I venture to conclude that the typical anatomical arrangement of a nervous mechanism is not a *cord with two ends—a point of origin and a terminal extremity*, but a cord *without an end—a continuous circuit.*

The peculiar structure of the caudate nerve-cells, which I have described, renders it, I think, very improbable that these cells are *sources* of nervous power, while, on the other hand, the structure, mode of growth, and indeed the whole life-history of the rounded ganglion-cells render it very probable that they perform such an office. These two distinct classes of nerve-cells, in connexion with the nervous system, which are very closely related, and probably, through nerve-fibres, structurally continuous, seem to perform very different functions,—the one *originating* currents, while the other is concerned more particularly with the distribution of these, and of secondary currents induced by them, in very many different directions. A current originating in a *ganglion-cell* would probably give rise to many induced currents as it traversed a *caudate nerve-cell.* It seems probable that nerve-currents emanating from the rounded ganglion-cells may be constantly traversing the innumerable circuits in every part of the nervous system, and that nervous actions are due to a disturbance, perhaps a variation in the intensity of the currents, which must immediately result from the slightest change occurring in any part of the nerve-fibre, as well as from any physical or chemical alteration taking place in the nerve-centres, or in peripheral nervous organs.

XXIII. "On the Physical Constitution and Relations of Musical Chords." By Alexander J. Ellis, F.R.S., F.C.P.S.* Received June 8, 1864.

When the motion of the particles of air follows the law of oscillation of a simple pendulum, the resulting sound may be called a *simple tone.* The *pitch* of a simple tone is taken to be the number of *double* vibrations which the particles of air perform in one second. The greatest elongation of a particle from its position of rest may be termed the *extent* of the tone. The *intensity* or loudness is assumed to vary as the square of the extent. The tone heard when a tuning-fork is held before a proper resonance-box is simple. The tone of wide covered organ-pipes and of flutes is nearly simple.

Professor G. S. Ohm has shown mathematically that all musical tones whatever may be considered as the algebraical sum of a number of simple tones of different intensities, having their pitches in the proportion of the numerical series 1, 2, 3, 4, 5, 6, 7, 8, &c. Professor Helmholtz has established that this mathematical composition corresponds to a fact in nature, that the ear can be taught to hear each one of these simple tones separately, and that the character or *quality* of the tone depends on the law of the intensity of the constituent simple tones.

These constituent simple tones will here be termed indifferently *partial*

* The Tables belonging to this Paper will be found after p. 422.

tones or *harmonics*, and the result of their combination a *compound tone*. By the pitch of a compound tone will be meant the pitch of the lowest partial tone or *primary*.

When two simple tones which are not of the same pitch are sounded together, they will alternately reinforce and enfeeble each other's effect, producing a libration of sound, termed a *beat*. The number of these beats in one second will necessarily be the difference of the pitches of the two simple tones, which may be termed the beat *number*. As for some time the two sets of vibrations concur, and for some time they are nearly opposite, the compound extent will be for some time nearly the sum, and for some time nearly the difference of the two simple extents, and the *intensity* of the beat may be measured by the ratio of the greater intensity to the less.

But the beat will not be audible unless the ratio of the greater to the smaller pitch is less than 6 : 5, according to Professor Helmholtz. This is a convenient limit to fix, but it is probably not quite exact. To try the experiment, I have had two sliding pipes, each stopped at the end, and having each a continuous range of an octave, connected to one mouthpiece. The tones are nearly simple; and when the ratio approaches to 6 : 5, or the interval of a minor third, the beats become faint, finally vanish, and do not reappear. But the exact moment of their disappearance is difficult to fix, and indeed seems to vary, probably with the condition of the ear. The ear appears to be most sensitive to the beats when the ratio is about 16 : 15. After this the beats again diminish in sharpness; and when the ratio is very near to unity, the ear is apt to overlook them altogether. The effect is almost that of a broken line of sound, as —— —— —— ——, the spaces representing the silences.

Slow beats are not disagreeable; for example, when they do not exceed 3 or 4 in a second. At 8 or 10 they become harsh; from 15 to 40 they thoroughly destroy the continuity of tone, and are discordant. After 40 they become less annoying. Professor Helmholtz thinks 33 the beat number of maximum disagreeableness. As the beats become very rapid, from 60 to 80 or 100 in a second, they become almost insensible. Professor Helmholtz considers 132 as the limiting number of beats which can be heard. They are certainly still to be distinguished even at that rate, but become more and more like a scream. Though $f^3\sharp$ and g^3 should give 198 beats in a second if $c=264$, and the interval is that for which the ear is most sensitive, I can detect no beats when these tones are played on two flageolet-fifes. Hence beats from 10 to 70 may be considered as discordant, and as the source of all discord in music. Beyond these limits they produce a certain amount of harshness, but are not properly discordant.

When the extent of the tones is not infinitesimal, Professor Helmholtz has proved that on two simple tones being sounded together, many other tones will be generated. The pitch of the principal and only one of these *combinational* tones necessary to be considered, is the difference of the pitch of its generating tones. It will therefore be termed the *differential*

tone. Its intensity is generally very small, but it becomes distinctly audible in beats. The differential tone is frequently acuter than the lower generator, and hence the ordinary name "grave harmonic" is inapplicable. As its pitch is the beat number of the combination, Dr. T. Young attributed its generation to the beats having become too rapid to be distinguished. This theory is disproved, first, by the existence of differential tones for intervals which do not beat, and secondly, by the simultaneous presence of distinct beats and differential tones, as I have frequently heard on sounding f^4, $f^4\sharp$, or even f^2, $f^2\sharp$ together on the concertina, when the beats form a distinct rattle, and the differential tone is a peculiar penetrating but very deep hum.

The object of this paper is to apply these laws, partly physical and partly physiological, to explain the constitution and relations of musical chords. It is a continuation of my former paper on a *Perfect Musical Scale**, and the Tables are numbered accordingly.

Two simple tones which make a greater interval than 6 : 5, and therefore never beat, will be termed *disjunct*. Simple tones making a smaller interval, and therefore generally beating, will be termed *pulsative*. The unreduced ratio of the pitch of the lower pulsative tone for which the beat number is 70 to that for which it is only 10, will be termed the *range* of the beat. The fraction by which the pitch of the lower pulsative tone must be multiplied to produce the beat number, will be termed the beat *factor*. The ratio of the pitches of the pulsative tones, on which the sharpness of the dissonance depends, will be termed the beat *interval*.

A compound tone will be represented by the absolute pitch of its primary and the relative pitches of its partial tones, as C (1, 2, 3, 4, ...). As generally only the relative pitch of two compound tones has to be considered, the pitches will be all reduced accordingly. Thus, if the two primaries are as 2 : 3, the two compound tones will be represented by 2, 4, 6, 8, 10, ..., and 3, 6, 9, 12, 15 . . . . The intensity of the various partial tones differs so much in different cases, that any assumption which can be made respecting them is only approximative. In a well-bowed violin we may assume the extent of the harmonics to vary inversely as the number of their order. Hence, putting the extent and intensity of the primary each equal to 100, we shall have, with sufficient accuracy—

Harmonics...	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Extent...	100,	50,	33,	25,	20,	17,	14,	12,	11,	10.
Intensity...	100,	25,	11,	6,	4,	3,	2,	1,	1,	1.

It will be assumed that this law holds for all combining compound

* Proceedings of the Royal Society, vol. xiii. p. 93. The following misprints require correction:—P. 97, line 7 from bottom, for c^2 read b. Table I., p. 105, diminished 5th, example, *read* f : B; minor 6th, logarithm, *read* ·20412; Pythagorean Major 6th, *read* 27 : 16, $3^3 : 2^4$; Table V., col. VI., last line, *read* †$e\sharp$ †$g\sharp$ †$b\sharp$.

tones, the intensity of the primary in each case being the same. The results will be sufficient to explain the nature of chords on a quartett of bowed instruments, but may be much modified by varying the relative intensities of the combining tones.

On examining a single compound tone, we may separate its partial tones into two groups: the first *disjunct*, which will never beat with each other; the second *pulsative*, which will beat with the neighbouring disjunct tones. Thus

Disjunct . .	1,	2,	3,	4,	5,	6,	–,	8,	–,	10,	–,	12,	–,	–,	–,	16,
Pulsative .	–,	–,	–,	–,	–,	–,	7,	–,	9,	–,	11,	–,	13,	14,	15,	–,
Disjunct . .	–,	–,	–,	20,	–,	–,	–,	24,	–,	–,	–,	–,	–,	30.		
Pulsative .	17,	18,	19,	–,	21,	22,	23,	–,	25,	26,	27,	28,	29,	–.		

When any compound tone therefore developes any of the harmonics above the 6th, there may, and probably will, be beats, producing various degrees of harshness or shrillness, jarring or tinkling. These, however, are all *natural* qualities of tone, that is, they are produced at once by the natural mode of vibration of the substances employed. But if we were to take a series of *simple* tones having their pitches in the above ratios, and to vary their intensities at pleasure, we should produce a variety of *artificial* qualities of tone, some of which might be coincident with natural qualities, but most of which would be new. This method of producing artificial qualities of tone is difficult to apply, but has been used with success by Professor Helmholtz to imitate vowel-sounds, &c.

If, however, instead of using so many simple tones, we combine a few compound tones, the pitches of which are such that their primaries might be harmonics of some other compound tone, then the two sets of partial tones will necessarily combine into a single set, which may, or rather must be considered by the ear as the partial tones of some new compound tone, having very different intensities from those possessed by the partial tones of either of the combining compound tones. That is, an artificial quality of tone will have been created by the production of these *joint* harmonics. *Such an artificial quality of tone constitutes what is called a musical chord.* The two or more compound tones from which it is built up are its *constituents.* The primary joint harmonic is the real *root* or *fundamental bass* of the chord, which often differs materially from the supposititious root assigned by musicians.

If the primaries of the constituents are disjunct, and all their partial tones are disjunct, then the joint harmonics will be also disjunct, unless some pulsative differential tones have been introduced. If, however, the constituents have pulsative partial tones, the chord will also have them. Such chords, which are generally without beats, and are only exceptionally accompanied by beats, are termed *concords,* and they are *unisonant* or *dissonant* according as the beats are absent or present. Their character therefore consists in having the pitches of their constituents as 1, 3, 5, or as

these numbers multiplied by various powers of 2, that is, as 1, 3, 5, or their octaves.

