Again,—

$$(a, b, \ldots \between \Xi, \Xi_1)^n = (a_n \Xi + \beta \Xi_1) \ldots (a_2 \Xi + \beta_2 \Xi)(a_1 \Xi + \beta_1 \Xi)$$

$$= \left| \begin{array}{lll} (\nabla \nabla_1 \between a_n \beta_n) & (a_{n-1} \beta_{n-1}) & \cdots \\ (\nabla \nabla_1 \ldots \between a_n \beta_n \between (\nabla \nabla_1 \between a_{n-1} \beta_{n-1}) & & \cdots \\ \cdot & & \cdots \end{array} \right|$$

where $(\nabla \nabla_1 \nabla \between a_2 \beta_2 \between \times \between a_1 \beta_1) = (\nabla \nabla_1 \nabla \between a_2 \beta_2 \between a_1 \beta_1)$.

It is further shown that the effect of the operations ∇, ∇_1 on a given function, $u = \Sigma a_i x^{n-i} y^i$, may be represented by

$$F(\nabla, \nabla_1)\, u = \Sigma \left\{ F\left(u, (u - i + 1)\, \epsilon^{-\frac{d}{di}} + (i+1)\, \epsilon^{\frac{d}{di}}\right) \right\} a_i x^{n-i} y^i;$$

and the case of $\dfrac{u}{F(\nabla, \nabla_1)}$ is examined in detail.

The value of

$$s_1 s_2 \ldots s_i \frac{d^i}{dx^{i-j}\, dy^j}$$

in terms of $s_1 \dfrac{d}{dx}$, $s_2 \dfrac{d}{dx}$, .. $s_1 \dfrac{d}{dy}$, $s_2 \dfrac{d}{dy}$, .. are calculated, (1) when s_1, s_2, .. are any linear functions of x, y, (2) when they are any functions whatever; and, in case (1), the effect of the above operation on a given function is determined.

IV. "Problem on the Divisibility of Numbers." By Francis Elefanti, Esq. Communicated by Arthur Cayley, Esq.

Received November 24, 1859.

Problem. To find a proceeding by which the divisibility of a proposed integer N by 7 or 13, or by both 7 and 13, may be determined through the same rule.

Solution. We can designate the number N by $abcd \ldots mn$, so that (a) be the first or highest, and (n) the last or lowest digit in it, therefore we may put

$$N = abcde \ldots mn.$$

1. Take the first digit, and having placed it below the fourth, make the subtraction in the usual way, thus:

$$\begin{array}{l} N'=bcde\ldots mn \\ \quad\;\; -a \end{array}$$

2. Take the first digit of N′, and having placed it below the fourth, effect the subtraction, thus:

$$\begin{array}{l} N''=cde\ldots mn \\ \quad\;\;\; -b \end{array}$$

3. Continue in the same way till you have effected the subtraction upon (n), and you will have the rule:

Is $N'''^{\cdots}=\ldots l'm'n'$ divisible by 7 or 9, or by 7 and 9 at the same time? then is the number N divisible too by 7 or 13, or by 7 and 13 at the same time.

Ex. I.

$$N\ =71491$$

$$N'=\begin{array}{r}1491\\-7\end{array}=1421$$

$$N''=\begin{array}{r}421\\-1\end{array}=420=7\cdot 60\,;$$

therefore N is divisible by 7.

Ex. II.

$$N\ =246571$$

$$N'=\begin{array}{r}46571\\-2\end{array}=46371$$

$$N''=\begin{array}{r}6371\\-4\end{array}=6331$$

$$N'''=\begin{array}{r}331\\-6\end{array}=325=5^2\cdot 13\,;$$

the number N is divisible by 13.

Ex. III.

$$\begin{array}{l}
N=1{,}1\;8\;3\;5\;3\;0\;8\;0\;3\\
\qquad\;\; 1{,}8\;2\;5\\
\qquad\quad\;\; 8{,}2\;4\;3\\
\qquad\qquad\; 2{,}3\;5\;0\\
\qquad\qquad\quad\; 3{,}4\;8\;8\\
\qquad\qquad\qquad 4{,}8\;5\;0\\
\qquad\qquad\qquad\;\; 8{,}4\;6\;3\\
\qquad\qquad\qquad\qquad 4\;5\;5=5\cdot 7\cdot 13\,;
\end{array}$$

the proposed number N is divisible both by 7 and by 13.

