sidence of electricity in the wire when any one key is touched, and to let the different strengths of current, in one direction or the other, be produced by the different keys. Thus without a condensed code, thirty words per minute could be telegraphed through subterranean or submarine lines of 500 miles; and from thirty to fifty or sixty words per minute through such lines, of lengths of from 500 miles to 100 miles.

The rate of from fifty to sixty words per minute could be attained through almost any length of air line, were it not for the defects of insulation to which such lines are exposed. If the imperfection of the insulation remained constant, or only varied slowly from day to day with the humidity of the atmosphere, the method I have indicated might probably, with suitable adjustments, be made successful; and I think it possible that it may be found to answer for air lines of hundreds of miles' length. But in a short air line, the strengths of the currents received, at one extremity, from graduated operations performed at the other, might suddenly, in the middle of a message, become so much changed as to throw all the indications into confusion, in consequence of a shower of rain, or a trickling of water along a spider's web.

VI. "On the Equation of Laplace's Functions," &c. By W. F. Donkin, M.A., F.R.S., F.R.A.S., Savilian Professor of Astronomy, Oxford. Received December 3, 1856.

(Abstract.)

The equation $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$, when transformed by putting $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, may be written in the form

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\phi}\right)^2+(\sin\theta)^2 r\frac{d}{dr}\left(r\frac{d}{dr}+1\right)\right\}u=0; \quad (1)$$

and if $u=u_0+u_1r+u_2r^2+\ldots+u_nr^n+\ldots$, we find on substituting this value in (1), and equating to zero the coefficient of r^n, that u_n satisfies the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\phi}\right)^2+n(n+1)(\sin\theta)^2\right\}u_n=0, \quad . \quad . \quad (2)$$

commonly called the equation of Laplace's functions. If we put $\sin\theta\frac{d}{d\theta}+n\cos\theta=\varpi_n$, then the equation (2) may be written

$$\left(\varpi_n\varpi_{-n}+n^2+\left(\frac{d}{d\phi}\right)^2\right)u_n=0\,;$$

and the operation ϖ_n possesses the following property, namely

$$\varpi_{-n}\varpi_n+n^2=\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2\,;$$

hence it is easily shown, that in general the complete solution of (2) is

$$u_n=\varpi_n\varpi_{n-1}\ldots\varpi_2\varpi_1\,.\,u_0,$$

where u_0 is the solution of

$$\left(\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\phi}\right)^2\right)u_0=0,$$

namely

$$u_0=f\left(e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)+\mathrm{F}\left(e^{-\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)\,;$$

and the operation $\varpi_n\varpi_{n-1}\ldots\varpi_2\varpi_1$ is easily seen to be equivalent to

$$(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n.$$

(This result is compared with that obtained in a different way by Professor Boole (Cambridge and Dublin Journal, vol. i. p. 18), to which it bears a general resemblance, but the author has not succeeded at present in reducing the one form to the other.)

In the case in which u_n does not contain ϕ, we have

$$u_0=\mathrm{C}_1+\mathrm{C}_2\log\tan\frac{\theta}{2}.$$

The general expression for a "Laplace's coefficient" of the nth order, not containing ϕ, is therefore $(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n.\,\mathrm{C}$; and if this be called v_n when $\mathrm{C}=1$, the development of $(1-2r\cos\theta+r^2)^{-\frac{1}{2}}$ is

$$v_0+v_1r+v_2\frac{r^2}{1\,.\,2}+\ldots+v_n\frac{r^n}{1\,.\,2\ldots n}+\ldots\,;$$

and it is shown that the coefficient of $\frac{r^n}{1\,.\,2\ldots n}$ in the development of $(1-2r\cos\theta+r^2)^{-\frac{i+1}{2}}$ is

$$(\sin\theta)^{-n-i}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n(\sin\theta)^i.$$