If any of the constituents is pulsative the chord will generally have beats, but may be exceptionally without beats. Such chords are termed *discords*. Their character consists in having *two* or more of the pitches of their constituents as 1, 3, 5, or their octaves, and at least *one* of them as 7, 9, or some other pulsative tones, or their octaves. What pulsative tones should be selected depends on the sharpness of the dissonance which it is intended to produce, and therefore on the interval of the beat which is created. Thus, since $7:6=1{\cdot}16667$ and $8:7=1{\cdot}14286$ are both near the limit $6:5=1{\cdot}2$, the discord arising from 7 would be slight. Some writers have even considered the chord 1, 3, 5, 7 to be concordant. Again, $9:8=1{\cdot}125$ is rather rough, but $10:9=1{\cdot}11111$ is much rougher. Hence, if 9 is introduced, 10 should be avoided, that is, the octave of 5 should be omitted, which generally necessitates the omission of 5 itself, as in the chord 1, 3, 9. But $11:10=1{\cdot}1$ and $12:11=1{\cdot}09091$ are both so sharply dissonant, that if 11 is used neither 10 nor 12 should be employed. Now 10 is the octave of 5, and 12 is both the 3rd harmonic of 4 and the 4th harmonic of 3, and would therefore be produced from 3 and 4. Hence the use of 11 would forbid the use of 3, 4, and 5, that is, of the best disjunct tones. Hence 11 cannot be employed at all. Similarly, $13:12=1{\cdot}08333$ and $14:13=1{\cdot}07692$ are both extremely harsh. The latter is of no consequence, because 7 can be easily omitted. But even $15:13=1{\cdot}15384$ is more dissonant than $7:6$. Hence 13 would also beat with the harmonics of 3, 4, and 5. Consequently 13 must be also excluded. All combinations in which the differential tones 11 and 13 are developed will also be extremely harsh. As we therefore suppose that $14:13=1{\cdot}07692$ never occurs, and as $14:12=7:6$, the mildest of the dissonances, 14 may be used if 15 is absent, and thus $15:14=1{\cdot}07143$ avoided. When 14 and 15 are developed as harmonics of 7 and 5, and not as the primaries of constituent tones, their intensity will be so much diminished that the discord will not generally be too harsh. When 15 is used as a constituent, 14 and 16 should be avoided; that is, 7, and 1, 2 and 4, of which 14 and 16 are upper harmonics, should be omitted to avoid $15:14=1{\cdot}07143$ and $16:15=1{\cdot}06667$, which may be esteemed the maximum dissonance. By omitting 16 and 18, and thus avoiding $17:16=1{\cdot}0625$ and $18:17=1{\cdot}05882$ (that is, by not using 4, 8, or 9 as constituent tones), 17 becomes useful; for $17:15=1{\cdot}13333$ is milder than $9:8=1{\cdot}125$, which is by no means too rough for occasional use. The other pulsative harmonics, which are represented by prime numbers, are not sufficiently harmonious for use; but those produced from 2, 3, 5 (such as 25, 27, 45) may be sometimes useful, provided that the tones with which they form sharp dissonances are omitted.

The result of the above investigation is that the only pulsative tones suitable for constituents are 7, 9, 15, 17, 25, 27, 45, and their octaves.

The introduction of any one of these tones in conjunction with 1, 3, 5 and their octaves will therefore form a discord, the harshness of which may be frequently much diminished by the omission of 1 and its octaves for the constituents 7, 15, 17, by the omission of 5 for the constituent 9, and by the omission of 24 for the constituents 25, 27, 45.

Using the notation of my former paper, where ʒ=63 : 64, and putting in addition vij=84 : 85, xj=33 : 32, xiij=39 : 40, 1ʒ=255 : 256, and xvij=135 : 136, the tones 1 to 18 may be represented by the following notes in terms of C^4 :—

1,	2,	3,	4,	5,	6,	7,	8,	9,	10,
C^4,	C,	G,	c,	e,	g,	ʒ*b*♭ or vij *a*♯,	c^2,	d^2,	e^2,

11,	12,	13,	14,	15,	16,	17,
xj f^2,	g^2,	xiij a^2,	ʒb^2♭ or vij a^2♯,	b^2,	c^4,	1ʒd^4♭ or xvij c^4♯,

18,	20,	24,	25,	27,	45,
d^4,	e^4,	g^4,	‡g^4♯,	†a^4,	f^4♯.

This notation will show what are the musical names of the constituents of musical chords, and how they may be approximately produced on an organ, harmonium, or pianoforte.

By the *type* of a musical chord is meant the numbers which express the relative pitches of its constituents, after such octaves below them have been taken as to leave only uneven numbers, which are then called the *elements* of the type. By the *form* of the chord is meant the numbers before such reduction. Thus the *type* 1, 3, 5 embraces, among others, the *forms* 1, 3, 5; 1, 2, 3, 5; 2, 3, 5; 4, 3, 5; 3, 8, 10; 6, 10, 16; 2, 5, 6, 8, and so on; hence the types of musical chords consist of groups of the elements 1, 3, 5, 7, 9, 15, 17, 25, 27, 45. The type of a concord is 1, 3, 5, and of a discord 1, 3, 5, P, or 1, 3, 5, P, P', where P, P' are any of the numbers 7, 9, 15, 17, 25, 27, 45. Discords may be divided into *strong* and *weak*, according as those disjunct tones with which the pulsative tones principally beat are retained or omitted. These discords again may be distinguished into those which have one or two pulsative constituents. The chords may also be grouped according to the number of elements in their type, *dyads* containing two, *triads* three, *tetrads* four, and *pentads* five. The number of elements in the type by no means limits the numbers of constituents, as any octaves above any of the elements may be added.

Hence it is possible to classify all the suitable chords of music according to their type, as in Table VI., where the notes corresponding to each type are added in the typical form only. A simple systematic nomenclature is proposed in an adjoining column, and the names by which the true chords or their substitutes are known to musicians are added for reference. Occasionally two forms of substitution are given, as they are of theoretical importance, although confounded on some tempered instruments. A mode of symbolizing the chords is subjoined, in which several types are classed under one family. A capital letter shows the root of major chords, either

complete or imperfect, and of strong discords, and a smaller letter gives the root of weak discords, a number pointing out the family. In the minor triad the characteristic number is omitted; thus *c* is written for 15 *c*, meaning the minor triad *g e b*, which is the major tetrad 15 *C*, or *C G E B*, with its root *C* omitted, and is usually called "the minor chord of *e*," a nomenclature which conceals its derivation.

Although chords of the same type have the same general character, this is so much modified by the particular forms which they can assume, that it is necessary to examine these forms in detail. They may be distinguished as *simple* and *duplicated*. In the former the number of constituents is the same as in the type; thus 4, 5, 6; 2, 3, 5 are simple forms of the type 1, 3, 5. In the latter, the number of constituents is increased by the higher octaves of some or all of them; thus 1, 2, 3, 5; 2, 4, 5, 6 are duplicated forms of 1, 3, 5 and 2, 5, 6, as they contain the octaves 1, 2 and 2, 4.

The mode in which the effect of any or all of these combinations may be calculated is shown in Table VII., which consists of two corresponding parts, each commencing with a column containing the "No. of J. H.," or of the joint harmonics resulting from the combination of the harmonics of the constituent compound tones. The next columns are headed by the relative pitch of the constituent tones, and contain their harmonics, never extending beyond the 8th, arranged so that their pitch is opposite to the corresponding number of the joint harmonic. It is thus seen at a glance which harmonics of the constituents are *conjunct* or tend to reinforce each other, and produce a louder joint harmonic, and also which are disjunct and pulsative. In the second part of the Table the extent of each harmonic of each constituent is given on the assumptions already explained. To find the extent of the joint harmonic, we add the extents of the generating conjunct harmonics, and thence find the intensity by squaring and dividing by 100. The differential tones must then be found by subtracting the pitches of the primaries (or in exceptional cases of higher and louder harmonics). The intensity of these differential tones may be called 1 for a single tone, and 4 for two concurrent tones, and this number may be subscribed to the intensity of the corresponding joint harmonic, as 0_1, 25_4.

The beat intervals have next to be noted, and the beat factors, which are usually the reciprocal of the relative pitch of the lower pulsative harmonic. Thus for the dyad 3, 4 the beat interval is $\frac{9}{8}$, and the beat factor $\frac{1}{8}$. From this factor, or $1:f$, we calculate the range $P:p=70f:10f=210:30$ in the present case. This must not be reduced, as it shows that the interval is dissonant when the pitch of the lower tone is between 30 and 210. To find the intensity, we add and subtract the extents of the pulsative joint harmonics; in this case 50 and 33 are the extents of the 8th and 9th joint harmonics, and their sum and difference are 83 and 17. Then we take the ratio of their squares, each divided by 100, which gives 69 : 3. This result must not be reduced, as it gives not only the relative loudness of the swell and fall, but also the loudness of these in relation to the other

Table VI.—Classification of Musical Chords. (See p. 397.)

Type.	Example.	Symbol.	Syst. Name.	Ordinary Name.
I. Concords.				
1, 1	C^4 C^4	C	1 Dyad	Unison (Octave 1, 2).
1, 3	C^4 G	C	3 Dyad	Twelfth (Fifth 2, 3; Fourth 3, 4).
1, 5	C^4 e	C	5 Dyad	Major 17th (Ma. 10th 2, 5; Ma. 3rd 4, 5; Mi. 6th 5, 8).
3, 5	G e	c	3, 5 Dyad	Ma. 6th (Mi. 3rd 5, 6).
1, 3, 5	C^4 G e	C	Major Triad	Common Major Chord.
II. Strong Discords.				
1. One Pulsative Constituent.				
1, 7	C^4 ƺb♭	7C	7 Dyad	Perfect 7th 4, 7; Extended tone 7, 8.
3, 7	G ƺb♭	7C	3, 7 Dyad	Contracted 3rd 6, 7; Ext. 6th 7, 12.
5, 7	e ƺb♭	7C	5, 7 Dyad	Contracted 5th 5, 7.
1, 3, 7	C^4 G ƺb♭	7C	3, 7 Triad	Imp. Ch. of Dominant 7th.
1, 5, 7	C^4 e ƺb♭	7C	5, 7 Triad	„ „ „
1, 5, 7	C^4 e vij a♯	7′C	5, 7′ Triad	**Chord of the Italian 6th.**
1, 5, 9	C^4 e d	9C	5, 9 Triad	Imp. Ch. of the 9th.
1, 3, 15	C^4 G b^2	15C	3, 15 Triad	Imp. Ch. of the Mi. 6th, Minor mode.
1, 5, 15	C^4 e b^2	15C	5, 15 Triad	
1, 3, 17	C^4 G 1ƺd^4♭	17C	3, 17 Triad	Imp. Ch. of the Minor 9th.
1, 5, 17	C^4 e 1ƺd^4♭	17C	5, 17 Triad	
1, 3, 5, 7	C^4 G e ƺb♭	7C	7 Tetrad	Ch. of the Dominant 7th.
1, 3, 5, 7	C^4 G e vij a♯	7′C	7′ Tetrad	„ „ German 6th.
1, 3, 5, 9	C^4 G e d^2	9C	9 Tetrad	„ „ 9th.
1, 3, 5, 15	C^4 G e b^2	15C	Major Tetrad	„ „ Ma. 7th.
1, 3, 5, 17	C^4 G e 1ƺd^4♭	17C	17 Tetrad	„ „ Mi. 9th (imp.)
1, 3, 5, 27	C^4 G e †a^4	27C	27 Tetrad	„ „ add. 6th, ma. m.
2. Two Pulsative Constituents.				
1, 3, 5, 7, 9	C^4 G e ƺb♭ d^2	’9C	7, 9 Pentad	Ch. of the added 9th.
1, 3, 5, 7, 17	C^4 G e ƺb♭ 1ƺd^4♭	’17C	7, 17 Pentad	„ „ Mi. 9th.
1, 3, 5, 9, 15	C^4 G e d^2 b^4	’15C	Major Pentad	„ „ Ma. 9th.
1, 3, 5, 15, 17	C^4 G e b^2 xvij c^4♯	15, 17C	15, 17 Pentad	„ „ augmented 8th.
III. Weak Discords.				
1. One Pulsative Constituent.				
3, 5, 7	G e ƺb♭	7c	7 Triad	Ch. of the Diminished 5th.
1, 3, 9	C^4 G d	9c	9 Triad	„ „ 9th (imp.)
3, 5, 15	**G e b**	**c**	**Minor Triad**	Common Minor Chord.
3, 5, 17	G e 1ƺd^4♭	17c	17 Triad	Ch. of the Dim. 7th (imp.).
3, 5, 17	G e xvij c^4♯	17′c	17′ Triad	„ „ Minor added 6th (imperfect).

