II. The force per unit of area is given in magnitude and direction at each point of each surface.

The formulæ are applied to determine the deformation produced in the earth by the tide-generating forces of the moon and sun, on certain definite hypotheses as to the rigidity of the earth. Thus the theoretical results used in a previous communication by the same author, "On the Rigidity of the Earth" (Proceedings of the Royal Society, May 15, 1862), are proved.

II. "On the Exact Form and Motion of Waves at and near the Surface of Deep Water." By Professor W. J. Macquorn Rankine, C.E., F.R.S. &c. Received September 27, 1862.

(Abstract.)

The investigations of the Astronomer Royal and of other mathematicians on the question of straight-crested parallel waves in a liquid, are based on the supposition that the displacements of the particles are small compared with the length of a wave. Hence it has been very generally inferred that the results of those investigations are approximate only, when applied to waves in which the displacements, as compared with the length of a wave, are considerable.

In the present paper, the author proves that one of those results, viz., that in very deep water the particles move with a uniform velocity in vertical circles whose radii diminish in geometrical progression with increased depth, and consequently that surfaces of equal pressure, including the upper surface, are trochoidal,—is exact for all displacements, how great soever.

The trochoidal form of waves was first explicitly described by Mr. Scott Russell; but no demonstration of its exactly fulfilling the cinematical and dynamical conditions of the question has yet been published.

In 'A Manual of Applied Mechanics' (first published in 1858), the author stated that the theory of rolling waves might be deduced from that of the positions assumed by the surface of a mass of water revolving in a vertical plane about a horizontal axis; but as the theory of such waves was foreign to the subject of the book, he deferred until now the publication of the investigation on which that statement was founded.

Having communicated some of the leading principles of that investigation to Mr. William Froude in April 1862, the author was informed by that gentleman that he had arrived independently at similar results by a similar process, although he had not published them.

The following is a summary of the leading results demonstrated in the paper.

Proposition I.—In a mass of gravitating liquid whose particles revolve uniformly in vertical circles, a wavy surface of trochoidal profile fulfils the conditions of uniformity of pressure; such trochoidal profile being generated by rolling, on the under side of a horizontal straight line, a circle whose radius is equal to the height of a conical pendulum that revolves in the same period with the particles of liquid.

Proposition II.—Let another surface of uniform pressure be conceived to exist indefinitely near to the first surface; then, if the first surface is a surface of continuity (that is, a surface always traversing identical particles), so also is the second surface. (Those surfaces contain between them a continuous layer of liquid.)

Corollary.—The surfaces of uniform pressure are identical with surfaces of continuity throughout the whole mass of liquid.

Proposition III.—The profile of the lower surface of the layer referred to in Proposition II., is a trochoid generated by a rolling circle of the same radius with that which generates the upper surface; and the tracing-arm of the second trochoid is shorter than that of the first trochoid by a quantity bearing the same proportion to the depth of the centre of the second rolling circle below the centre of the first rolling circle, which the tracing-arm of the first rolling circle bears to the radius of that circle.

Corollaries.—The profiles of the surfaces of uniform pressure and of continuity form an indefinite series of trochoids, described by equal rolling circles, rolling with equal speed below an indefinite series of horizontal straight lines.

The tracing-arms of those circles (each of which is the radius of the circular orbits of the particles contained in the trochoidal surface which it traces) diminish in geometrical progression with a uniform increase of the vertical depth at which the centre of the rolling circle is situated.

The preceding propositions agree with the existing theory, except that they are more comprehensive, being applicable to large as well as small displacements. The following proposition is entirely new.

Proposition IV.—The centres of the orbits of the particles in a given surface of equal pressure stand at a higher level than the same particles do when the liquid is still, by a height which is a third proportional to the diameter of the rolling circle and the length of the tracing-arm, or radius of the orbits of the particles, and which is equal to the height due to the velocity of revolution of the particles.

Corollaries.—The mechanical energy of a wave is half actual and half potential; half being due to motion, and half to elevation. The crests of the waves rise higher above the level of still water than their hollows fall below it; and the difference between the elevation of the crests and the depression of the hollows is double of the quantity mentioned in Proposition IV.

The hydrostatic pressure at each individual particle during the wave-motion is the same as if the liquid were still.

Friction between a Wave and a Wave-shaped Solid.

In an Appendix is given the investigation of the problem, to find approximately the amount of the pressure required to overcome the friction between a trochoidal wave-surface and a wave-shaped solid in contact with it. The application of the result of this investigation to the resistance of ships was explained in a paper read to the British Association in 1861, and published in various Engineering Journals in October of that year. The following is the most useful of the formulæ arrived at. Let w be the heaviness of the liquid; f, the coefficient of friction; g, gravity; v, the velocity of advance of the solid; L, its length, being that of a wave; z, the breadth of the surface of contact of the solid and liquid; β, the greatest angle of obliquity of that surface to the direction of advance; P, the force required to overcome the friction: then

$$\mathrm{P}=\frac{fwv^2}{2g}\,.\,\mathrm{L}z(1+4\sin^2\beta+\sin^4\beta).$$

In ordinary cases the value of f for water sliding over painted iron is about ·0036. The quantity $\mathrm{L}z(1+4\sin^2\beta+\sin^4\beta)$ is what has been called the "augmented surface." In practice, $\sin^4\beta$ may in general be neglected on account of its smallness.