there are enlargements resembling ganglia, and that the hepatic ducts and gall-bladder are largely supplied with these gangliform plexuses of nerves, which all arise from the semilunar ganglion and solar plexus."

In a Postscript, received October 4, 1862, the author adds, that from an elaborate dissection which he has made since the date of the paper, "it is demonstrated—

"1. That the nerves of the liver take their origin from ganglia situated around the root of the hepatic artery, which are intimately connected with, or actually form a part of, the semilunar ganglion of the great sympathetic.

"2. That the hepatic nerves, thus originating, proceed to the liver along with the hepatic artery, hepatic veins, the vena portæ, and the hepatic ducts.

"3. That the hepatic nerves, on reaching the liver, send numerous branches to the different lobes, along with the ramifications of the hepatic artery to every part of the organ, and that plexuses of nerves accompany the most minute branches of the arteries.

"4. That the hepatic and cystic ducts are surrounded with plexuses of ganglia and nerves, and that nerves accompany the arteries of the gall-bladder throughout their distribution.

"5. That besides these nerves, accompanying the trunk and branches of the hepatic artery and surrounding the cystic and hepatic ducts, there is a great system of ganglionic nerves distributed to the walls of the vena portæ."

V. "On the Volumes of Pedal Surfaces." By T. A. Hirst, F.R.S. Received 28th August, 1862.

(Abstract.)

Since the term "pedal surface" has but recently been definitively adopted*, it may be well to state that it indicates, simply, the locus of the feet of perpendiculars let fall from a fixed point, the *pedal origin*, upon all the tangent planes of a given surface. It is sometimes convenient, too, to regard the pedal surface as the envelope of a sphere, whose diameter is the radius vector from the pedal origin

* A Treatise on the Analytic Geometry of Three Dimensions. By George Salmon, D.D. 1862.

to any point on the primitive surface. The primitive surface remaining unaltered, the form and magnitude of its pedal vary, of course, with the position of the pedal origin.

In the first part of the memoir, of which the present note is an abstract, the volumes of pedals derived from the same primitive surface, but corresponding to different origins, are investigated, and the general formula found by means of which the volume of any pedal whatever may be calculated when that of any other is known. From this formula are deduced the following new and very general properties of pedal surfaces :—

Whatever may be the nature of the primitive surface, the origins of pedals of the same volume lie on a surface of the third order.

It should be observed that the volume of the pedal is here understood to be that of the conical space swept by the perpendicular, as the tangent plane of the primitive takes all possible positions. In this sense the term volume may clearly be applied to the pedals of *unclosed* surfaces. It is in fact to such surfaces that the above theorem applies ; for *when the primitive is a closed surface, but in other respects perfectly arbitrary, the locus of the origins of pedals of constant volume is a quadric, or surface of the second order. The whole series of quadric loci, corresponding to all possible volumes, constitutes a system of similar, similarly placed, and concentric quadrics, the common centre of all being the origin of the pedal of least volume.*

From the three equations which determine the position of the origin of the pedal of least volume, it follows that this origin always coincides with the centre of the primitive, whenever the latter possesses such a point ; when, moreover, the primitive, besides being closed, is everywhere convex in curvature, and symmetrical with respect to three rectangular planes, each origin-locus is an ellipsoid whose principal diametral planes coincide with the planes of symmetry.

This is the case with the pedals of the ellipsoid, which, ever since the researches of Fresnel on light, have been regarded with especial interest. Their properties form the subject of the second part of the memoir.

It is shown that the volume of any ellipsoid-pedal, the coordinates of whose origin are given, may be found by simple differentiation of

the expression for the volume of the least or central pedal. Amongst the new properties of such pedals the following may be here cited:—

The volume of the pedal whose origin is at a corner of the rectangular parallelopiped described about the primitive ellipsoid is equal to four times the volume of the central pedal, and to twice the volume of the pedal at any one of the eight points where the ellipsoid is pierced by the diagonals of the parallelopiped.

Again, *the algebraical sum of the volumes of the three ellipsoid-pedals whose origins are at the extremities of any three conjugate diameters of a concentric and co-axal quadric is constant, and equal to three times the volume of the pedal at any one of the eight points where this quadric is pierced by the diagonals of its circumscribed rectangular parallelopiped.*

From this theorem several others are deduced by assuming, for the quadric in question, particular forms. For instance, when it coincides with the primitive surface itself, we learn that *the sum of the volumes of the three ellipsoid-pedals whose origins are at the extremities of any three conjugate diameters of the primitive surface is constant, and equal to six times the volume of the central or least pedal.*

In this theorem is included, of course, the special case where the origins of the three pedals coincide with the vertices of the primitive ellipsoid.

If, for convenience of enunciation, we define the *pedal-altitude* at any point to be the altitude of a parallelopiped whose base is the square on the line joining that point to the centre of the ellipsoid, and whose volume is equal to that of the pedal having the point in question for origin, it is found that *the algebraical sum of the three pedal-altitudes at the extremities of any three orthogonal diameters of a quadric, concentric and co-axal with the primitive ellipsoid, is constant, and equal to three times the pedal-altitude at any one of the eight points on this quadric which are equidistant from its axes.* It follows, consequently, that *this sum is not only invariable for one and the same quadric, but for all concentric and co-axal quadrics which pass through one and the same point equidistant from the principal diametral planes of the primitive ellipsoid.*

In the third part of the memoir, the volume of any pedal of the ellipsoid

$$\frac{x^2}{a_1}+\frac{y^2}{a_2}+\frac{z^2}{a_3}=1$$

is expressed by means of the three first partial differential coefficients of the symmetrical integral

$$V=\int_0^\infty \frac{dv}{\sqrt{(v+a_1)(v+a_2)(v+a_3)}}.$$

If P denote the volume of the pedal whose origin has the coordinates x, y, z, the expression in question is

$$P=-\frac{\pi}{2}\left[a_1 M_1 \frac{dV}{da_1}+a_2 M_2 \frac{dV}{da_2}+a_3 M_3 \frac{dV}{da_3}\right],$$

where

$$3M_1=(a_2+a_3)(3r^2+a)+3(a_2y^2+a_3z^2)+a_2^2+a_3^2,$$
$$3M_2=(a_3+a_1)(3r^2+a)+3(a_3z^2+a_1x^2)+a_3^2+a_1^2,$$
$$3M_3=(a_1+a_2)(3r^2+a)+3(a_1x^2+a_2y^2)+a_1^2+a_2^2;$$

r^2 and a being abbreviations for $x^2+y^2+z^2$ and $a_1+a_2+a_3$, respectively.

The memoir concludes with the expression of the volume P by means of ordinary elliptic functions, and the consideration of the special cases when the primitive is an ellipsoid of rotation. The expression in question may be readily obtained on observing that the integral V is reducible to the form

$$V=2\frac{F(\theta,k)}{\sqrt{a_1-a_3}},$$

where the amplitude θ and modulus k of the elliptic function F of the first kind are determined by the relations

$$\cos^2\theta=\frac{a_3}{a_1},\ k^2=\frac{a_1-a_2}{a_1-a_3}.$$

By the introduction of elliptic functions, however, the great advantages of symmetry are necessarily lost; and in investigating the properties of pedal-volumes, the above symmetrical expressions will in general be preferred. An opportunity thus presents itself, however, of verifying an expression for the volume of the central pedal, the only one hitherto calculated, which was first given in 1844 by Prof. Tortolini in vol. xxxi. of Crelle's Journal.