XII. "On the Laws of Operation, and the Systematization of Mathematics." By Alexander J. Ellis, Esq., B.A., F.C.P.S. Communicated by Archibald Smith, Esq., M.A. Received May 26, 1859.

(Abstract.)

The object of the following investigation is to give a firmer basis to the calculus of operations, to assign the strict limits and connexion of the mathematical sciences, and to found them upon purely inductive considerations, without any metaphysical or *a priori* reasoning.

Starting with the indemonstrable but verifiable hypothesis, that objects exist external to the subject, we recognize equality as existing between objects with common and peculiar properties, in respect of their common properties. Operations, which, when performed on equal objects, produce equal objects as their result, are recognized as equal, in respect to the common properties considered in the equalities of the objects. When one operation is performed on an object, and another on the resultant object, the single operation by which the first object is transformable into the last is regarded as the *product* of the other two, the order of succession being important. When the resultant object is the same as the original operand, the product of the operations is termed *unity*. When two operations performed on the same object produce different resultant objects, the operation of transforming one of these resultant objects into the other, is regarded as the *quotient* of the two former operations. Two operations are termed *reciprocal* when their product is unity. Hence the quotient of two operations is the product of the one and of the reciprocal of the other. When two objects are combined in any manner so as to produce a third, and the two first are formable from any fourth by two known operations, the single operation by which the third object can be also formed from the fourth, is termed the same combination of the two first operations. From this we gain the conception of *null* or *zero*, as the operation of annihilating any object in respect to any place. The product of a combination of two operations and a third operation, is the same combination of the products

of each of the combined operations severally and the third operation, in the particular order thus specified, provided all the operations and products are performable on the same operand.

The above general conceptions and laws of combined operations hold for any operations whatsoever with their appropriate operand objects; but the nature of the operations and operands requires especial study. In *mathematics*, objects are only considered with respect to their three most general properties: first, as contemplatable in discontinuous succession, whence number and *Arithmetic*; secondly, as contemplatable in continuous succession, whence extension and *Geometry*; and thirdly, as contemplatable in a continuous succession bearing a relation to another continuous succession, whence motion in time and *Mechanics*. The problem of mathematics is, first, to discover the laws of these successions as respects results (that is, statically), by means of considerations drawn from contemplating operations (that is, dynamical); secondly, to investigate the relations of these laws, giving rise to statical algebra; thirdly, to reduce all dynamical to statical laws, as in dynamical algebra; and fourthly, to make the expression of all the results dependent on the most simple, viz. those of common arithmetic. The purpose of the problem is to prepare the mind for the further investigation of nature, and to increase practical power immediately.

In *Arithmetic* we conceive objects spread out in a *scale*, and by aggregating those contained between any one and the beginning of the scale, form statical groups, whose distinctive character is derived from the scale. The operation by which any group is formed from the first object is termed an *integer*, the especial laws of which are next investigated. All objects being interchangeable in respect to discontinuous succession, an aggregate is not changed by altering the disposition of its parts. This leads to the first two *laws of commutation and association in addition*. The possibility of arranging objects at once in two horizontal directions, and a third vertical direction, leads to the *laws of commutation and association in multiplication*. Combining these with the two former, we have the *law of commutative distribution*. From the laws of association in multiplication is immediately deduced the *law of repetition* or indices.

Having obtained these laws, we proceed to study their relations in the *algebra of integers*, first, statically, in order to reduce all results

to the form of a numerical integer; secondly, dynamically, considering the effect of a variation in the integer employed. This leads to the conception of a *formation* (Lagrange's "analytical function"), as a combination of a fixed and independently variable integer. Such a combination is, therefore, also itself dependently variable. The *inversion of formations*, whereby the independent variable is expressed as a formation of the dependent variable, immediately engages our attention. The inversion of a sum leads to a difference, with the limitation that the minuend should be greater than the subtrahend. The inversion of a product leads to a quotient, with the limitation that the dividend should be a multiple of the divisor. The inversions of a power lead to the root and logarithm, with increasing limitations. The study of discontinuous objects then allows the application of these inversions to the solution of problems in common life.

