"On an Application of the Theory of Scalar and Clinant Radical Loci." By ALEXANDER J. ELLIS, Esq., B.A., F.C.P.S. Communicated by A. CAYLEY, Esq. Received February 20, 1861. Read March 14, 1861*.

1. The following investigation, which contains a correction and extension of that in Plücker's 'System der Geometrie' (§ 3. art. 64), is a direct application of the theories in the writer's paper "On Scalar and Clinant Algebraical Coordinate Geometry" (Proceedings, vol. x. pp. 415–426), to which reference must be made for an explanation of the notation and terminology.

2. Let $f(x, y)$ be an algebraical formation (function) of x, y and of $n+2m$ dimensions, such that when any scalar value is attributed to x, $f=0$ has n scalar (possible) and $2m$ clinant (imaginary) roots, and let λ be the coefficient of y^{n+2m}. Then

$$f(x, y)=\lambda(y-f_1x)\dots(y-f_nx)\times(y-f_1'x)(y-f_2'x)\dots$$
$$(y-f_1^{(m)}x)(y-f_2^{(m)}x).$$

3. Now let PQ be a line which, when produced, will cut the curve, whose equations are

$$\mathrm{OM}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OB}, \qquad f(x, y)=0,$$

(where x and y are scalar, and OI is in the same straight line with OP, and OB is a line equal in length to OI, and parallel to and in the same direction with PQ) in the n points $M_1, M_2, \dots M_n$. Then, if $\mathrm{OP}=x_1\,.\,\mathrm{OI}$, where x_1 is scalar, $\frac{\mathrm{PM}_1}{\mathrm{OB}}, \frac{\mathrm{PM}_2}{\mathrm{OB}}, \dots \frac{\mathrm{PM}^n}{\mathrm{OB}}$ will represent the n scalar roots $f_1x_1, f_2x_1, \dots f_nx_1$. Hence if $\mathrm{PQ}=y_1\,.\,\mathrm{OB}$, the point Q not being necessarily a point in the curve, any one of the factors $y_1-f_rx_1$ will be represented by $\frac{\mathrm{PQ}-\mathrm{PM}_r}{\mathrm{OB}}=\frac{\mathrm{M}_r\mathrm{Q}}{\mathrm{OB}}$.

Now for the clinant roots†, put $y=r+\mathrm{i}\,.\,s$, where the Roman $\mathrm{i}=\sqrt{-1}$, and r and s are scalar, then must

$$f(x, y)=\mathrm{F}_1(x, r, s)+\mathrm{i}\,.\,s^n\,.\,\mathrm{F}_2(x, r, s),$$

because there are n scalar roots for each of which $s=0$. For the

* An abstract of this Paper has already appeared in the "Proceedings," page 141. It is now printed in full by order of the Council.

† Plücker only says that "no imaginary point of intersection must be neglected" (System, p. 45), but he gives no means for taking them into consideration.

clinant roots, then, we have $F_1=0$, and $F_2=0$; whence we find, by alternately eliminating s and r, $F_3(x, r)=0$, and $F_4(x, s)=0$.

Now describe the loci of the points R and S, where

$$OR=x\,.\,OI+r\,.\,OB, \qquad F_3(x, r)=0,$$

and

$$OS=x\,.\,OI+s\,.\,OB, \qquad F_4(x, s)=0.$$

Then from the nature of the clinant roots, which will be in pairs of the form $r_1 \pm i\,.\,s_1$, if we produce PQ to cut these curves, for every point R' in which it cuts the first there will be two points S'_1, S'_2, in which it cuts the second. If, then, from the point R' we draw $R'M'_1=i\,.\,PS'_1$ and $R'M'_2=i\,.\,PS'_2$, two corresponding clinant roots of the equation will be $\frac{PM'_1}{OB}$, $\frac{PM'_2}{OB}$, so that

$$y_1-f_1^{(r)}x_1=\frac{PQ-PM_1^{(r)}}{OB}=\frac{M_1^{(r)}Q}{OB},$$

and

$$y_1-f_2^{(r)}x_1=\frac{PQ-PM_2^{(r)}}{OB}=\frac{M_2^{(r)}Q}{OB}.$$

See the example and figure in (11).