3, 5, 15	G e b	c	Minor Triad	Common Minor Chord.
3, 5, 17	G e 1ʒd^4♭	17c	17 Triad	Ch. of the Dim. 7th (imp.).
3, 5, 17	G e xvij c^4♯	17′c	17′Triad	„ „ Minor added 6th (imperfect).
1, 5, 25	C e ‡g♯	25c	25 Triad	Superfluous Triad.
1, 5, 27	C^4 e †a	27c	27 Triad	Ch. of the Ma. added 6th (imperfect).
		2. Two Pulsative Constituents.		
3, 5, 7, 9	G e ʒb♭ d^2	’9c	7, 9 Tetrad	Ch. of the Mi. 7th, ma. m.
3, 5, 7, 17	G e ʒb♭ 1ʒd^4♭	’17c	7, 17 Tetrad	„ „ Diminished 7th.
3, 5, 9, 15	G e d b	’c	Minor Tetrad	„ „ Mi. 7th (mi. m.).
3, 5, 15, 17	G e b xvij c^4♯	15, 17c	15, 17 Tetrad	„ „ added 6th (mi. mode).
1, 5, 15, 25	C e b ‡g♯	25c	25 Tetrad	„ „ augmented 5th.
3, 5, 15, 45	G e b f^4♯	’45c	45 Tetrad	„ „ added 9th (mi. mode).

Table VII.—Construction of Musical Chords from Compound Tones. (See p. 398.)

No. of J. H.	Harmonics of the Constituent Tones. 1	2	3	4	5	6	7	8	9	10	12	14	15	16	17
1	1														
2	2	2													
3	3	..	3												
4	4	4	..	4											
5	5	..	..	..	5										
6	6	6	6	..	..	6									
7	7	..	..	..	..	..	7								
8	8	8	..	8	..	..	..	8							
9	..	..	9	..	..	..	..	..	9						
10	..	10	..	..	10	..	..	..	..	10					
11	..	..	..	..	..	..	..	..	..	..					
12	..	12	12	12	..	12	..	..	..	..	12				
13	..	..	..	..	..	..	..	..	..	..	..				
14	..	14	..	..	..	..	14	..	..	..	..	14			
15	..	..	15	..	15	..	..	..	..	..	..	..	15		
16	..	16	..	16	..	..	..	16	..	..	..	..	..	16	
17	..	..	..	..	..	..	..	..	..	..	..	..	..	..	17
18	..	..	18	..	..	18	..	..	18	..	..	..	..	..	..
19	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
20	..	..	..	20	20	..	..	..	..	20	..	..	..	..	..
21	..	..	21	..	..	..	21	..	..	..	..	..	..	..	..
24	..	..	24	24	..	24	..	24	..	..	24	..	..	..	..

No. of J. H.	Extent of the Harmonics of the Constituent Tones. 1	2	3	4	5	6	7	8	9	10	12	14	15	16	17
1	100														
2	50	100													
3	33	....	100												
4	25	50	....	100											
5	20	....	....	....	100										
6	17	33	50	....	....	100									
7	14	....	....	....	....	....	100								
8	12	25	....	50	....	....	....	100							
9	....	....	33	....	....	....	....	....	100						
10	....	20	....	....	50	....	....	....	....	100					
11	....	....	....	....	....	....	....	....	....	....					
12	....	17	25	33	....	50	....	....	....	....	100				
13	....	....	....	....	....	....	....	....	....	....	....				
14	....	14	....	....	....	....	50	....	....	....	....	100			
15	....	....	20	....	33	....	....	....	....	....	....	....	100		
16	....	12	....	25	....	....	....	50	....	....	....	....	....	100	
17	....	....	....	....	....	....	....	....	....	....	....	....	....	....	100
18	....	....	17	....	....	33	....	....	50	....	....	....	....	....	....
19	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
20	....	....	....	20	25	....	....	....	....	50	....	....	....	....	....
21	....	....	14	....	....	....	33	....	....	....	....	....	....	....	....
24	....	....	12	17	....	25	....	33	....	....	50	....	....	....	....

Table VIII.—Qualities of Concordant Dyads. (See p. 399.)

No.	1	2	3	4	5	6	7	8	9	10	11	12	13
Form.	2/1	3/2	5/2	3/1	4/3	5/3	5/4	6/5	8/5	12/5	8/3	10/3	16/5
Name. / No. of Joint Harmonics.	Octave.	Fifth.	Ma. Tenth.	Twelfth.	Fourth.	Ma. Sixth.	Ma. Third.	Mi. Third.	Mi. Sixth.	Mi. Tenth.	Eleventh.	Ma. Thirteenth.	Mi. Thirteenth.
1	100_1	0_1	....	100	0_1	....	0_1	0_1	....	....	....	....	....
2	225	100	100	25_1	....	0_1	....	....	....	....	....	....	....
3	11	100	0_1	177	100	100	....	....	0_1	....	100	100	....
4	56	25	25	6	100	....	100	....	....	....	....	....	....
5	4	....	100	4	....	100	100	100	100	100	0_1	....	100
6	25	69	11	45	25	25	....	100	....	....	25	25	....
7	2	....	....	2	....	....	....	....	....	0_1	....	0_1	....

monics.

Index. Form.	Number.
1, 2	1
1, 3	4
2, 3	2
2, 5	3
3, 4	5
3, 5	6
3, 8	11
3, 10	12

5	4	....	100	4	....	100	100	100	100	100	0_1	....	100	Intensities of the Joint Harmonics.
6	25	69	11	45	25	25	....	100	....	....	25	25	....	
7	2	....	....	2	....	....	....	....	....	0_1	....	0_1	....	
8	14	6	6	2	25	....	25	....	100	....	100	....	....	
9	....	11	....	11	11	11	....	....	....	....	11	11	....	
10	4	4	49	....	....	25	25	25	25	25	....	100	25	
11	....	....	....	....	....	....	....	....	....	....	....	....	0_1	
12	3	18	3	6	56	6	11	25	....	100	6	6	....	
13	....	....	....	....	....	....	....	....	....	....	....	....	....	
14	2	2	2	....	....	....	....	....	....	....	....	....	....	
15	....	4	11	4	4	28	11	11	11	11	4	4	11	
16	2	2	2	....	6	....	6	....	25	....	25	....	100	
17	....	....	....	....	....	....	....	....	....	....	....	....	....	
18	....	3	....	3	3	3	....	11	....	....	3	3	....	
19	....	....	....	....	....	....	....	....	....	....	....	....	....	
20	....	....	6	....	4	6	20	6	6	6	....	25	6	
21	....	2	....	2	2	2	....	....	....	....	2	2	....	
24	....	2	....	2	9	2	3	6	11	25	21	2	....	
Beats.	....	$\frac{9}{8}, \frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{20}{18}$	$\frac{16}{15}$	$\frac{25}{24}$	$\frac{9}{8}, \frac{16}{15}$	$\frac{10}{9}, \frac{21}{20}$	$\frac{11}{10}, \frac{16}{15}$	Interval.
	....	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	Factor.
	....	$\frac{140}{20}$	$\frac{140}{20}$	$\frac{70}{10}$	$\frac{210}{30}$	$\frac{210}{30}$	$\frac{280}{40}$	$\frac{175}{25}$	$\frac{350}{50}$	$\frac{350}{50}$	$\frac{210}{30}$	$\frac{210}{30}$	$\frac{350}{50}$	Range.
	....	$\frac{34}{1}, \frac{28}{2}$	$\frac{21}{4}$	$\frac{21}{4}$	$\frac{69}{3}$	$\frac{69}{3}$	$\frac{34}{1}$	$\frac{34}{1}$	$\frac{69}{3}$	$\frac{49}{9}$	$\frac{177}{45}, \frac{49}{9}$	$\frac{177}{45}, \frac{41}{13}$	.., $\frac{177}{45}$	Intensity.

3, 4	5
3, 5	6
3, 8	11
3, 10	12
4, 5	7
5, 6	8
5, 8	9
5, 12	10
5, 16	13

joint harmonics. It must be remembered that when there are several disjunct harmonics, their unbroken sound tends to obliterate the action of the beats. There is no sensible silence between the beats unless the tones are simple and the intensities nearly equal. The intensities of the beats between joint harmonics and differential tones cannot be reduced to figures. It is not large. The history of a beat is therefore given by four fractions, which in this case are the interval 9 : 8, the factor 1 : 3, the range 210 : 30, and the intensity 69 : 3.

These calculations have been made for concordant dyads in Table VIII., and for concordant or major triads in Table IX. An attempt has been made to arrange the 13 forms of the first, and the 20 forms of the second in order of sonorousness, by considering the distribution of the intensities among the several joint harmonics, the development of pulsative differential tones, and the nature of the beats, omitting those due to the seventh harmonic of an isolated constituent. It has not been thought necessary to give the history of every beat. The intervals of all the beats are seen at a glance by the list of intensities of the joint harmonics.