The operation by which any group in the arithmetical scale already described is formable from any other group in the same scale, leads to the conception of a *fraction*, necessarily expressible, according to the general laws of operation, as the quotient of two integers. The operands of such operations must admit of being separated into certain numbers of equal parts, or rather, in order that they may admit of *any* fractional operation, into *any* number of equal parts. Thus discontinuous approaches continuous succession. The *laws of fractions* are the same as the laws of integers, provided the indices used are all integers. The object of the statical *algebra of fractions* is to reduce all combinations of numerical fractions to numerical fractions. The inversion of formations is less limited than before. There is the same limitation respecting differences, but none respecting quotients. The attempt to convert all fractions into radical fractions (whose denominators are some powers of the radix of the system of numeration), leads to the conception of convergent infinite series, and hence allows an approximation to the inversion of a power with a constant index.

In *Geometry*, the notion of continuous succession or extension is derived from the motion of the hand, which recognizes separable but not separated parts. This motion gives the conception of surfaces, which by their intersections two and two, or three and three, give lines and points. Recognizing a line as the simplest form of exten-

sion, we distinguish the straight lines, which coincide when rotated about two common points, from the curves, which do not. These straight lines are shown to be fit operands for the integer and fraction operations. By moving one coinciding line over another so as to continue to coincide (by sliding), or to have one point only in common (by rotating), or no points in common (by translation), we obtain the conceptions of angles and parallels, which suffice to show that the exterior angle of a triangle is equal to the two interior and opposite, and that two straight lines meet or not according as the exterior angle they make with a third is not or is equal to, the interior angle. Angles are then considered statically as amounts of rotation not exceeding a semi-revolution. Proceeding to examine the relations of triangles and parallelograms, we discover the operation of taking a fraction of a straight line, and therefore of a triangle and of any rectilineal figure. We see that this operation is, in fact, the same as that of altering a third line into a fourth, so that the multiples of the third and fourth, when arranged in order of magnitude, should lie in the same order as those of the first and second when similarly arranged. The relation of two magnitudes, with respect to this order, we term their *ratio*, and the equality of ratios *proportion*. The inversion and alternation of the four terms of a proportion are now investigated. The operation of changing any magnitude into one which bears a given ratio to it, is called a *tensor*. The *laws of tensors*, being investigated, are shown to be the same as those of fractions. They, however, furnish the complete conception of infinite and infinitesimal tensors, by letting one or other of the magnitudes by which the ratio is given become infinite or infinitesimal. Thence is developed the law, that tensors differing infinitesimally are equal for all assignables. Consequently tensors may be represented by convergent series of fractions. The *algebra of tensors* allows of the inversion of a sum with the same limitation as in the case of fractions, the complete inversion of a product of tensors, and the practical inversion of a power with a constant integral index. This algebra applied to geometry allows of the investigation of all statical relations, that is, of all the *geometry of the ancients*, in which magnitudes alone were considered, without direction. In respect to areas, the consideration of the parallelogram swept out by one straight line translated so as to keep one point on another straight line, leads

to an independent *algebra of areas,* in which the generating lines are considered immediately. The laws of the relations of lines thus discovered, are shown to be identical with the laws of the relations of tensors. Consequently, with certain limitations, the whole of the algebra of tensors may be interpreted as results in the algebra of areas. This leads to a perfect conception of the principle of *homonomy,* or dissimilar operations having the same laws, and consequently the same algebra.

In *dynamical* or modern *geometry,* all lines are considered as in construction, having initial and final points. If the initial points of any two straight lines are joined to a third, not on either, and the two parallelograms be completed, the lines drawn from the point parallel to the given lines are dynamically equal to them; if these last lie on each other, the first two lines have the *same direction*; if the last have only one point in common and lie in the same straight line, the first have *opposite directions*; and if the last do not lie in the same straight line, the first have *different directions,* and the angle between the last is the angle between the first lines. Similar definitions can be given of direction in the case of angles and circular arcs. If from the final point of any line we draw a line equal to a second, and join the initial point of the first with the final point of the line thus drawn, we are said to *append* the second to the first, and the joining line is called the *appense* of the other two. The *laws of appension* are shown to be the same as those of addition, and are hence expressible by the same signs of combination, the difference in the objects combined preventing any ambiguity. We thus get the conception of a point as an annihilated line.