With respect to the development of

$$\left(1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi)+r^2\right)^{-\frac{1}{2}},$$

it is shown that the coefficient of $r^n\cos i\phi$ may be put in either of the two forms,

$$\frac{2}{1\,.\,2\,\ldots\,(n-i)\,.\,1\,.\,2\,\ldots\,(n+i)}(\sin\theta)^{-n}(\sin\theta')^{-n}\Theta^n\Theta'\left(\tan\frac{\theta}{2}\right)^i\left(\tan\frac{\theta'}{2}\right),$$

or

$$\frac{2\,.\,1^2\,.\,3^2\,\ldots\,(2i-1)^2}{1\,.\,2\,\ldots\,(n-i)\,.\,1\,.\,2\,\ldots\,(n+i)}(\sin\theta)^{-n}(\sin\theta')^{-n}\Theta^{n-i}\Theta'^{\,-i}(\sin\theta)^{2i}(\sin\theta')^{2i},$$

where Θ represents the operation $\sin\theta\frac{d}{d\theta}\sin\theta$, and the factor 2 is in each case to be omitted when $i=0$. (This coefficient is a solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2-i^2\right\}u=0,$$

of which the complete integral may be expressed in the form

$$(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{n-i}(\sin\theta)^{2i}\left(\mathrm{C}_1+\mathrm{C}_2\int d\theta(\sin\theta)_{-2i-1}\right),$$

at least in the case in which i is an integer not greater than n, for which case this form is here demonstrated.)

If it be assumed that the solution of (2), obtained on the supposition that n is an integer, may be extended to the case in which n is a general symbol, it follows that the solution of (1) will be obtained from it by changing n into $r\frac{d}{dr}$. This would give

$$u=(\sin\theta)^{-r\frac{d}{dr}}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{r\frac{d}{dr}}\left\{f\left(r,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)\right.$$
$$\left.+\mathrm{F}\left(r,\ e^{-\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)\right\},$$

which is easily shown to be equivalent to

$$u=f\left(\rho\sin\theta\frac{d}{d\theta}\sin\theta,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)+\mathrm{F}\left(\rho\sin\theta\frac{d}{d\theta}\sin\theta,\ e^{-\phi\sqrt{-1}}\tan\frac{\theta}{2}\right),$$

where $\rho=r(\sin\theta)^{-1}$, but ρ is to be treated as a constant till after all operations.

This expression is shown to give known particular integrals, such as $(1-2r\cos\theta+r^2)^{-\frac{1}{2}}$, and

$$r^n (\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\tan\frac{\theta}{2}\right)\cos i\phi.$$

It appears probable, therefore, that the generalization of the result obtained for the limited value of n is legitimate; but the author does not profess to demonstrate this conclusion, believing that the principle of the "permanence of equivalent forms" is not at present established in such a sense as to amount to a demonstration.

VII. "A Memoir on Curves of the Third Order." By Arthur Cayley, Esq., F.R.S. Received Oct. 30, 1856.

(Abstract.)

A curve of the third order, or cubic curve, is the locus represented by an equation such as $U=(*\between x, y, z)^3=0$; and it appears by my "Third Memoir on Quantics," that it is proper to consider, in connexion with the curve of the third order, $U=0$, and its Hessian $HU=0$ (which is also a curve of the third order), two curves of the third class, viz. the curves represented by the equations $PU=0$ and $QU=0$. These equations, I say, represent curves of the third class; in fact, PU and QU are contravariants of U, and therefore, when the variables x, y, z of U are considered as point coordinates, the variables ξ, η, ζ of PU, QU must be considered as line coordinates, and the curves will be curves of the third class. I propose (in analogy with the form of the word Hessian) to call the two curves in question the Pippian and Quippian respectively. A geometrical definition of the Pippian was readily found; the curve is in fact Steiner's curve R_0 mentioned in the memoir "Allgemeine Eigenschaften der algebraischen Curven," Crelle, t. xlvii. pp. 1–6, in the particular case of a basis-curve of the third order; and I also found that the Pippian might be considered as occurring implicitly in my "Mémoire sur les Courbes du Troisième Ordre," Liouville, t. ix. p. 285, and "Nouvelles Remarques sur les Courbes du Troisième Ordre," Liouv. t. x. p. 102. As regards the Quippian, I