chords have been arranged in the forms 4, 5, 6 and 10, 12, 15, in accordance with the usual practice of musicians. In the present paper the typical 1, 3, 5 and 3, 5, 15 have, for obvious reasons, been made the basis of the arrangement.

XXIV. "On the Temperament of Musical Instruments with Fixed Tones." By Alexander J. Ellis, F.R.S., F.C.P.S.* Received June 8, 1864.

In the preceding paper on the Physical Constitution of Musical Chords (Proceedings, vol. xiii. p. 392), of which the present is a continuation, I drew attention to the importance of abolishing the distinction between tones which differ by the comma 81 : 80, on account of the number of fresh relations between chords that would be thus introduced. The contrivances necessary for this purpose have long been known under the name of Temperament. I have shown that the musical scale which introduces the comma consists of tones whose pitch is formed from the numbers 1, 3, 5, by multiplying continually by 2, 3, and 5. Hence to abolish the comma it will be necessary to use other numbers in place of these. But this alteration will necessarily change the physical constitution of musical chords, which will now become approximate, instead of exact representatives of qualities of tone with a precisely defined root. It is also evident that all the conjunct harmonics will be thus rendered pulsative, and that therefore all the concords will be decidedly dissonant at all available pitches. The result would be intolerable if the beats were rapid. Temperament, therefore, only becomes possible because very slow beats are not distressing to the ear. Hence temperament may be defined to consist in slightly altering the perfect ratios of the pitch of the constituents of a chord, for the purpose of increasing the number of relations between chords, and facilitating musical performance and composition by the reduction of the number of tones required for harmonious combinations.

The subject has been frequently treated †, but the laws of beats and

* The Tables belonging to this Paper will be found after p. 422.

† I have consulted the following works and memoirs. *Huyghens*, Cosmotheoreos, lib. i.; Cyclus Harmonicus. *Sauveur*, Mémoires de l'Académie, 1701, 1702, 1707, 1717. *Henfling*, Miscellanea Berolinensia, 1710, vol. i. pp. 265–294. *Smith*, Harmonics, 2nd edit. 1759. *Marpurg*, Anfangsgruende der theoretischen Musik, 1757. *Estève*, Mém. de Math. présentés à l'Acad. par divers Savans, 1755, vol. ii. pp. 113–136. *Cavallo*, Phil. Trans. vol. lxxviii. *Romieu*, Mém. de l'Acad., 1758. *Lambert*, Nouveaux Mém. de l'Acad. de Berlin, 1774, pp. 55–73. *Dr. T. Young*, Phil. Trans. 1800, p. 143; Lectures, xxxiii. *Robison*, Mechanics, vol. iv. p. 412. *Farey*, Philosophical Magazine, 1810, vol. xxxvi. pp. 39 and 374. *Delezenne*, Recueil des Travaux de la Société des Sciences, &c. de Lille, 1826–27. *Woolhouse*, Essay on Musical Intervals, 1835. *De Morgan*, On the Beats of Imperfect Consonances, Cam. Phil. Trans. vol. x. p. 129. *Drobisch*, Ueber musikalische Tonbestimmung und Temperatur, Abhandlungen

composition of tones discovered by Prof. Helmholtz have enabled me to present it in an entirely new form, and to determine with some degree of certainty what is the best possible form of temperament.

Let the compound tones P and Q, of which P is the sharper, form the concordant interval $p:q$. Then $P:Q=p:q$, or $qP=pQ$, that is, the qth harmonic of P and the pth harmonic of Q are conjunct. Now let P be changed into $P.(1+t)$, where t is small, and rarely or never exceeds $\frac{1}{80}=\cdot 0125$. Then the qth harmonic of $P.(1+t)$ will be $qP.(1+t)$ and will no longer $=pQ$. The difference between the pitch of these harmonics is $qP.(1+t)-pQ=qt.P=pt.Q$. Hence the number of beats in a second produced by this change in P will be found by multiplying the lower pitch Q by pt, which is therefore the beat factor, and will be positive or negative according as the pitch of P is increased or diminished, or the interval is sharpened or flattened. The other beats which existed between the joint harmonics of the dyad P, Q may be increased or diminished by this change, but in either case so slightly that they may be left out of consideration in comparison with the beats thus introduced. But the differential tone which was $P-Q$ becomes $Pt+P-Q$, and is therefore a tone which is entirely unrelated to the original chord, and which may become prominently dissonant. This is an evil which cannot be avoided by any system of temperament, and is about equally objectionable in all systems. It may therefore be also left out of consideration in selecting a temperament.

The melody will also suffer from the alteration in the perfect ratios. An interval is best measured by the difference of the tabular logarithms of the pitches of the two tones which form it. Hence the *interval error* $\epsilon=\log[P.(1+t)\div Q]-\log[P:Q]=\log(1+t)=\mu t$, if the square and higher powers of t be neglected, and μ be the modulus. Hence the beat factor which $=pt$, will $=p\epsilon\div\mu$, or $\propto p\epsilon$. I call $p\epsilon$ the *beat meter*, and represent it by β.