By Table VIII. we see that the only unisonant dyad is the octave 1, 2*, which will be as unisonant as the constituents themselves. All other dyads are occasionally dissonant. Thus the fifth itself is decidedly dissonant when the pitch of the lower constituent lies between 20 and 140. On a bass concertina tuned justly, I find the fifth, $C^4 G^4$, quite intolerable, the fifth, *C G*, rough, but *D* †*A* nearly smooth, and at higher pitches there is no perceptible dissonance. The beat interval of the major third is 16 : 15, and the range of dissonance is much greater. The roughness can be distinctly heard as high as *c e*; in the lower octaves *CE* is quite discordant, and $C^4 E^4$ intolerable. This Table VIII., therefore, establishes the fact that concordance does *not* depend on simplicity of ratio alone; but when the denominator of the beat factor is small the range is lower, and therefore the dissonance less felt. Dissonance also arises from the pulsative differential tones 7 and 11, so that if the relative pitches are expressed in terms high enough to differ by 7 and 11, the combination will be dissonant. The ear is also not satisfied with forms in which great intensities of joint harmonics are widely separated by many small intensities. The four last forms in Table VIII., namely, the minor tenth 5, 12, the eleventh 3, 8, and the two thirteenths 3, 10 and 5, 16, should therefore be treated as discords. The Table also suggests how defects may be remedied by introducing new constituents to fill up gaps, or by duplications.

Similar observations apply to the triads in Table IX. None of them can be unisonant at all pitches. Some of them, as the last seven, are really discordant. The gaps may be generally filled up by duplication. Thus

* That is, within the limits of the Table. Dyads such as 1, 2; 1, 3; 1, 4; 1, 5; 1, 6 are all unisonant; but when the interval is very large, the want of connexion between the tones renders them unpleasant. The dyad 1, 8 which developes the differential tone 7 is dissonant.

1, 3, 5 may be converted into 1, 2, 3, 5, and by thus strengthening the 2, 4, and 8 joint harmonics the finest form of concord is produced. In this way the series of duplications in Table X. was produced. In this Table an example has been added to each form to facilitate trial; but the great imperfection of the major third in the ordinary system of tuning pianos and harmoniums materially deteriorates the effect of the chords, which ought to be taken on some justly tuned instrument.

The discords may be deduced from Tables VII. and VIII., when properly extended, by supposing 7, 9, 15, 17, 25, 27, 45 to be used in the first, and their effect allowed for in the second. The additional discordant effect of 7 will be necessarily least felt where 7 occurs as a differential tone, but these are not the best forms of either triad or tetrad. In the better forms the dissonances 6, 7 and 7, 8 will always be well developed, and as the latter is sharper, the omission of 8, at least as a constituent tone, is suggested. If $7\frac{1}{9}$ is used instead of 7, the omission of 8 becomes more urgent, while 6, $7\frac{1}{9}$ will beat less sharply than 6, 7, and therefore almost inaudibly. The real beats of the constituents 6, $7\frac{1}{9}$ arise from the harmonics $6 . 6$, $5 . 7\frac{1}{9}$, or 36, $35\frac{5}{9}$, which are, however, not so much felt as those of $6 . 6$, $5 . 7$ arising from 6, 7, because $36 : 35\frac{5}{9} = 1{\cdot}0125$ is further from $16 : 15 = 1{\cdot}0667$ than is $36 : 35 = 1{\cdot}0286$. Hence, when 8 is omitted, the dissonance arising from $7\frac{1}{9}$ is less than that arising from 7 itself. When 8 is present, 7 or $7\frac{1}{32}$ is superior to $7\frac{1}{9}$. The use of $17\frac{1}{15}$ for 17 would hardly create any perceptible alteration of roughness when 18 is absent, and when 18 is present $18 : 17\frac{1}{15} = 1{\cdot}0548$ is further from $16 : 15$ than is $18 : 17 = 1{\cdot}0588$, and therefore the roughness is not quite so great.

Of all discords the least dissonant is the minor triad 3, 5, 15, which is formed from the tetrad 1, 3, 5, 15 by omitting the root 1, to avoid the dissonance 15, 16. When the differential tones derived from the primaries of the constituents are deeper than the primaries, and therefore merely indicate the presence of a pulsative tone, which is only faintly realized by the differential tones derived from the upper harmonics of the primaries, and when the dissonant intervals of the minor tenth and major thirteenth 5, 12 and 3, 10 are not present in the constituent tones, this chord may be treated as a concord. But in most positions the minor triad is sensibly dissonant, as shown in Table XI., where an attempt has been made to arrange its 20 forms in order of sonorousness. The pitches of the differential tones are added, and examples subjoined. The effect of the minor chord is very much injured by the usual tuning of harmoniums, &c. A peculiar character of these and other discords, when the pulsative constituent is not the highest, consists in the quality of tone being due to very high joint harmonics, except such as are due to differential tones. The root will be consequently extremely deep when the constituent tones are taken at a moderate absolute pitch. This great depth renders its recognition by the ear difficult. Hence probably the disputes of musicians concerning the roots of certain discords, and their error in considering 5 to be

TABLE IX.—Qualities of Concordant Triads. (See p. 399.)

No. of J. H.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
	5	5	5	6	8	6	8	10	6	10	10	12	10	12	12	12	16	16	16	16	
	3	3	4	5	5	5	6	3	5	4	3	5	8	5	5	8	10	5	6	12	
	2	1	3	2	3	4	5	1	1	3	2	2	3	1	4	5	3	3	5	5	
1	0_1	100	0_4	0_1	....	0_4	0_1	100_1	100_1	0_1	0_1	....	....	100	0_1	....	....	....	0_1	....	
2	100_1	25_4	0_1	100	0_1	0_1	0_1	25_1	25	....	100	100	0_1	25	....	....	....	0	....	....	
3	100_1	177	100	0_1	100_1	....	0_1	177	11	100	100	0_1	100	11	....	0_1	100	100	....	....	
4	25	6_1	100	25	....	100	....	6	6_1	100	25	25	....	6_1	100	0_1	....	....	....	0_1	
5	100	144	100	100	100_1	100	100	4	144	....	....	100	0_1	144	100	100	....	100	100	100	
6	67	45	25	177	25	100	100	45	137	25_1	67	11	25	3	....	....	25	25	100	....	
7	....	2	....	....	....	....	....	2_1	2	0_1	0_1	0_1	0_1	0_1	0_1	0_1	0_1	....	....	0_1	
8	6	2	25	6	100	25	100	2	2	25	6_1	6	100	2	25	100	....	....	....	....	
9	11	11	11	....	11	....	....	33_1	....	11	11	....	11	....	....	....	11	11	....	....	
10	49	25	25	49	25	25	25	100	25	100	144	19_1	100	25	25	25	100	25	50	25	
11	....	....	....	....	....	....	....	....	....	....	....	....	....	0_1	....	....	....	0_1	0_1	0_1	
12	16	6	34	45	6	69	25	6	25	34	16	137	6	225	177	100	6	6	50	100	
13	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	0_1	0_1	....	....	
14	2	....	....	2	....	....	....	....	....	....	2	2	....	....	....	....	....	....	....	....	
15	20	28	28	11	28	11	11	4	11	4	4	11	4	11	11	11	4	28	33	11	
16	2	....	6	....	25	6	25	....	....	6	2	....	25	....	6	25	100	100	100	100	
17	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	
18	3	3	3	21	3	11	11	3	11	3	3	2	3	....	....	....	3	3	33	....	
19	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	
20	4	6	20	6	6	20	6	25	6	20	4	6	25	6	20	6	25	6	6	25	
21	2	2	2	....	2	....	....	2	....	2	....	....	2	....	....	....	....	....	....	....	
24	....	....	8	6	20	18	34	1	6	8	1	25	20	25	45	69	1	1	6	25	
Beats.	$\frac{16}{5}$	$\frac{10}{9}$	$\frac{9}{8}, \frac{10}{9}$	$\frac{15}{14}$	$\frac{10}{9}, \frac{16}{15}$	$\frac{16}{15}$	$\frac{16}{15}$	$\frac{10}{9}$	$\frac{8}{7}, \frac{20}{18}$	$\frac{9}{8}, \frac{16}{15}$	$\frac{10}{9}$	$\frac{7}{6}, \frac{8}{7}$	$\frac{10}{9}$	$\frac{11}{10}, \frac{12}{11}$	$\frac{8}{7}, \frac{16}{15}$	$\frac{16}{15}$	$\frac{13}{12}$	$\frac{13}{12}$	$\frac{12}{11}$	$\frac{16}{15}$	Int
	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	1	1, 2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	1	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	Fa
	$\frac{140}{20}$	$\frac{70}{10}$	$\frac{210}{30}$	$\frac{140}{20}$	$\frac{210}{30}$	$\frac{280}{40}$	$\frac{350}{50}$	$\frac{70}{10}$	$\frac{70}{10}, \frac{35}{5}$	$\frac{210}{30}$	$\frac{140}{20}$	$\frac{140}{20}$	$\frac{210}{30}$	$\frac{70}{10}$	$\frac{280}{40}$	$\frac{350}{50}$	$\frac{210}{30}$	$\frac{210}{30}$	$\frac{350}{50}$	$\frac{350}{50}$	Ra
	$\frac{34}{10}$	$\frac{69}{3}$	$\frac{69}{3}$	$\frac{22}{5}$	$\frac{69}{3}, \frac{106}{1}$	$\frac{34}{1}$	$\frac{69}{3}$	$\frac{177}{45}$	$\frac{7}{1}, \frac{34}{1}$	$\frac{177}{45}, \frac{20}{1}$	$\frac{234}{76}$	....	$\frac{177}{45}$	....	.., $\frac{34}{1}$	$\frac{69}{3}$	....	....	....	$\frac{177}{45}$	Int

20	No.
16 12 5	Form.
.... 0_1 100^1 .... 0_1 .... 25 0_1 100^1 .... 11 100 .... 25 25	Intensity of Joint Harmonics.
$\frac{16}{15}$ $\frac{1}{5}$ $\frac{350}{50}$ $\frac{177}{45}$	Interval. Factor. Range. Intensity.

Index.

Form.	No.
1, 3, 5	2
1, 3, 10	8
1, 5, 6	9
1, 5, 12	14
2, 3, 5	1
2, 3, 10	11
2, 5, 6	4
2, 5, 12	12
3, 4, 5	3
3, 4, 10	10
3, 5, 8	5
3, 5, 16	18
3, 8, 10	13
3, 10, 16	17
4, 5, 6	6
4, 5, 12	15
5, 6, 8	7
5, 6, 16	19
5, 8, 12	16
5, 12, 16	20

TABLE X.—Duplicated Forms of the Concordant Triad. (See p. 400.)