The tensor operation, considered dynamically, leads to the operation of changing a line dynamically so that it should bear the same relation to the result as two given lines bear to each other in magnitude and direction. This assumes three principal forms according to the difference of direction. If there is no difference of direction, the operation is purely a tensor. If the directions differ by a semi-revolution, the rotation of one line into the position of the other may take place on any plane. The operation is then termed a *negative scalar*; the tensor, which includes the operation of turning through any number of revolutions, is distinguished as a *positive scalar.* If the rotation be through any angle, but always on the same plane,

the operation is here termed a *clinant.* If the rotation may take place on any variable plane, the operation is a *quaternion.*

The *laws of scalars* are immediately proved to be the same as those of tensors, but in addition they introduce the idea of negativity. This enables us in the *algebra of scalars,* to invert a sum generally, and thus allows of a perfect inversion of the first two formations. But a power with a fixed integral exponent can only be inverted on certain conditions. This partial inversion, however, leads to a solution of quadratic equations, and to a proof that formations consisting of a sum of integral powers, cannot be reduced to null by more scalar values of the variable than are marked by its highest exponent. Hence if such a formation is always equal to null, all the coefficients of the variable must be null. We thus obtain the method of indeterminate coefficients, by which we are enabled to discover a series which obeys the laws of repetition with respect to its variable, and becomes equal to a power when its variable is an integer. This enables us to define a power with any index, as this series, and hence to attempt the inversion of powers with variable indices, which we succeed in accomplishing under certain conditions. This investigation introduces the logarithm of a tensor, powers with fractional and negative exponents, and the binomial theorem for these powers. It also induces us to consider the *laws of formators,* or the operations by which a formation of any variable is constructed. They are shown to be commutative and associative in addition, associative in multiplication, directly distributive and repetitive, but not generally commutative in multiplication, nor even inversely distributive. When formators are commutative in multiplication and distribution, they are entirely homonomous with scalars, which may even be considered as a species of formators. The results of the former investigation, therefore, show that logarithms, fractional and negative powers, and the binomial theorem hold for these commutative formators.

The necessity of tabulating logarithms and of approximating to the solutions of equations, leads to the consideration of a method of deriving consecutive values of formations for known differences of the variable, and of interpolating values of the same formation for intermediate values of the variable; that is, the *algebra of differences.* Considering the two operations of altering a formation by increasing the variable, and taking the difference between two different values

of the formation (of which operations the first is necessarily unity added to the second), we regard them as formators, and immediately apply the results of that algebra, which furnishes all the necessary formulæ. For approximating to the roots of equations, we require to consider the case where the variable changes infinitesimally, thus founding the *algebra of differentials*, which is, in fact, a mere simplification of that of differences, owing to all the results being ultimately calculated for assignables only. Finally, to find the alteration in a formation of commutative formators, when the variable formator is increased by any other formator, we found the *algebra of derivatives*.

In applying the results of *scalar* algebra to *geometry*, we start with the fundamental propositions that the appense of the sides of an enclosed figure taken in order is a point, and that when the magnitude and direction of the diagonal of a parallelogram or parallelopipedon, and lines parallel the sides which have the same initial point as the diagonal, are given, the whole figures are completely determined. In order to introduce scalars, a unit-sphere is imagined, with its radii parallel to the lines in any figure, and in known directions. Any line can then be represented as the result of performing a scalar operation on the corresponding radius.

The first object is to reduce the consideration of angles to that of straight lines, by the introduction of cosines and sines, which are strictly defined as the scalars represented by the relation of the abscissa to the abscissal radius, and the ordinate to the ordinate radius respectively. These definitions immediately lead to the relations between the cosines and sines of the sums of two angles, and those of the angles themselves, whatever be their magnitude or direction, and thus found *goniometry*.

Defining a *projection* of any figure on any plane to be that formed by joining the points on that plane corresponding according to any law with those of the figure, we have the fundamental relation that, if the first, and therefore the second figure is enclosed, the appense of the sides of the second in the order indicated by the sides of the first, is a point. The orthogonal projection of any figure, by means of planes drawn perpendicular to any line, being all in one line, each projection can be represented as the result of a scalar operation performed on the same unit radius, and hence this projection leads to one

invariable relation between scalars. By choosing three lines at right angles to each other on which to project, we obtain three scalar relations from every solid figure. If the figure is plane, then by projecting on a line and on a perpendicular to that line, we get two scalar relations.

Applying these results to *transversals*, where a line parallel to one unit radius cuts several other unit radii, produced either way if necessary, we obtain, by considering *two* intersected radii, the results of *trigonometry*, and by considering *three* or *four* intersected radii, those of *anharmonic ratios*.