Ex. IV.

$$N = 7429$$

$$N' = \begin{array}{r}429\\-7\end{array} = 422 = 2\cdot 211,$$

the number N is divisible neither by 7 nor by 13.

Remark.—The above proceeding is but the application of a general principle, inherent to our decadic system. The author is in possession of similar proceedings for all prime numbers up to 107; above that limit (except a few cases) the rules become more complicated, and lose the high value of easy application.

Taking again $N = abcde \ldots mn$, we have the following Table for operating by the different prime numbers up to 109:—

Divisors.	Operation.
7, 13	$d-a$
17, 59	$d-3a$
19, 53	$d-7a$
23, 43	$+\begin{array}{c}c\,d\\a\,a\end{array}$
29	$e-5a$
31	$+\begin{array}{c}d\\8a\end{array}$
37	$+\begin{array}{c}d\\a\end{array}$
41, 61	$e-4a$
47, 71	$\begin{array}{c}de\\-aa\end{array}$
67	$d-sa$
73, 137	$e-a$
79, 127	$\begin{array}{c}de\\-3(aa)\end{array}$
83	$\begin{array}{c}d\\+4a\end{array}$
89	$\begin{array}{c}c\quad d\\+(2a),\,a\end{array}$
97, 103	$\begin{array}{c}e\\+9a\end{array}$
101	$c-a$
107	$c-7a$

Explanation.—17, 59 : $d-3a$ means, take three times the first digit, and subtract it from the preceding bcd.

Ex. gr. I.

$$
\begin{array}{r}
N=3{,}2\,4\,6\,3\,4\,9\,2\,9\,1 \\
-9 \\
\hline
2{,}3\,7\,3 \\
-6 \\
\hline
3{,}6\,7\,4 \\
-9 \\
\hline
6{,}6\,5\,9 \\
-1\,8 \\
\hline
6{,}4\,1\,2 \\
-1\,8 \\
\hline
3{,}9\,4\,9 \\
-9 \\
\hline
9{,}4\,0\,1 \\
-2\,7 \\
\hline
3\,7\,4
\end{array}
$$

Note to *Ex.* I.

$15=\ 17-2$

$18=\ 17+1$

$9=\ 17-8$

$21=\ 17+4$

$27=2\cdot17-7$

Therefore we can operate thus:—

$$
\begin{array}{l}
3{,}2\,4\,6\,3\,4\,9\,2\,9\,1 \\
\quad +8 \\
\hline
2{,}5\,4\,3 \\
\quad\ \ -6 \\
\hline
5{,}3\,7\,4 \\
\quad\ \ \ +2 \\
\hline
3{,}7\,6\,9 \\
\quad\ \ \ -9 \\
\hline
7{,}6\,0\,2 \\
\quad\ \ \ \ -4 \\
\hline
5{,}9\,8\,9 \\
\quad\ \ \ \ +2 \\
\hline
9{,}\,9\,1\,1 \\
\quad\ \ \ \ \ +7 \\
\hline
9\,1\,8
\end{array}
$$

Hence the rule:—If a multiple of the digit to be subtracted be *below* the divisor, we can convert subtraction into addition. If it be beyond the divisor, we can subtract the excess instead of the multiple.

II. The symbol for 89 was $+\overset{c\ d}{2a\ a}$, which means that the first

ought to be placed below the fourth, and the double of the first below the third, thus :—

N=6,5 9 1 9 7 4 5 7
+1 2 6
7,1 7 9
1 4 7
3,2 6 7
6 3
3,3 0 4
6 3
3,6 7 5
6 3
7,3 8 7
1 4 7
5 3 4=6·89

But the work can be done more easily in the following way :—

$$a=\left\{\begin{array}{l}1\ldots\ldots\\2\ldots\ldots\\3\ldots\ldots\\4\ \ 2aa=\\5\ldots\ldots\\6\ldots\ldots\\7\ldots\ldots\\8\ldots\ldots\\9\ldots\ldots\end{array}\right\}\begin{array}{l}\ldots\ldots21\\\ldots\ldots42\\\ldots\ldots63\\\ldots\ldots84=-5\\\ldots\ldots16\\\ldots\ldots37\\\ldots\ldots58\\\ldots-10\\\ldots\ldots11\end{array}$$