No.	Simple.	Duplicated.			Simple.	Duplicated.		
1	2, 3, 5	2, 3, 4, 5	2, 3, 5, 8	2, 3, 5, 6	C G e	C G c e	C G e c^2	C G e g
2	1, 3, 5	1, 2, 3, 5	1, 3, 4, 5	1, 3, 5, 6	C^4 G e	C^4 C G e	C^4 G c e	C^4 G e g
3	3, 4, 5	3, 4, 5, 6	3, 4, 5, 8	3, 4, 5, 10	G c e	G c e g	G c e c^2	G c e e^2
4	2, 5, 6	2, 4, 5, 6	2, 5, 6, 8		C e g	C c e g	C e g c^2	
5	3, 5, 8	3, 5, 6, 8	3, 5, 8, 10		G e c^2	G e g c^2	G e c^2 e^2	
6	4, 5, 6	4, 5, 6, 8	4, 5, 6, 10	4, 5, 6, 12	c e g	c e g c^2	c e g e^2	c e g g^2
7	5, 6, 8	5, 6, 8, 10	5, 6, 8, 12		e g c^2	e g c^2 e^2	e g c^2 g^2	
8	1, 3, 10	1, 2, 3, 10	1, 3, 4, 10	1, 3, 6, 10	C^4 G e^2	C^4 C G e^2	C^4 G c e^2	C^4 G g e^2
9	1, 5, 6	1, 2, 5, 6	1, 4, 5, 6		C^4 e g	C^4 C e g	C^4 c e g	
10	3, 4, 10	3, 4, 6, 10			G c e^2	G c g e^2		
11	2, 3, 10	2, 3, 4, 10	2, 3, 8, 10		C G e^2	C G c e^2	C G c^2 e^2	
12	2, 5, 12	2, 4, 5, 12	2, 5, 8, 12		C e g^2	C c e g^2	C e c^2 g^2	
13	3, 8, 10	3, 6, 8, 10			G c^2 e^2	C g c^2 e^2		
14	1, 5, 12	1, 2, 5, 12	1, 4, 5, 12	1, 5, 8, 12	C^4 e g^2	C^4 C e g^2	C^4 c e g^2	C^4 e c^2 g^2
15	4, 5, 12	4, 5, 8, 12	4, 5, 10, 12	4, 5, 8, 10	c e g^2	c e c^2 g^2	c e c^2 g^2	c e c^2 e^2
16	5, 8, 12	5, 8, 10, 12			e c^2 g^2	e c^2 e^2 g^2		
17	3, 10, 16	3, 6, 10, 16	3, 10, 12, 16		G e^2 c^4	G g e^2 c^4	G e^2 g^2 c^4	
18	8, 5, 16	3, 5, 6, 16	3, 5, 10, 16		G e c^4	G e g c^4	G e e^2 c^4	
19	5, 6, 16	5, 6, 10, 16	5, 6, 12, 16		e g c^4	e g e^2 c^4	e g g^2 c^4	
20	5, 12, 16	5, 10, 12, 16			e g^2 c^4	e e^2 g^2 c^4		

TABLE XI. (See p. 400.)

Forms of the Minor Triad.					Index.	
No.	Form.	Diff. Tones.	Form.	Differential Tones.	Form.	No.
1	3, 5, 15	2, 10, 12	G^4 E b	C^4, e, g	3, 5, 15	1
2	12, 15, 20	3, 5, 8	g b e^2	G^4, E, c	3, 10, 15	8
3	10, 12, 15	2, 3, 5	e g b	C^4, G^4, E	3, 15, 20	12
4	5, 12, 15	3, 7, 10	E g b	G^4, ʒB♭, e	3, 15, 40	18
5	6, 10, 15	4, 5, 9	G e b	C, E, d	5, 6, 15	11
6	15, 20, 24	4, 5, 9	b e^2 g^2	C, E, d	5, 12, 15	4
7	6, 15, 20	5, 9, 14	G b e^2	E, d, ʒb♭	5, 15, 24	16
8	3, 10, 15	5, 7, 12	G^4 e b	E, ʒB♭, g	5, 15, 48	19
9	12, 15, 40	3, 25, 28	g b e^4	G^4, ‡g^2♯, ʒb^2♭	6, 10, 15	5
10	10, 15, 24	5, 9, 14	e b g^2	E, d, ʒb♭	6, 15, 20	7
11	5, 6, 15	1, 9, 10	E G e	C^8, d, e	6, 15, 40	17
12	3, 15, 20	5, 12, 17	G^4 b e^2	E, g, 1ʒd^2♭	10, 12, 15	3
13	15, 20, 48	5, 28, 33	b e^2 g^4	E, ʒb^2♭, xj c^4	10, 15, 24	10
14	15, 24, 40	9, 16, 25	b g^2 e^4	d, c^2, ‡g^2♯	10, 15, 48	20
15	15, 40, 48	8, 25, 33	b e^4 g^4	c, ‡g^2♯, xj c^4	12, 15, 20	2
16	5, 15, 24	9, 10, 19	E b g^2	d, e, $\frac{1216}{1215}$ d^2♯	12, 15, 40	9
17	6, 15, 40	9, 25, 34	G b e^4	d, ‡g^2♯, 1ʒd^4♭	15, 20, 24	6
18	3, 15, 40	12, 25, 37	G^4 b e^4	g, ‡g^2♯, ‡‡e^4♭	15, 20, 48	13
19	5, 15, 48	10, 33, 43	E b g^4	e, xjc^4, $\frac{1419}{1408}$ f^4	15, 24, 40	14
20	10, 15, 48	5, 23, 38	e b g^4	E, $\frac{736}{729}$ †f^2♯, $\frac{1216}{1215}$ d^4♯	15, 40, 48	15

TABLE XII.—Gene

Just note.	Tempered note.	log pere
c, †c	c	·0000
‡c♯	c♯	·0285
d♭	d♭	·0226
	c×	·0570
‡d, d	d	·0511
	e♭♭	·0452
‡d♯	d♯	·0796
†e♭	e♭	·0737
e	e	·1023
	f♭	·0964
	e♯	·1308
f, †f	f	·1249
‡f♯, f♯	f♯	·1534
	g♭	·1475
	f×	·1819
‡g, g	g	·1760
	a♭♭	·1702
‡g♯	g♯	·2046
†a♭	a♭	·1987
	g×	·2331
a, †a	a	·2272
	b♭♭	·2213
	a♯	·2557
b♭, †b♭	b♭	·2498
‡b, b	b	·2783
	c♭	·2725
	b♯	·0058
c^2	c^2	·3010

Whe

I.—General Table of Equal Temperament. (See p. 407.)

n- ed e.	T, log of tempered pitch.	J, log of just pitch.	ϵ, T − J.	β, beat meter.	Interval.
	·0000000	·0000000	0		I
		·0053950	−k		†I
♯	·0285191 − 7x	·0177287	2k − 7x		‡I♯
♭	·0226335 + 5x	·0280285	−k + 5x		2
×	·0570382 − 14x				
	·0511526 − 2x	·0457574	k − 2x		‡II
		·0511526	−2x		II
♭♭	·0452670 + 10x				
♯	·0796717 − 9x	·0688813	2k − 9x		‡II♯
♭	·0737861 + 3x	·0791812	−k + 3x	−6k + 18x	3
	·1023052 − 4x	·0969100	k − 4x	5k − 20x	III
♭	·0964196 + 8x				
♯	·1308243 − 11x				
	·1249387 + x	·1249386	x	4x	4
		·1303338	−k + x		†4
♯	·1534578 − 6x	·1426675	2k − 6x		‡IV
		·1480626	k − 6x		IV
♭	·1475722 + 6x				5
×	·1819769 − 13x				
	·1760913 − x	·1706961	k − x		‡V
		·1760913	−x	−3x	V
♭♭	·1702057 + 11x				
♯	·2046104 − 8x	·1938200	2k − 8x		‡V♯
♭	·1987248 + 4x	·2041199	−k + 4x	−8k + 32x	6
×	·2331295 − 15x				
	·2272439 − 3x	·2218486	k − 3x	5k − 15x	VI
		·2272438	−3x		†VI
♭♭	·2213583 + 9x				
♯	·2557630 − 10x				
♭	·2498774 + 2x	·2498773	2x		7
		·2552725	−k + 2x		†7
	·2783965 − 5x	·2676061	2k − 5x		‡VII
		·2730013	k − 5x		VII
♭	·2725109 + 7x				
♯	·0058851 − 12x				
2	·3010300	·3010300	0		VIII
			$\Sigma\epsilon^2 = 32k^2 - 212kx + 420x^2$	$\Sigma\beta^2 = 150k^2 - 1078kx + 1998x^2$	

Where $k = \cdot 0053950$ and x is arbitrary.

an octave of the root of the minor triad, so that *e, g, b* or 10, 12, 15 is considered by them as derived from E^4 instead of C^8.

Chords will evidently be related to each other when one or more of their constituents are identical, and natural qualities of tone will be related which have one or more identical harmonics, or which form parts of related chords. Transitions between related chords and compound tones will be easy and pleasing. Hence, in forming a collection of compound tones for use either as natural qualities of tone (in melody) or as constituents of artificial qualities of tone, that is, chords (in harmony), it is important to select such as will have numerous relations, and will produce the concordant dyads and triads, and the least dissonant discord, the minor triad. Hence, taking the concordant major triad 1, 3, 5 as a basis, we must possess its products by 2, 3, and 5. There must be abundant multiples by 2 in order to take the several forms of the triad and to introduce the duplications. The products by 3 and 5 give 3, 9, 15 and 5, 15, 25. We have then the tones 1, 3, 5, 9, 15, 25, and their octaves. These give three concordant major triads, 1, 3, 5 ; 3, 9, 15 ; and 5, 15, 25, each of which has one constituent in common with each of the others. We have also the major pentad 1, 3, 5, 9, 15, the nine-tetrad 1, 3, 5, 9, the major tetrad 1, 3, 5, 15, and the minor tetrad 3, 5, 9, 15, whence, by omissions, result the nine-triad 1, 3, 9 and minor triad 3, 5, 15. Each of these triads is related to two of the three major triads. The minor triad is intimately related to all three major triads by having two constituents in common with each of them. The discords involving 7 and 17 would evidently require 1, 3, 5, 7, 17 to be taken as a basis. Neglecting these discords for the present, the above results show that we should obtain a useful series of tones by multiplying 1, 3, 5 successively by 3, and each product by 5, taking octaves above and below all the tones thus introduced. We thus find

MAJOR.	*Minor.*	MAJOR.	MAJOR.	*Minor.*	MAJOR.
			1, 3, 5	3, 5, 15	5, 15, 25
1, 3, 5			F C A		
3, 9, 15	3, 5, 15	5, 15, 25	C G E	*c a e*	A E ‡C♯
9, 27, 45	9, 15, 45	15, 45, 75	G D B	*g e b*	E B ‡G♯
27, 81, 135	27, 45, 135	45, 135, 225	D †A F♯	*d b f*♯	B F♯ ‡D♯
81, 243, 405			†A †E C♯		

Any of the smaller numbers may be multiplied by 2, 4, 8, 16, 32, 64, 128, 256, in order to compare them with the larger numbers. Such multiplications are presumed to have been made in the columns of notes. Hence †*A* : *A*=81 : 5.16, or †=81 : 80,

F♯ : *F*=135 : 1.128, or ♯=135 : 128,

‡*C*♯ : *C*=25 : 3.8, or ‡♯=25 : 24, whence ‡=80 : 81.

