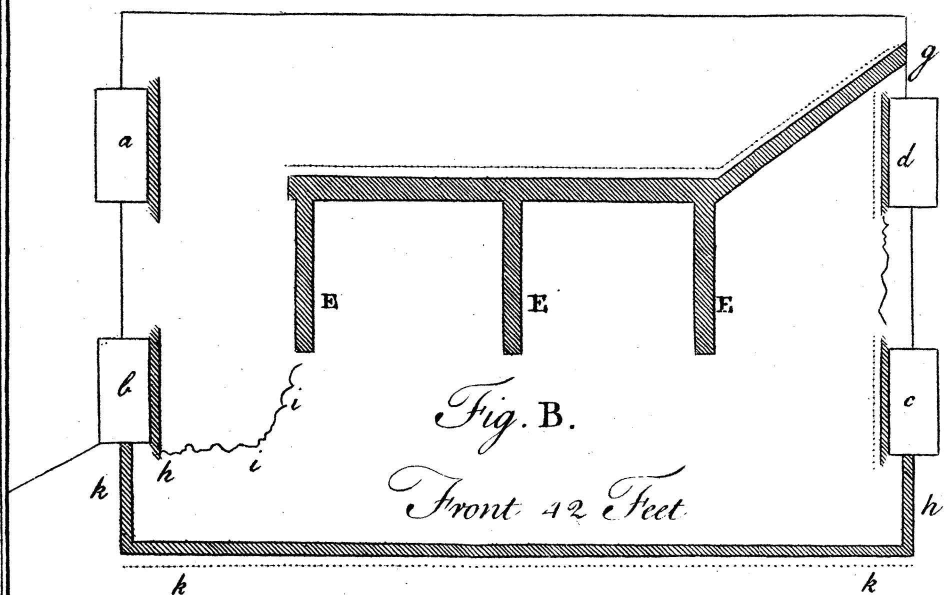
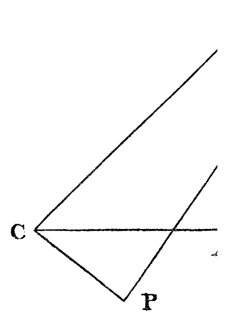


Fig. B.

Front 42 Feet

Fig. 1.



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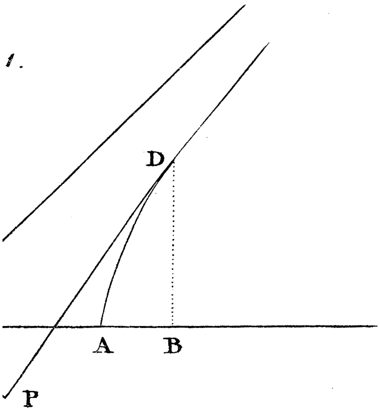
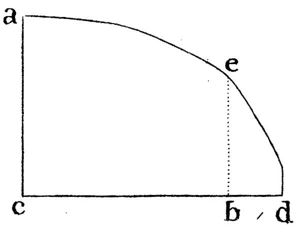


Fig. 2.



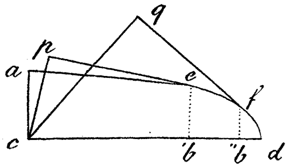


Fig. 3.

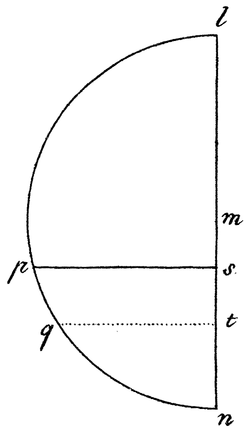


Fig. 4.

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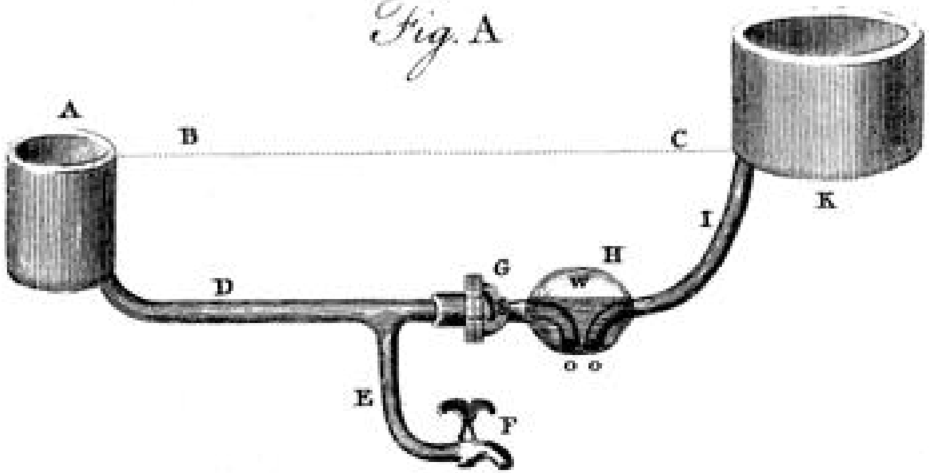
Fig. A