Hence we shall have for the geometrical representation of any algebraical formation, by (2),

$$f(x_1, y_1)=\lambda\,.\,\frac{M_1Q}{OB}\cdots\frac{M_nQ}{OB}\times\frac{M'_1Q}{OB}\,.\,\frac{M'_2Q}{OB}\cdots\frac{M_1^{(m)}Q}{OB}\,.\,\frac{M_2^{(m)}Q}{OB}.$$

4. Next suppose that the origin of the coordinates be altered to O', so that $O'I'=OI$, and $O'I'$, when produced, passes through Q. This will be equivalent to putting $y \pm b$ instead of y, and the result will not disturb the coefficient of y^{n+2m}. Also since one of the radical loci for the scalar values remains unaltered, and the others (which are the axes) are parallel to the former, the points $M_1, M_2, \ldots M_n$ in which PQ, when produced, will cut the curve, remain the same, and consequently there are the same number of scalar roots as before.

5. But as the terms themselves of the equation $f(x, y)=0$ are altered, the radical loci of R and S in (3) will be altered, and as the distances are now measured from Q and not from P, the lines $QR'_1+i\,.\,QS'_3$, $QR'_1+i\,.\,QS'_4$ will now be different and $=Qm'_1$, Qm'_2. If then $O'Q=x_2\,.\,O'I'$, and $QP=y_2\,.\,BO=-y_2\,.\,OB$, then

$$f(x_2, -y_2)=\lambda\,.\,\frac{M_1P}{OB}\cdots\frac{M_nP}{OB}\times\frac{m'_1P}{OB}\,.\,\frac{m'_2P}{OB}\cdots\frac{m_1^{(m)}P}{OB}\,.\,\frac{m_2^{(m)}}{OB}.$$

Or, replacing OB in each case by BO, we have

$$f(x_2, y_2)=\lambda' . \frac{M_1P}{BO} \dots \frac{M_nP}{BO} \times \frac{m'_1 P}{BO} \dots \frac{m_2^{(m)}P}{BO};$$

where, since there are $n+2m$ factors,

$$\lambda'=(-1)^{n+2m} . \lambda.$$

6. But if, instead of simply changing the origin, we change the inclination of PQ and make it parallel to OC, then if OB and OC make the angles β and γ with OI, and OJ be at right angles to OI, the lengths of OI, OJ, OB, OC being the same, and M be any point, we have

$$OM=x . OI+y . OB=(x+y \cos \beta) . OI+y \sin \beta . OJ,$$

and

$$OM=x' . OI+y' . OC=(x'+y' \cos \gamma) . OI+y' \sin \gamma . OJ,$$

which give two simple equations to determine x, y in terms of x', y', so that on substitution $f(x, y)$ becomes $\phi(x', y')$, a formation of the same number of dimensions. But the coefficient of y'^{n+2m} will be different from that of y^{n+2m}. Let it $=\mu$. The curve for the scalar values of x', y' in

$$OM=x' . OI+y' . OC, \qquad \phi(x', y')=0$$

will be precisely the same as that in (3); but as PQ, having a different inclination, will cut it in very different places or not at all, the numbers of scalar and clinant roots of $\phi=0$ may be individually different from, although their combined number will be the same as, those of $f=0$. If then there be h scalar and $2k$ clinant roots of $\phi=0$, where $h+2k=n+2m$, and we determine N_1, $N_2 \dots N'_1$, $N'_2 \dots$ in the same way as we previously determined M_1, M_2, ... M'_1, M'_2, ... we find

$$f(x_1, y_1)=\phi(x_1, y_1)=\mu . \frac{N_1Q}{OC} \dots \frac{N_hQ}{OC} \times \frac{N'_1 Q}{OC} \dots \frac{N_2^{(k)} Q}{OC}.$$

7. This investigation shows that if there be any number of lines drawn from a point, cutting those radical loci of an equation for scalar and clinant roots which correspond to the inclination of these lines to the axis and to the position of the origin, coefficients λ, μ, &c. can always be assigned such as to make the product of these ratios each equal to the other; and that these coefficients have only to be multiplied by $(-1)^{n+2m}$, in order to give the coefficients corresponding to

such a change of origin as would make the axis pass through the other extremities of the lines.

8. Now let $O_1O_2 \ldots O_nO_1$ be a completely enclosed polygon, the sides of which, when produced, cut the radical loci of a known algebraical equation of n dimensions in known points. Let the product of the series of ratios formed as above (3), corresponding to lines drawn from the point O_r in the direction $O_r O_{r+1}$ (on which last-named line the unit line $O_r O'_{r+1}$, corresponding to OI in (3), is measured), be represented by

$$[O_{r,\, r+1}M \div O_r O'_{r+1}].$$

And let $\lambda_{r,\, r+1}$ be the coefficient necessary to make this product a correct representation of the formation on the left-hand side of the given equation for the values of x and y, corresponding to the point O_r and the direction $O_r O_{r+1}$.