And in the same way the other ratios in 'Proceedings,' vol. xiii. p. 95, are reproduced.

In addition to the chords already noticed, we have now the twenty-seven tetrad, 1, 3, 5, 27, or *F C A D*, and the twenty-seven triad, 1, 5, 27, or *F A D*, and all the discords derived from 1, 3, 5, 9, 15, 25, 27, 45. But for those derived from 7 and 17 substitutes must be employed. These are obtained as follows. The chord 9, 27, 45, 1 . 64 is 9 times 1, 3, 5, $7\frac{1}{9}$, so that *G D B F* approximates to 1, 3, 5, 7 in a manner already tested. Again, 1 . 32, 3 . 32, 5 . 32, 225 is 32 times 1, 3, 5, $7\frac{1}{32}$, whence *F C A* ‡*D*♯ gives the second and closer approximation to 1, 3, 5, 7 already considered. When $7\frac{1}{9}$ is used for 7 it will be better to use 1, 3, 5, $7\frac{1}{9}$, $8\frac{8}{9}$, or one-ninth of 9, 27, 45, 1 . 64, 5 . 16, that is *G B D F A*, in place of 1, 3, 5, $7\frac{1}{9}$, 9 or one-ninth of 9, 27, 44, 1 . 64, 81, that is *G B D F* †*A*, to avoid the dissonance 5 . $7\frac{1}{9}$, 4 . 9, or $35\frac{5}{9}$, 36. This will therefore replace the seven-nine pentad 1, 3, 5, 7, 9.

The chord 45, 135, 225, 5 . 64, 3 . 256 is 45 times 1, 3, 5, $7\frac{1}{9}$, $17\frac{1}{15}$, or *B F*♯ ‡*D*♯ *A C*, and it forms an excellent substitute for the seven-seventeen pentad 1, 3, 5, 7, 17. Again, the chord 3 . 16, 5 . 16, 15 . 16, 135, or 16 times 3, 5, 15, $16\frac{7}{8}$, that is *C A E F*♯, is a sufficiently close approximation to the rough discord 3, 5, 15, 17.

It has already been shown that the alterations in the discords thus produced will be slight, and, under certain circumstances, improvements. The omission of 7, 17 in the base 1, 3, 5 is therefore justified. Their insertion would embarrass the performer and composer by an immense variety of tones very slightly differing from each other, as 64, 63 ; 135, 136 ; 255, 256. As it is, the distinction between 81, 80 is the source of much difficulty, and separates chords such as 81, 243, 405, and 5 . 16, 15 . 16, 25 . 16, or 80, 240, 400, that is, †*A* †*E C*♯ and *A E* ‡*C*♯, which composers desire to consider as identical. It was shown in my former paper (Proceedings, vol. xiii. p. 98) that the use of 1, 3, 5 as a basis requires 72 different tones, exclusive of octaves. The introduction of 7 in the base would increase this number by 45, and the introduction of 17 by 30, while the mental effect produced would be very slightly different. On the other hand, if instead of 1, 3, 5 as a base, we took 1, $2v$, $4T$, where v, T are ratios differing slightly from 3 : 2 and 5 : 4, we might avoid the ratio 81 : 80, reduce the number of tones to 27, and greatly increase the relations of chords. How to effect this important result with the least dissonant effect will be considered in the following paper on *Temperament*.

The three major triads 1, 3, 5 ; 3, 9, 15 ; 9, 27, 45 are so related as to form two major pentads, 1, 3, 5, 9, 15 and 3 . 1, 3 . 3, 3 . 5, 3 . 9, 3 . 15. Hence the middle triad forming part of both pentads connects the three triads into a whole, closely related to the middle triad, and therefore to its root. These are called the *tonic* chord and *tonic* tone, and the connexion itself is termed *tonality*. If octaves of these tones be taken, thus,

1 . 32,	3 . 8,	5 . 8 or	*F C A*,
3 . 8,	9 . 4,	15 . 2	*C G E*,
9 . 4,	27,	45	*G D B*,

and the results be taken in order of pitch, we find, on supplying the second octave 3 . 16,

24,	27,	30,	32,	36,	40,	45,	48
C,	*D,*	*E,*	*F,*	*G,*	*A,*	*B,*	*c.*

In this series any two consecutive tones, except 40, 45 or *A, B,* belong to the same major pentad, and these are therefore eminently adapted for successions of chords. Even 40, 45, or 5 . 8, 45, belong to two related discords; for 1, 3, 5, 9, or *F, C, A, G,* and 1, 3, 5, 27, or *F C A D,* have each two constituents in common with 9, 27, 45, 1 . 64, or *G D B F.* The discord 3, 5, 15, 45, or *C A E B,* contains both the tones in question. These considerations justify the major diatonic scale.

The last discord contains a minor triad, 3, 5, 15. These minor triads, from their relations to three major triads, are evidently peculiarly adapted to introduce successions of harmonies. Taking then the three minor triads and forming their octaves, thus

3 . 64,	5 . 32,	15.8 or	c^2 *a e,*
9 . 16,	15 . 8,	45.4	*g e b,*
27 . 8,	45 . 2,	135	d^2 *b f*♯,

we may extend them into a scale,

120,	135,	144,	160,	180,	192,	216,	240
e,	*f*♯,	*g,*	*a,*	*b,*	c^2,	d^2,	e^2,

where the chordal relations are even more intimate than before, and by means of the chord 45, 135, 225, 5 . 64, 3 . 256, or *B F*♯ ‡*D*♯ *a c,* already noticed, the major triad, 45, 135, 225, or *B F*♯ ‡*D*♯, is brought into close connexion with the minor triad, 3, 5, 15, or *c a e.* Practically the use of the minor scale consists of a union of four major triads, 1, 3, 5; 3, 9, 15; 9, 27, 45; 27, 81, 135, forming two major scales, with three other major triads, 5, 15, 25; 15, 45, 75; 45, 135, 225, forming a third major scale, by means of three minor triads, 3, 5, 15; 9, 15, 45; 27, 45, 135, the roots of which, 1, 3, 9, are the same as the roots of the first three major triads. There are therefore seven roots to all the chords introduced, namely 1, 3, 9, 27, and 5, 15, 45, or *F, C, G, D* and *A, E, B,* and these seven roots form a major diatonic scale. From these relations spring all the others which distinguish the minor scale together with all its various forms and its uncertain tonality, which is generally assumed to be the relation of the chords to 15 or *E,* the tonic of the last three major triads, but which evidently wavers between this and 3, 9 or *C, G,* the tonics of the first four major triads, and these three tonics, 3, 9, 15, or *C G E,* form a major triad.

By extending this system of chords up and down, right and left, we arrive at the perfect musical scale in Table V. (Proceedings, vol. xiii. p. 108), which is therefore entirely justified on physical and physiological grounds, without any of those metaphysical assumptions or mystical attributes of numbers which have hitherto disfigured musical science. In that Table the

chords have been arranged in the forms 4, 5, 6 and 10, 12, 15, in accordance with the usual practice of musicians. In the present paper the typical 1, 3, 5 and 3, 5, 15 have, for obvious reasons, been made the basis of the arrangement.

XXIV. "On the Temperament of Musical Instruments with Fixed Tones." By Alexander J. Ellis, F.R.S., F.C.P.S.* Received June 8, 1864.

In the preceding paper on the Physical Constitution of Musical Chords (Proceedings, vol. xiii. p. 392), of which the present is a continuation, I drew attention to the importance of abolishing the distinction between tones which differ by the comma 81 : 80, on account of the number of fresh relations between chords that would be thus introduced. The contrivances necessary for this purpose have long been known under the name of Temperament. I have shown that the musical scale which introduces the comma consists of tones whose pitch is formed from the numbers 1, 3, 5, by multiplying continually by 2, 3, and 5. Hence to abolish the comma it will be necessary to use other numbers in place of these. But this alteration will necessarily change the physical constitution of musical chords, which will now become approximate, instead of exact representatives of qualities of tone with a precisely defined root. It is also evident that all the conjunct harmonics will be thus rendered pulsative, and that therefore all the concords will be decidedly dissonant at all available pitches. The result would be intolerable if the beats were rapid. Temperament, therefore, only becomes possible because very slow beats are not distressing to the ear. Hence temperament may be defined to consist in slightly altering the perfect ratios of the pitch of the constituents of a chord, for the purpose of increasing the number of relations between chords, and facilitating musical performance and composition by the reduction of the number of tones required for harmonious combinations.

The subject has been frequently treated †, but the laws of beats and

* The Tables belonging to this Paper will be found after p. 422.

† I have consulted the following works and memoirs. *Huyghens*, Cosmotheoreos, lib. i.; Cyclus Harmonicus. *Sauveur*, Mémoires de l'Académie, 1701, 1702, 1707, 1717. *Henfling*, Miscellanea Berolinensia, 1710, vol. i. pp. 265–294. *Smith*, Harmonics, 2nd edit. 1759. *Marpurg*, Anfangsgruende der theoretischen Musik, 1757. *Estève*, Mém. de Math. présentés à l'Acad. par divers Savans, 1755, vol. ii. pp. 113–136. *Cavallo*, Phil. Trans. vol. lxxviii. *Romieu*, Mém. de l'Acad., 1758. *Lambert*, Nouveaux Mém. de l'Acad. de Berlin, 1774, pp. 55–73. *Dr. T. Young*, Phil. Trans. 1800, p. 143; Lectures, xxxiii. *Robison*, Mechanics, vol. iv. p. 412. *Farey*, Philosophical Magazine, 1810, vol. xxxvi. pp. 39 and 374. *Delezenne*, Recueil des Travaux de la Société des Sciences, &c. de Lille, 1826–27. *Woolhouse*, Essay on Musical Intervals, 1835. *De Morgan*, On the Beats of Imperfect Consonances, Cam. Phil. Trans. vol. x. p. 129. *Drobisch*, Ueber musikalische Tonbestimmung und Temperatur, Abhandlungen

Table VI.—Classification of Musical Chords. (See p. 397.)