Ex. gr. I. N=6,5 9 1 9 7 4 5 7
+3 7
6,2 8 9
+3 7
3,2 6 7
6 3
3,3 0 4
6 3
3,6 7 5
6 3
7,3 8 7
5 8
4 5=5·89

II.

$$\begin{array}{l}
N = 9{,}9\,7\,6\,0\,9\,0\,9\,9 \\
\quad +1\,1 \\
\hline
9{,}8\,7\,0 \\
\quad 1\,1 \\
\hline
8{,}8\,1\,9 \\
\quad -1\,0 \\
\hline
8{,}0\,9\,0 \\
\quad -1\,0 \\
\hline
8{,}0\,9\,9 \\
\quad -1\,0 \\
\hline
\quad 8\,9
\end{array}$$

III. If we search separately for 79 (neglecting for the while 127), we can proceed thus :—

$$a = \left\{ \begin{array}{ll}
1 \ldots\ldots & -33 \\
2 \ldots\ldots & +13 \\
3 \ldots\ldots & -20 \\
4 \ldots\ldots & +26 \\
5 \ldots\ldots & -\ \ 7 \\
6 \ldots\ldots & -40 \\
7 \ldots\ldots & +\ \ 6 \\
8 \ldots\ldots & -27 \\
9 \ldots\ldots & -60
\end{array} \right\} = -3\,(aa)$$

Ex. 1.

$$\begin{array}{l}
7{,}9\,5\,5\,0\,6\,3 = N \\
\quad +6 \\
\hline
9{,}5\,5\,6\,6 \\
\quad -6\,0 \\
\hline
5{,}5\,0\,6\,3 \\
\quad -7 \\
\hline
5{,}0\,5\,6\,0 \\
\quad -7 \\
\hline
\quad 5\,5\,3 = 7 \cdot 79
\end{array}$$

Ex. 2.

$$
\begin{array}{l}
2{,}2\,7\,0\,8\,9\,7\,0\,2\,2\,4\,7\\
\quad 2{,}7\,2\,1\,9\\
\qquad 7{,}2\,3\,2\,7\\
\qquad\quad 2{,}3\,3\,3\,0\\
\qquad\qquad 3{,}3\,4\,3\,2\\
\qquad\qquad\quad 3{,}4\,1\,2\,2\\
\qquad\qquad\qquad 4{,}1\,0\,2\,4\\
\qquad\qquad\qquad\quad 1{,}0\,5\,0\,7\\
\qquad\qquad\qquad\qquad\quad 4\,7\,4=6{\cdot}79
\end{array}
$$

Remark.—The principle which I have thus developed is touched upon in some manuals of arithmetic, when we are shown that the same remainder in the expressions

$$\mathrm{R}\left(\frac{1000}{7}\right)=\mathrm{R}\left(\frac{1000}{13}\right)=\mathrm{R}\left(\frac{1000}{11}\right)=-1$$

leads to the identity

$$7{\cdot}11{\cdot}13=1001.$$

In the rules given in this Paper, I have shown the high *analytical* value of the principle; but the properties of numbers, to which we are led when we apply the same (principle) in a *synthetical* way, are not less remarkable. As an instance I may state that the symbol

$37 \ldots\ldots \overset{d}{+a}$ leads to a curious relation among the members 7, 11, and 37.

V. "On the Structure of the *Chorda Dorsalis* of the Plagiostomes and some other Fishes, and on the relation of its proper Sheath to the development of the Vertebræ." By Professor ALBERT KÖLLIKER, of Würzburg. Communicated by Dr. SHARPEY, Sec. R.S. Received December 3, 1859.

I take the liberty to present to the Royal Society the results of an extended series of investigations into the development of the vertebræ of the plagiostomous and some other fishes.