Then for the point O_{r+1} and the direction $O_{r+1} O_r$ (for which the corresponding unit line is $O_{r+1} O'_r$), we shall have the product of the corresponding series of ratios represented by

$$\lambda_{r+1,\, r}[O_{r+1,\, r}\, M \div O_{r+1}\, O'_r],$$

where

$$\lambda_{r+1,\, r} = (-1)^n \,.\, \lambda_{r,\, r+1}.$$

Proceeding then round the polygon, and forming the products for each of the two sides terminating at each point, we have by (7),

$$\begin{aligned}
\lambda_{1,\, n}[O_{1,\, n}\, M \div O_1 O'_n] &= \lambda_{1,\, 2}\, [O_{1,\, 2}\, M \div O_1 O'_2] \\
(-1)^n \lambda_{1,\, 2}[O_{2,\, 1} M \div O_2 O'_1] &= \lambda_{2,\, 3}\, [O_{2,\, 3} M \div O_2 O'_3], \\
\ldots &= \ldots \\
(-1)^n \lambda_{n-2,\, n-1}\, [O_{n-1,\, n-2} M \div O_{n-1}\, O'_{n-2}] &= \lambda_{n-1,\, n}[O_{n-1,\, n} M \div O_{n-1}\, O'_n] \\
(-1)^n \lambda_{n-1,\, n}\, [O_{n,\, n-1}\, M \div O_n\, O'_n] &= \lambda_{n,\, 1}\, [O_{n,\, 1}\, M \div O_n O'_1].
\end{aligned}$$

Hence multiplying all these n equations together, and remembering that $\lambda_{1,\, n} = (-1)^n \,.\, \lambda_{n,\, 1}$, and that $(-1)^{nn} = (-1)^n$, since nn and n are both odd or both even, we have

$$\begin{aligned}
&[O_{1,\, n}\, M \div O_1 O'_n] \,.\, [O_{2,\, 1}\, M \div O_2 O'_1] \ldots [O_{n,\, n-1}\, M \div O_n O'_{n-1}] \\
&= (-1)^n \,.\, [O_{1,\, 2}\, M \div O_1\, O'_2] \,.\, [O_{2,\, 3}\, M \div O_2 O'_3] \ldots [O_{n,\, 1} M \div O_n O'_1].
\end{aligned}$$

Now since $O_{r,\, r+1}\, M$ and $O_{r+1,\, r} M$ are on the same straight line, their ratio is scalar; and since $O_1 O'_n = -1 \,.\, O_n O'_1$, we have, by an

obvious extension of the notation, and by altering the order of the factors,

$$[O_{1,2}M \div O_{2,1}M] . [O_{2,3}M \div O_{3,2}M] \ldots [O_{n,1}M \div O_{1,n}M]$$
$$= (-1)^n . [O_1O'_2 \div O_2O'_1] . [O_2O'_3 \div O_3O'_2] \ldots [O_nO'_1 \div O_1O'_n]$$
$$= (-1)^n . [(-1)^n . (-1)^n . \ldots (n \text{ factors}) \ldots (-1)^n]$$
$$= (-1)^n . (-1)^{nn} = (-1)^n . (-1)^n = (-1)^{2n} = 1 \, *.$$

9. Hence if all the roots are scalar, that is, if each side of the polygon cuts the scalar radical locus in as many points as its abstract equation has dimensions, we have the following fundamental proposition of the theory of transversals:—

"If we take the points at the extremities of a side of a polygon, and find the ratios of the distances of these points from the intersections of that side with a curve, and proceed in this manner regularly round the polygon in one direction, the product of all these ratios will be unity."