Type.	Example.	Symbol.	Syst. Name.	Ordinary Name.
		I. Concords.		
1, 1	C^1 C^1	C	1 Dyad	Unison (Octave 1, 2).
1, 3	C^1 G	C	3 Dyad	Twelfth (Fifth 2, 3; Fourth 3, 4).
1, 5	C^1 e	C	5 Dyad	Major 17th (Ma. 10th 2, 5; Ma. 3rd 4, 5; Mi. 6th 5, 8).
3, 5	G e	c	3, 5 Dyad	Ma. 6th (Mi. 3rd 5, 6).
1, 3, 5	C^1 G e	C	Major Triad	Common Major Chord.
		II. Strong Discords.		
		1. One Pulsative Constituent.		
1, 7	C^1 g♭	7C	7 Dyad	Perfect 7th 4, 7; Extended tone 7, 8.
3, 7	G g♭	7C	3, 7 Dyad	Contracted 3rd 6, 7; Ext. 6th 7, 12.
5, 7	e g♭	7C	5, 7 Dyad	Contracted 5th 5, 7.
1, 3, 7	C^1 G g♭	7C	3, 7 Triad	Imp. Ch. of Dominant 7th.
1, 5, 7	C^1 e g♭	7C	5, 7 Triad	„ „ „
1, 5, 7	C^1 e vij a♯	7'C	5, 7' Triad	Chord of the Italian 6th.
1, 5, 9	C^1 e d	9C	5, 9 Triad	Imp. Ch. of the 9th.
1, 3, 15	C^1 G b^1	15C	3, 15 Triad	Imp. Ch. of the Mi. 6th, Minor mode.
1, 5, 15	C^1 e b^1	15C	5, 15 Triad	
1, 3, 17	C^1 G lgd♭	17C	3, 17 Triad	Imp. Ch. of the Minor 9th.
1, 5, 17	C^1 e lgd♭	17C	5, 17 Triad	
1, 3, 5, 7	C^1 G e g♭	7C	7 Tetrad	Ch. of the Dominant 7th.
1, 3, 5, 7	C^1 G e vij a♯	7'C	7' Tetrad	„ „ German 6th.
1, 3, 5, 9	C^1 G e d^1	9C	9 Tetrad	„ „ 9th.
1, 3, 5, 15	C^1 G e b^1	15C	Major Tetrad	„ „ Ma. 7th.
1, 3, 5, 17	C^1 G e lgd♭	17C	17 Tetrad	„ „ Mi. 9th (imp.)
1, 3, 5, 27	C^1 G e †a^1	27C	27 Tetrad	„ „ add. 6th, ma. m.
		2. Two Pulsative Constituents.		
1, 3, 5, 7, 9	C^1 G e g♭ d^1	'9C	7, 9 Pentad	Ch. of the added 9th.
1, 3, 5, 7, 17	C^1 G e g♭ lgd♭	'17C	7, 17 Pentad	„ „ Mi. 9th.
1, 3, 5, 9, 15	C^1 G e d^1 b^1	'15C	Major Pentad	„ „ Ma. 9th.
1, 3, 5, 15, 17	C^1 G e b^1 xvij c♯	15, 17C	15, 17 Pentad	„ „ augmented 8th.
		III. Weak Discords.		
		1. One Pulsative Constituent.		
3, 5, 7	G e g♭	7c	7 Triad	Ch. of the Diminished 5th.
1, 3, 9	C^1 G d	9c	9 Triad	„ „ 9th (imp.)
3, 5, 15	G e b	c	Minor Triad	Common Minor Chord.
3, 5, 17	G e lgd♭	17c	17 Triad	Ch. of the Dim. 7th (imp.)
3, 5, 17	G e xvij c♯	17'c	17' Triad	„ „ Minor added 6th (imperfect).
1, 5, 25	C e †g♯	25c	25 Triad	Superfluous Triad.
1, 5, 27	C^1 e †a	27c	27 Triad	Ch. of the Ma. added 6th (imperfect).
		2. Two Pulsative Constituents.		
3, 5, 7, 9	G e g♭ d^1	'9c	7, 9 Tetrad	Ch. of the Mi. 7th, ma. m.
3, 5, 7, 17	G e g♭ lgd♭	'17c	7, 17 Tetrad	„ „ Diminished 7th.
3, 5, 9, 15	G e d b	'c	Minor Tetrad	„ „ Mi. 7th (mi. m.).
3, 5, 15, 17	G e b xvij c♯	15, 17c	15, 17 Tetrad	„ „ added 6th (mi. mode).
1, 5, 15, 25	C e b †g♯	25c	25 Tetrad	„ „ augmented 5th.
3, 5, 15, 45	G e b f♯	'45c	45 Tetrad	„ „ added 9th (mi. mode).

Table VII.—Construction of Musical Chords from Compound Tones. (See p. 398.)

No. of J. H.	Harmonics of the Constituent Tones.														
	1	2	3	4	5	6	7	8	9	10	12	14	15	16	17
1	1														
2	2	2													
3	3	..	3												
4	4	4	..	4											
5	5	..	..	..	5										
6	6	6	6	..	..	6									
7	7	..	..	..	..	..	7								
8	8	8	..	8	..	..	..	8							
9	..	..	9	..	..	..	..	..	9						
10	..	10	..	..	10	..	..	..	..	10					
11	..	..	..	..	..	..	..	..	..	..					
12	..	12	12	12	..	12	..	..	..	..	12				
13	..	..	..	..	..	..	..	..	..	..	..				
14	..	14	..	..	..	..	14	..	..	..	..	14			
15	..	..	15	..	15	..	..	..	..	..	..	..	15		
16	..	16	..	16	..	..	..	16	..	..	..	..	..	16	
17	..	..	..	..	..	..	..	..	..	..	..	..	..	..	17
18	..	..	18	..	..	18	..	..	18	..	..	..	..	..	..
19	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
20	..	..	..	20	20	..	..	..	..	20	..	..	..	..	..
21	..	..	21	..	..	..	21	..	..	..	..	..	..	..	..
24	..	..	24	24	..	24	..	24	..	..	24	..	..	..	..

No. of J. H.	Extent of the Harmonics of the Constituent Tones.														
	1	2	3	4	5	6	7	8	9	10	12	14	15	16	17
1	100														
2	50	100													
3	33	....	100												
4	25	50	....	100											
5	20	....	....	....	100										
6	17	33	50	....	....	100									
7	14	....	....	....	....	....	100								
8	12	25	....	50	....	....	....	100							
9	....	....	33	....	....	....	....	....	100						
10	....	20	....	....	50	....	....	....	....	100					
11	....	....	....	....	....	....	....	....	....	....					
12	....	17	25	33	....	50	....	....	....	....	100				
13	....	....	....	....	....	....	....	....	....	....	....				
14	....	14	....	....	....	....	50	....	....	....	....	100			
15	....	....	20	....	33	....	....	....	....	....	....	....	100		
16	....	12	....	25	....	....	....	50	....	....	....	....	....	100	
17	....	....	....	....	....	....	....	....	....	....	....	....	....	....	100
18	....	....	17	....	....	33	....	....	50	....	....	....	....	....	....
19	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
20	....	....	....	20	25	....	....	....	....	50	....	....	....	....	....
21	....	....	14	....	....	....	33	....	....	....	....	....	....	....	....
24	....	....	12	17	....	25	....	33	....	....	50	....	....	....	....

Table VIII.—Qualities of Concordant Dyads. (See p. 399.)

No. of Joint Harmonics.	1	2	3	4	5	6	7	8	9	10	11	12	13	No.
	2/1	3/2	5/2	3/1	4/3	5/3	5/4	6/5	8/5	12/5	8/3	10/3	16/5	Form.
	Octave.	Fifth.	Ma. Tenth.	Twelfth.	Fourth.	Ma. Sixth.	Ma. Third.	Mi. Third.	Mi. Sixth.	Mi. Tenth.	Eleventh.	Ma. Thirteenth.	Mi. Thirteenth.	Name.
1	100₁	0₁	....	100	0₁	....	0₁	0₁	....	....	....	....	....	Intensities of the Joint Harmonics.
2	225	100	100	25₁	....	0₁	....	....	....	....	....	....	....	
3	11	100	0₁	177	100	100	....	....	0₁	....	100	100	....	
4	56	25	25	6	100	....	100	....	....	....	....	....	....	
5	4	....	100	4	....	100	100	100	100	100	0₁	....	100	
6	25	60	11	45	25	25	....	100	....	....	25	25	....	
7	2	....	....	2	....	....	....	....	....	0₁	....	0₁	....	
8	14	6	6	2	25	....	25	....	100	....	100	....	....	
9	....	11	....	11	11	11	....	....	....	....	11	11	....	
10	4	4	49	....	....	25	25	25	25	25	....	100	25	
11	....	....	....	....	....	....	....	....	....	....	....	....	0₁	
12	3	18	3	6	56	6	11	25	....	100	6	6	....	
13	....	....	....	....	....	....	....	....	....	....	....	....	....	
14	2	2	2	....	....	....	....	....	....	....	....	....	....	
15	....	4	11	4	4	28	11	11	11	11	4	4	11	
16	2	2	2	....	6	....	6	....	25	....	25	....	100	
17	....	....	....	....	....	....	....	....	....	....	....	....	....	
18	....	3	....	3	3	3	....	11	....	....	3	3	....	
19	....	....	....	....	....	....	....	....	....	....	....	....	....	
20	....	....	6	....	4	6	20	6	6	6	....	25	6	
21	....	2	....	2	2	2	....	....	....	....	2	2	....	
24	....	2	....	2	9	2	3	6	11	25	21	2	....	
Beats.	....	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Interval. Factor. Range. Intensity.

Index.	
Form.	Number.
1, 2	1
1, 3	4
2, 3	2
2, 5	3
3, 4	5
3, 5	6
3, 8	11
3, 10	12
4, 5	7
5, 6	8
5, 8	9
5, 12	10
5, 16	13

TABLE IX.—Qualities of Concordant Triads. (See p. 399.)

