III. "On the Calculus of Functions." By W. H. L. Russell, Esq., A.B. Communicated by A. Cayley, F.R.S. Received October 31, 1861.

(Abstract.)

One of the first efforts toward the formation of the calculus of functions is due to Laplace, whose solution of the functional equation of the first order, by means of two equations in finite differences, is well known. Functional equations were afterwards treated systematically by Mr. Babbage; his memoirs were published in the Transactions of this Society, and there is some account of them in Professor Boole's Treatise on the Calculus of Finite Differences. A very important functional equation was solved by Poisson in his memoirs on Electricity; which suggested to me the investigations I have now the honour to lay before the Society.

I have commenced by discussing the linear functional equation of the first order with constant coefficients, where the subjects of the unknown functions are rational functions of the independent variable, and have shown how the solution of such equations may in a variety of cases be effected by series, or by definite integrals. I have then considered functional equations with constant coefficients of the higher orders, and have proved that they may be solved by methods similar to those used for equations of the first order. I have next proceeded with the solution of functional equations with variable coefficients. In connexion with functional equations, I have considered equations involving definite integrals, and containing an unknown function under the integral sign; the methods employed for their resolution depend chiefly upon the solution of functional equations, as effected in this paper. The calculus of functions has now for a long time engaged the attention of analysts; and I hope that the present investigations will be found to have extended its power and resources.

IV. "On Tschirnhausen's Transformation." By Arthur Cayley, Esq., F.R.S. Received November 7, 1861.

(Abstract.)

The memoir of M. Hermite, "Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré," Comptes Rendus,

t. xlvi. p. 961 (1858), contains a very important theorem in relation to Tschirnhausen's transformation of an equation $f(x)=0$ into another of the same degree in y, by means of the substitution $y=\phi x$, where ϕx is a rational and integral function of x. In fact, considering for greater simplicity a quartic equation,

$$(a, b, c, d, e \between x, 1)^4 = 0,$$

M. Hermite gives to the equation $y=\phi x$ the following form,

$$y=a\mathrm{T}+(ax+b)\,\mathrm{B}+(ax^2+4bx+6c)\,\mathrm{C}+(ax^3+4bx^2+6cx+d)\,\mathrm{D}$$

(I write B, C, D in the place of his T_0, T_1, T_2), and he shows that the transformed equation in y has the following property: viz., every function of the coefficients which, expressed as a funtion of a, b, c, d, e, T, B, C, D, does not contain T, is an *invariant*, that is, an invariant of the two quantics

$$(a, b, c, d, e \between \mathrm{X}, \mathrm{Y})^4, \ (\mathrm{B}, \mathrm{C}, \mathrm{D} \between \mathrm{Y}, -\mathrm{X})^2.$$

This comes to saying that if T be so determined that in the equation for y the coefficient of the second term (y^3) shall vanish, the other coefficients will be invariants; or if in the function of y which is equated to zero we consider y as an absolute constant, the function of y will be an invariant of the two quantics. It is easy to find the value of T; this is in fact given by the equation

$$0=a\mathrm{T}+3b\mathrm{B}+3c\mathrm{C}+d\mathrm{D};$$

and we have thence for the value of y,

$$y=(ax+b)\,\mathrm{B}+(ax^2+4bx+3c)\,\mathrm{C}+(ax^3+4bx^2+6cx+3d)\,\mathrm{D};$$

so that for this value of y the function of y which equated to zero gives the transformed equation will be an invariant of the two quantics. It is proper to notice that in the last-mentiond expression for y, all the coefficients except those of the term in x^0, or $b\mathrm{B}+3c\mathrm{C}+3d\mathrm{D}$ are those of the binomial $(1, 1)^4$, whereas the excepted coefficients are those of the binomial $(1, 1)^3$; this suffices to show what the expression for y is in the general case.

I have in the two papers, "Note sur la transformation de Tschirnhausen" and "Deuxième Note sur la transformation de Tschirnhausen" (Crelle, t. lviii. pp. 259 and 263, 1861), obtained the transformed equations for the cubic and quartic equations; and by means of a grant from the Government Grant Fund, I have been enabled to procure the calculation by Messrs. Davis and Otter, under my

superintendence, of the transformed equation for the quintic equation. The several results are given in the present memoir; and for greater completeness, I reproduce the demonstration which I have given in the former of the above-mentioned two notes, of the general property, that the function of y is an invariant. At the end of the memoir I consider the problem of the reduction of the general quintic equation to Mr. Jerrard's form $x^5+ax+b=0$.

December 12, 1861.

Major-General SABINE, R.A., President, in the Chair.

In accordance with the announcement made from the Chair at the last Meeting, the question of Mr. Sievier's readmission was put to the vote, and was decided in the affirmative. The President accordingly declared that Mr. Sievier was readmitted into the Society.

The following communications were read:—

I. "On a Series for calculating the Ratio of the Circumference of a Circle to its Diameter." By AMOS CLARKSON, Esq. Communicated by Professor STOKES, Sec. R.S. Received September 27, 1861.

The ratio (π) of the circumference to the diameter of a circle may be calculated by the following series:—

$$\pi=\frac{8}{3}\left\{1-\frac{1}{3.10}-\frac{2}{3.5.10^2}-\frac{2.4}{3.5.7.10^3}-\dots\right\}$$
$$+\frac{4}{7}\left\{1-\frac{2}{3.10^2}-\frac{2.2^2}{3.5.10^4}-\frac{2.4.2^3}{3.5.7.10^6}-\dots\right\}. \quad . \quad (1)$$

This series may be thus established. We have, as is well known,

$$\frac{\pi}{4}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7};$$

and denoting by c the arc of which the tangent is t,

$$c=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\dots$$

Put

$$t^2=\frac{1}{x-1},$$