10. Thus if the triangle $O_1O_2O_3$ cut a straight line in A, B, C so that these points lie on O_2O_3, O_1O_3, O_1O_2 respectively, then

$$\frac{O_1C}{O_2C} \cdot \frac{O_2A}{O_3A} \cdot \frac{O_3B}{O_1B} = 1 \, \dagger.$$

If the same triangle $O_1O_2O_3$ cut a conic in the points A_1, A_2 by O_2O_3; B_1, B_2 by O_1O_3; C_1, C_2 by O_1O_2 respectively, then

$$\frac{O_1C_1}{O_2C_1} \cdot \frac{O_1C_2}{O_2C_2} \times \frac{O_2A_1}{O_3A_1} \cdot \frac{O_2A_2}{O_3A_2} \times \frac{O_3B_1}{O_1B_1} \cdot \frac{O_3B_2}{O_1B_2} = 1.$$

* Plücker says, $= \pm 1$, and adds, that "it is immediately evident that when the curve is of an even order, the upper sign must be taken; and when it is of an uneven order, either the upper or the lower, according as the number of the angles of the polygon is even or odd" (System, p. 44). Accordingly most of his examples, as shown in (10), give an erroneous result. It must be observed that Plücker does not give the equation in the form above written, but places the product of all the antecedents of the ratios on one side, and the product of all the consequents on the other. As, however, these antecedents and consequents are *directed* lines, such products have no proper meaning. He has in fact considered those directed lines as scalars, but has neglected to assign the directed unit lines with respect to which they are to be considered as scalars. Hence his error.

† Plücker makes the second side of this equation $= -1$, which is manifestly erroneous. But if O_1a, O_2b, O_3c be drawn from O_1, O_2, O_3 through any point M to O_2O_3, O_1O_3, O_1O_2 respectively, then by considering the intersections of the triangles O_1O_2a, O_1O_3a with O_3c and O_2b respectively, we immediately deduce from the equation in the text,

$$\frac{O_1c}{O_2c} \cdot \frac{O_2a}{O_3a} \cdot \frac{O_3b}{O_1b} = -1.$$

If the triangle touches the conic in A, B, C, then A_1, A_2 coincide with A, &c., and the above equation becomes

$$\frac{O_1C}{O_2C}\cdot\frac{O_2A}{O_3A}\cdot\frac{O_3B}{O_1B}=1 \text{ or } -1.$$

The first case shows that the three points lie on a straight line; the second, that the three lines O_1A, O_2B, O_3C meet in a point*.

If the point C were to lie at an infinite distance from O_1 and O_2, then since $O_1C=O_1O_2+O_2C$, and $\frac{O_1C}{O_2C}=1+\frac{O_1O_2}{O_2C}$, where the last term is infinitesimal, we have merely to consider $\frac{O_1C}{O_2C}=1$, or omit it from the product†. Thus if we take as two sides of the triangle the asymptotes of an hyperbola meeting at O_1, and suppose the third side O_2O_3 to cut the curve in A_1 and A_2, then since both O_1O_2 and O_1O_3 cut the curve at an infinite distance, the equation reduces to

$$\frac{O_2A_1}{O_3A_1}\cdot\frac{O_2A_2}{O_3A_2}=1;$$

whence we readily find

$$O_3A_2=O_2A_2, \text{ or } =-O_2A_1.$$

The first value is impossible, because A_2 lies between O_2 and O_3; the second gives the well-known property of the hyperbola‡.

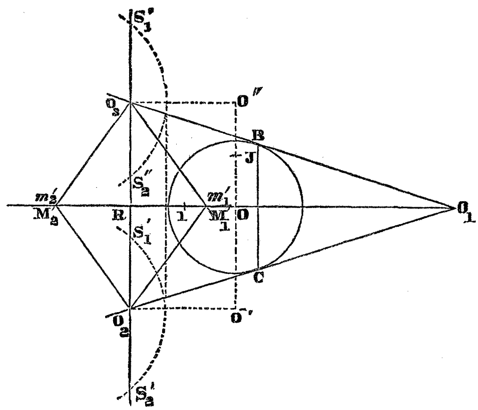
11. As regards the clinant roots, or the intersections of the sides of the polygon with the clinant radical loci, the wording of the proposition (9) requires to be changed to adapt it to the more complicated form of the product; and perhaps it may be considered sufficient to refer to the forms of the products in the final equation of (8),

* Plücker interchanges the two cases, and makes the positive result show the intersection of the three lines in a point, which is shown to be wrong in the last note. He rejects the first case, "because a conic can only be cut in two points by a straight line" (p. 45). But in this case both the conic and the triangle reduce to a straight line (that is, the points O_1, O_2, O_3, A, B, C all lie in the same straight line), and the points A, B, C do not indicate contacts, but double-intersections, which have the same analytical expression, owing to the rejection of infinitesimals in the above calculation of contact.