No. of J. H.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	No.
	5 3 2	5 3 1	5 4 3	6 5 2	8 5 3	6 5 4	8 6 5	10 3 1	6 5 1	10 4 3	10 3 2	12 5 2	10 8 3	12 5 1	12 5 4	12 8 5	16 10 3	16 5 3	16 6 5	16 12 5	Form.
1	0_1	100	0_1	0_1	....	0_1	0_1	100_1	100_1	0_1	0_1	....	....	100	0_1	....	....	....	0_1	....	Intensity of Joint Harmonics.
2	100_1	25_4	0_1	100	0_1	0_1	0_1	25_1	25	....	100	100	0_1	25	....	....	....	0	....	....	
3	100_1	177	100	0_1	100_1	....	0_1	177	11	100	100	0_1	100	11	....	0_1	100	100	....	....	
4	25	6_1	100	25	....	100	....	6	6_1	100	25	25	....	6_1	100	0_1	....	....	....	0_1	
5	100	144	100	100	100_1	100	100	4	144	....	....	100	0_1	144	100	100	....	100	100	100	
6	67	45	25	177	25	100	100	45	137	25_1	67	11	25	3	....	....	25	25	100	....	
7	....	2	....	....	....	....	....	2_1	2	0_1	0_1	0_1	0_1	0_1	0_1	0_1	0_1	....	....	0_1	
8	6	2	25	6	100	25	100	2	2	25	6_1	6	100	2	25	100	....	....	....	....	
9	11	11	11	....	11	....	....	33_1	....	11	11	....	11	....	....	....	11	11	....	....	
10	49	25	25	49	25	25	25	100	25	100	144	19_1	100	25	25	25	100	25	50	25	
11	....	....	....	....	....	....	....	....	....	....	....	....	....	0_1	....	....	....	0_1	0_1	0_1	
12	16	6	34	45	6	69	25	6	25	34	16	137	6	225	177	100	6	6	50	100	
13	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	0_1	0_1	....	....	
14	2	....	....	2	....	....	....	....	....	....	2	2	....	....	....	....	....	....	....	....	
15	20	28	28	11	28	11	11	4	11	4	4	11	4	11	11	11	4	28	33	11	
16	2	....	6	....	25	6	25	....	....	6	2	....	25	....	6	25	100	100	100	100	
17	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	
18	3	3	3	21	3	11	11	3	11	3	3	2	3	....	....	....	3	3	33	....	
19	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	
20	4	6	20	6	6	20	6	25	6	20	4	6	25	6	20	6	25	6	6	25	
21	2	2	2	....	2	....	....	2	....	2	....	....	2	....	....	....	....	....	....	....	
24	....	....	8	6	20	18	34	1	6	8	1	25	20	25	45	69	1	1	6	25	
Beats.	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Interval. Factor. Range. Intensity.

Index.

Form.	No.
1, 3, 5	2
1, 3, 10	8
1, 5, 6	9
1, 5, 12	14
2, 3, 5	1
2, 3, 10	11
2, 5, 6	4
2, 5, 12	12
3, 4, 5	3
3, 4, 10	10
3, 5, 8	5
3, 5, 16	18
3, 8, 10	13
3, 10, 16	17
4, 5, 6	6
4, 5, 12	15
5, 6, 8	7
5, 6, 16	19
5, 8, 12	16
5, 12, 16	20

Table X.—Duplicated Forms of the Concordant Triad. (See p. 400.)

No.	Simple.	Duplicated.			Simple.	Duplicated.		
1	2, 3, 5	2, 3, 4, 5	2, 3, 5, 8	2, 3, 5, 6	C G e	C G c e	C G e c^2	C G e g
2	1, 3, 5	1, 2, 3, 5	1, 3, 4, 5	1, 3, 5, 6	C^1 G e	C^1 C G e	C^1 G c e	C^1 G e g
3	3, 4, 5	3, 4, 5, 6	3, 4, 5, 8	3, 4, 5, 10	G c e	G c e g	G c e c^2	G c e e^2
4	2, 5, 6	2, 4, 5, 6	2, 5, 6, 8		C e g	C c e g	C e g c^2	
5	3, 5, 8	3, 5, 6, 8	3, 5, 8, 10		G e c^2	G e g c^2	G e c^2 e^2	
6	4, 5, 6	4, 5, 6, 8	4, 5, 6, 10	4, 5, 6, 12	c e g	c e g c^2	c e g e^2	c e g g^2
7	5, 6, 8	5, 6, 8, 10	5, 6, 8, 12		e g c^2	e g c^2 e^2	e g c^2 g^2	
8	1, 3, 10	1, 2, 3, 10	1, 3, 4, 10	1, 3, 6, 10	C^1 G e^2	C^1 C G e^2	C^1 G c e^2	C^1 G g e^2
9	1, 5, 6	1, 2, 5, 6	1, 4, 5, 6		C^1 e g	C^1 C e g	C^1 c e g	
10	3, 4, 10	3, 4, 6, 10			G c e^2	G c g e^2		
11	2, 3, 10	2, 3, 4, 10	2, 3, 8, 10		C G e^2	C G c e^2	C G c^2 e^2	
12	2, 5, 12	2, 4, 5, 12	2, 5, 8, 12		C e g^2	C c e g^2	C e c^2 g^2	
13	3, 8, 10	3, 6, 8, 10			G c^2 e^2	G g c^2 e^2		
14	1, 5, 12	1, 2, 5, 12	1, 4, 5, 12	1, 5, 8, 12	C^1 e g^2	C^1 C e g^2	C^1 c e g^2	C^1 e c^2 g^2
15	4, 5, 12	4, 5, 8, 12	4, 5, 10, 12	4, 5, 8, 10	c e g^2	c e c^2 g^2	c e e^2 g^2	c e c^2 e^2
16	5, 8, 12	5, 8, 10, 12			e c^2 g^2	e c^2 e^2 g^2		
17	3, 10, 16	3, 6, 10, 16	3, 10, 12, 16		G e^2 c^3	G g e^2 c^3	G e^2 g^2 c^3	
18	3, 5, 16	3, 5, 6, 16	3, 5, 10, 16		G e c^3	G e g c^3	G e e^2 c^3	
19	5, 6, 16	5, 6, 10, 16	5, 6, 12, 16		e g c^3	e g e^2 c^3	e g g^2 c^3	
20	5, 12, 16	5, 10, 12, 16			e g^2 c^3	e e^2 g^2 c^3		

Table XI. (See p. 400.)

Forms of the Minor Triad.					Index.	
No.	Form.	Diff. Tones.	Form.	Differential Tones.	Form.	No.
1	3, 5, 15	2, 10, 12	G^1 E b	C^1, e, g	3, 5, 15	1
2	12, 15, 20	3, 5, 8	g b e^2	G^1, E, e	3, 10, 15	8
3	10, 12, 15	2, 3, 5	e g b	C^1, G^1, E	3, 15, 20	12
4	5, 12, 15	3, 7, 10	E g b	G^1, ₇b♭, e	3, 15, 40	18
5	6, 10, 15	4, 5, 9	G e b	C, E, d	5, 6, 15	11
6	15, 20, 24	4, 5, 9	b e^2 g^2	C, E, d	5, 12, 15	4
7	6, 15, 20	5, 9, 14	G b e^2	E, d, ₇b♭	5, 15, 24	16
8	3, 10, 15	5, 7, 12	G^1 e b	E, ₇b♭, g	5, 15, 48	19
9	12, 15, 40	3, 25, 28	g b e^4	G^1, 1g♯, ₇b♭	6, 10, 15	5
10	10, 15, 24	5, 9, 14	e b g^2	E, d, ₇b♭	6, 15, 20	7
11	5, 6, 15	1, 9, 10	E G e	C^2, d, e	6, 15, 40	17
12	3, 15, 20	5, 12, 17	G^1 b e^2	E, g, 1₇d♭	10, 12, 15	3
13	15, 20, 48	5, 28, 33	b e^2 g^4	E, ₇b♭, xj e^4	10, 15, 24	10
14	15, 24, 40	9, 16, 25	b g^2 e^4	d, c^2, 1g♯	10, 15, 48	20
15	15, 40, 48	8, 25, 33	b e^4 g^4	c, 1g♯, xj e^4	12, 15, 20	2
16	5, 15, 24	9, 10, 19	E b g^2	d, e, [illegible] d♯	12, 15, 40	9
17	6, 15, 40	9, 25, 34	G b e^4	d, 1g♯, 1₇d♭	15, 20, 24	6
18	3, 15, 40	12, 25, 37	G^1 b e^4	g, 1g♯, 11e♭	15, 20, 48	13
19	5, 15, 48	10, 33, 43	E b g^4	e, xje^4, [illegible] f^4	15, 24, 40	14
20	10, 15, 48	5, 23, 38	e b g^4	E, [illegible] f♯, [illegible] d♯	15, 40, 48	15

Table XII.—General Table of Equal Temperament. (See p. 407.)

Just note.	Tempered note.	T, log of tempered pitch.	J, log of just pitch.	ε, T − J.	β, beat meter.	Interval.
c †c ‡c♯ d♭	c c♯ d♭ c×	·0000000 ·0285191 − 7x ·0293335 + 5x ·0570382 − 14x	·0000000 ·0053950 ·0177287 ·0280285	0 − k 2k − 7x − k + 5x		I †I ‡I♯ 2
‡d d ‡d♯ †e♭	d e♭♭ d♯ e♭	·0511526 − 2x ·0452670 + 10x ·0796717 − 9x ·0737861 + 3x	·0457574 ·0511526 ·0688813 ·0791812	k − 2x − 2x 2k − 9x − k + 3x	 − 6k + 18x	‡II II ‡II♯ 3
e	e f♭ e♯	·1023052 − 4x ·0964196 + 8x ·1308243 − 11x	·0969100	k − 4x	5k − 20x	III
f †f ‡f♯ f♯	f f♯ g♭ f×	·1249387 + x ·1534578 − 6x ·1475722 + 6x ·1819769 − 13x	·1249380 ·1303338 ·1426675 ·1480626	x − k + x 2k − 6x k − 6x	4x	4 †4 ‡IV IV 5
‡g g ‡g♯ †a♭	g a♭♭ g♯ a♭ g×	·1760913 − x ·1702057 + 11x ·2046104 − 8x ·1987248 + 4x ·2331295 − 15x	·1706961 ·1760913 ·1938200 ·2041199	k − x − x 2k − 8x − k + 4x	 − 3x − 8k + 32x	‡V V ‡V♯ 6
a †a b♭ †b♭	a b♭♭ a♯ b♭	·2272439 − 3x ·2213583 + 9x ·2557630 − 10x ·2498774 + 2x	·2218486 ·2272438 ·2498773 ·2552725	k − 3x − 3x 2x − k + 2x	5k − 15x	VI †VI 7 †7
‡b b c^2	b c♭ b♯ c^2	·2783665 − 5x ·2725109 + 7x ·0068851 − 12x ·3010300	·2676061 ·2730013 ·3010300	2k − 5x k − 5x 0		‡VII VII VIII
				$\Sigma\epsilon^2$ $=32k^2$ $-212kx$ $+420x^2$	$\Sigma\beta^2$ $=150k^2$ $-1078kx$ $+1008x^2$	

Where $k = {\cdot}0053950$ and x is arbitrary.