As any line drawn from the centre of the unit-sphere may be considered as the appense of three lines drawn along or parallel to three given unit radii, it may be expressed as the sum of the results of three scalar operations performed on these radii respectively. By properly varying these three scalars, the final point of the line may be made to coincide with any point in space. But if there be a given relation between the scalars, then the number of points will be limited, and the whole number of the points constitutes the locus of the original concrete equation referred to the accessory abstract equation. The consideration of this entirely new view of *coordinate geometry* is reserved for a second memoir.

Proceeding next to the *laws of clinants*, we readily demonstrate that they are the same as the laws of scalars; they introduce a new conception, however, that of rotating through an angle not necessarily the same as a semi-revolution, that is, of a plane versor. By the concrete equation of coordinate geometry, it is immediately shown that all clinants can be expressed as the sum of a scalar, and of the product of a scalar by a fixed, but arbitrarily chosen versor. The simplest versor to select is the quadrantal versor, which, under the name of quadrantation, is now studied. The two addends of a clinant, considered as a sum, are called its scalar and vector; its two factors, considered as a product, are its tensor and versor. The laws of these parts are then studied.

The statical *algebra of clinants* has for its object the reduction of all combinations of clinants given in the standard form of the sum of a scalar and vector, to a clinant of the same form. The application of this to the series obtained for a general scalar power, leads to two series, called cosines and sines of the variables, as distinguished from

the goniometrical cosines and sines of an angle, with which they are ultimately shown to have a close connexion, which can be rendered most evident by assuming as the unit-angle that subtended by a circular arc of the length of its radius. Studying these series quite independently of these relations to angles, we discover that they bear to each other the same relations as the goniometrical cosines and sines, and that if the least tensor value of the variable for which the cosine series becomes null, is known, all its other values can be found by multiplying this by four times any scalar integer. This last product must be added to the least tensor value of the variable for which both the cosine or the sine series become equal to given scalars, in order to find all the solutions of such equations. Supposing the values of such series tabulated by the method of differences for all scalar values of the variable, so that such least tensor values can always be found, we are now able to assign the meaning of any power whose base and index are both clinants, and the logarithm of any clinant. This enables us to invert completely all the simple formations, sum, product, power with variable base and constant index, or constant base and variable index; and hence to solve all equations of four dimensions with clinant coefficients, and to show that every formation consisting of a sum of integral powers with clinant coefficients, can be expressed as a product of as many simple formations as is determined by the highest index of the variable. The cosine and sine series can also be generally inverted. The versor of any clinant having a known angle (which is always equal to the cosine of its angle added to the product of the sine of its angle into a quadrantal versor), can now be shown to equal the cosine series added to the sine series multiplied by a quadrantal versor, when the variable of the series is the scalar ratio of the angle of the clinant to the angle subtended by a circular arc equal to its radius. From this the ratio of the circumference to the diameter of a circle is shown to be twice the least tensor value of the variable, for which the cosine series is equal to null; and as that value can be readily assigned in a convergent series, the former ratio is determined. The same investigation shows the relation already mentioned between the goniometrical cosines and sines, and the cosine and sine series.

Clinant algebraical geometry allows us to interpret all results of clinant algebra when referred to lines on one plane. It thus fur-

nishes a complete explanation of the "imaginary" points and lines in the theory of *anharmonic ratios*, when viewed in relation to the unit radii, as already explained. In the case of *coordinate geometry* of two, and even three dimensions, the possibility of interpreting the results of a clinant operation performed on a given unit radius in a given plane, allows us to understand the whole theory of "imaginary" intersections. The theory of *scalar and clinant algebraical coordinate geometry* will form the subject of a future memoir.

Proceeding to *quaternions*, we find their laws to be the same as those of clinants while the plane remains unaltered; but if the plane is alterable, they cease to be commutative in multiplication, that relation being replaced by one between certain related quaternions called their conjugates. This makes the *algebra of quaternions* (which is not here systematized, as being too recent) entirely different from that of scalars.

In *mechanics* the motion of any point is not considered absolutely as in dynamical geometry, but relatively to some external, constant, independent motion, as the apparent motion of the fixed stars; this gives the conception of time. But the necessity of considering the motion not merely of a point, but of a body, gives rise to the comparison of the motions of various bodies, and to a conception of their equality, when the products of their velocities, multiplied by a constant which is always the same for the same body, but different for different bodies, are equal. This constant is the mass, which in bodies of the same kind varies as the volume.

By considering the case of the mutual destruction of motion, we eliminate time and simplify the problem, thus founding *statics*; and by conceiving the motion of any body to be destroyed by the application of variable motions equal and opposite to those actually existent, we reduce *dynamics* to statics.