† Plücker says that "in every such case there must be a fresh change of sign" (p. 46). This apparently arises from his having neglected to particularize the directed unit, or to note the primary relation $O_1C=O_1O_2+O_2C$.

‡ Plücker makes $O_2A_1=O_3A_2$ (p. 46); that is, he neglects the relation of direction, which is all-important in such investigations.

and append a single simple example in which the product admits of being easily formed.



Let the triangle $O_1O_2O_3$ touch a circle BC in the points B and C, and let the side O_2O_3 be perpendicular to the line O_1O drawn through the centre of the circle, O, but be wholly exterior to the circle. Take O as the origin of coordinates, and suppose that the equations to the circle are

$$OP = x\,.\,OI + y\,.\,OJ, \quad x^2+y^2=c^2.$$

Now change the origin to O', where $OO' = \frac{1}{2}O_3O_2 = -b\,.\,OJ$. Then the equations to the circle become

$$O'P = x'\,.\,OI + y'\,.\,OJ, \quad x'^2+(y'-b)^2=c^2.$$

And for the radical loci, when y' is clinant, we have (3)

$$O'R = x'\,.\,OI + r'\,.\,OJ, \quad r'-b=0,$$

that is, the line O_1O, and

$$O'S = x'\,.\,OI + s'\,.\,OJ, \quad x'^2-s'^2=c^2,$$

that is, the rectangular hyperbola $S'_1\,S'_2$. The roots will then be

$$\frac{O_2R + i\,.\,O_2S'_1}{OJ} = \frac{O_2R + RM'_1}{OJ} = \frac{O_2M'_1}{OJ}$$

and

$$\frac{O_2R + i\,.\,O_2S'_2}{OJ} = \frac{O_2R + RM'_2}{OJ} = \frac{O_2M'_2}{OJ},$$

and the factors corresponding to them in the final product in (3) will be

$$\frac{O_2O_3 - O_2M'_1}{OJ}\,.\,\frac{O_2O_3 - O_2M'_2}{OJ} = \frac{M'_1O_3}{OJ}\,.\,\frac{M'_2O_3}{OJ}.$$

Similarly, by transferring the origin to O'', where $OO''=b \,.\, OJ$, the equations to the circle become

$$O''P=x''.\,OI+y''.\,OJ, \quad x''^2+(y''+b)^2=c^2.$$

The radical loci, when y'' is clinant, will have for their equations

$$O''R=x''.\,OI+r''.\,OJ, \quad r''+b=0,$$

that is, the line O_1O, and

$$O''S=x''.\,OI+s''.\,OJ, \quad x''^2-s''^2=c^2,$$

that is, the rectangular hyperbola S''_1, S''_2. Hence the roots will be

$$\frac{O_3R+i\,.\,O_3S''_1}{OJ}=\frac{O_3R+Rm'_1}{OJ}=\frac{O_3m'_1}{OJ}$$

and

$$\frac{O_3R+i\,.\,O_3S''_2}{OJ}=\frac{O_3R+Rm'_2}{OJ}=\frac{O_3m'_2}{OJ};$$

and the factors corresponding to them in the product (5) will be, after replacing OJ by JO, as there shown,

$$\frac{O_3O_2-O_3m'_1}{JO}\,.\,\frac{O_3O_2-O_3m'_2}{JO}=\frac{m'_1O_2}{JO}\,.\,\frac{m'_2O_2}{JO}.$$

Hence the final equation in (8) gives

$$\left(\frac{O_1C}{O_2C}\right)^2.\left(\frac{O_3M'_1}{O_2m'_2}\,.\,\frac{O_3M'_2}{O_2m'_1}\right).\left(\frac{O_3B}{O_1B}\right)^2=1.$$

Now the second factor of the product on the left-hand side of this equation, is itself a product of which each factor $=-1$, and it may consequently be omitted from the equation, so that

$$\frac{O_1C}{O_2C}=\pm\frac{O_1B}{O_3B}.$$

The under sign cannot be taken, because it would necessitate one of the points B or C lying without, and the other within the corresponding side of the triangle. Hence we must have

$$\frac{O_1C}{O_2C}=\frac{O_1B}{O_3B};$$

that is, BC will be parallel to O_3O_2, a result which readily follows from other considerations.

12. The great complexity of the process, and the necessity of knowing the algebraical form of the equation to the curve, will probably prevent this employment of clinant roots from having any practical value beyond the completion of the theoretical harmony between the abstract and the concrete, the algebraical formula and the geometrical figure.