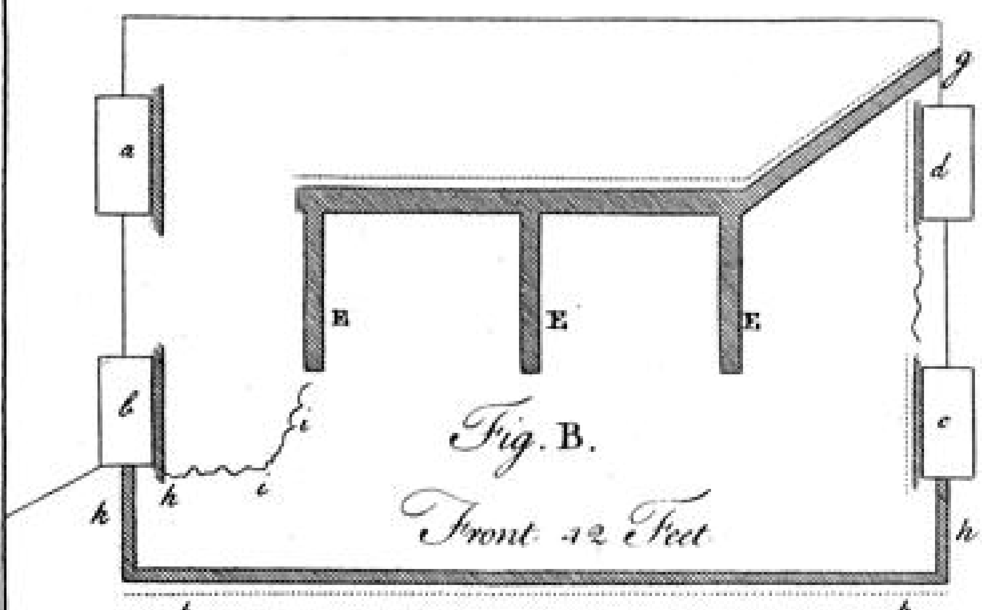


Fig. B.

Front 12 Feet

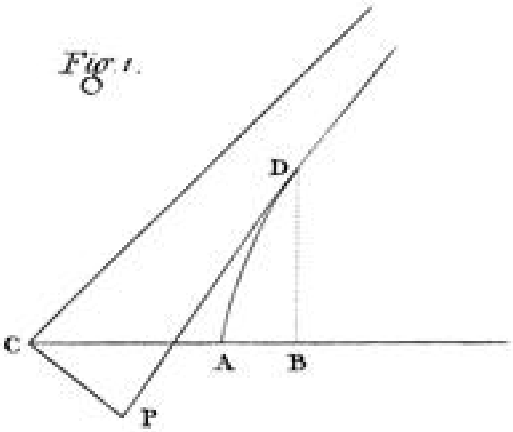


Fig. 1.

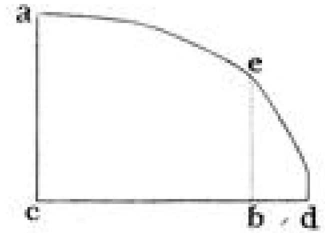


Fig. 2.

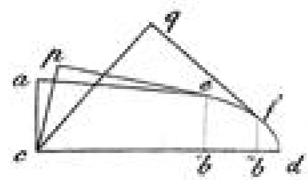


Fig. 3.

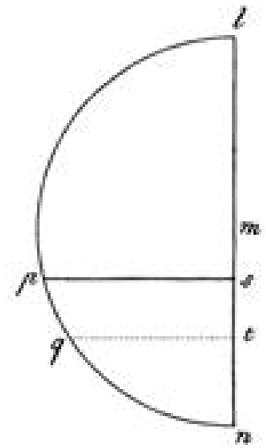


Fig. 4.

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