We may assume that the dissonance created by temperament $\propto\beta^2$. Hence for the same just interval $p:q$, variously represented in different temperaments, the dissonance $\propto\epsilon^2$. That is, the harmony varies inversely as β^2, and the melody varies inversely as ϵ^2. Hence for the same interval the harmony and melody both vary inversely as ϵ^2. The *general* harmony and melody may be assumed to be best when $\Sigma\beta^2$ and $\Sigma\epsilon^2$ are minima, which will not happen simultaneously.

The following contractions for the names of the principal intervals will

der k. Sächsischen Gesellschaft der Wissenschaften, vol. iv. Nachträge zur Theorie der musikalischen Tonverhältnisse, ibid. vol. v. Ueber die wissenschaftliche Bestimmung der musikalischen Temperatur, Poggendorff's Annalen, vol. xc. p. 353. *Naumann*, Ueber die verschiedene Bestimmung der Tonverhältnisse und die Bedeutung des Pythagoreischen oder reinen Quinten-Systems für unsere heutige Musik, 1858. *Helmholtz*, Die Lehre von den Tonempfindungen, 1863. I am most indebted to Smith, Drobisch, and Helmholtz.

be used throughout this paper. See also the last columns in Tables XII. and XIV.

Sign.	Interval.	Example.	Sign.	Interval.	Example.
Ist.	Unison	c c			
IInd.	Major Second	c d	2nd.	Minor Second	e f
IIIrd.	Major Third	c e	3rd.	Minor Third	e g
IVth.	Augmented Fourth	f b	4th.	Perfect Fourth ..	c f
Vth.	Perfect Fifth	c g	5th.	Diminished Fifth ..	b f^2
VIth.	Major Sixth	c a	6th.	Minor Sixth	e c^2
VIIth.	Major Seventh ..	c b	7th.	Minor Seventh ..	g f^2
VIIIve.	Octave	c c^2			
IXth.	Major Ninth	c d^2	9th.	Minor Ninth	e f^2
Xth.	Major Tenth	c e^2	10th.	Minor Tenth	e g^2

In no system of temperament will it be possible to interfere with the octave, the only unisonant concord. Hence 2 will remain unchanged. Let the ratios of the tempered IIIrd and Vth be T, v, which will replace 5 : 4 and 3 : 2 throughout the system of chords. Hence if we take four successive perfect major triads in the form 4, 5,6 as $C\,E\,G$, $G\,B\,d$, $d f\sharp\ \dagger a$, $\dagger a\ c^2\sharp\ \dagger e^2$, and suppose them to be tempered so that the distinction between E and $\dagger E$ no longer exists, but that in each chord the pitch of the second and third tones are T and v times that of the first tone respectively, while the ratio of the octave remains unchanged, the ratio of each of the above tones to C will be as under:—

$$\begin{array}{ccccccccc} C, & E, & G, & B, & d, & f\sharp, & a, & c^2\sharp, & e^2 \\ 1, & T, & v, & Tv, & v^2, & Tv^2, & v^3, & Tv^3, & v^4. \end{array}$$

Hence, since $e^2=4E$, we have $v^4=4T$ as the first condition of temperament, showing that we shall arrive at the same tone whether we take two VIIIves and a tempered IIIrd, or take four tempered Vths, as in Cc, cc^2, $c^2\,e^2$, and $C\,G$, Gd, da, ae^2. In this case the above ratios reduce to

$$\begin{array}{ccccccccc} C, & E, & G, & B, & d, & f\sharp, & a, & c^2\sharp, & e^2 \\ 1, & \frac{1}{4}v^4, & v, & \frac{1}{4}v^5, & v^2, & \frac{1}{4}v^6, & v^3, & \frac{1}{4}v^7, & v^4. \end{array}$$

If we further call the interval of the mean tone m, the limma l, the sharp $\sharp$, the flat $\flat$, and the diesis δ, the above ratios give

$$m=\frac{d}{c}=\frac{d}{2C}=\frac{v^2}{2},$$

$$l=\frac{c}{\mathrm{B}}=\frac{2C}{\mathrm{B}}=\frac{2^3}{v^5},$$

$$\sharp=\frac{c^2\sharp}{c^2}=\frac{c^2\sharp}{4C}=\frac{v^7}{2^4};\qquad \flat=\frac{1}{\sharp}=\frac{2^4}{v^7},$$

$$\delta=\frac{d^2\flat}{c^2\sharp}=\frac{2v^2\times(2^4\div v^7}{v^7\div 4}=\frac{2^7}{v^{12}}.$$

Whence $m=\sharp l$, $l=\delta\sharp$, $m=\sharp\delta\sharp$, $m^5\,l^2=2$.

Hence all intervals and pitches can be expressed in terms of v. This further appears from arranging the 27 different tones required in tempered scales, in order of Vths, thus

$$a\flat\flat,\ e\flat\flat,\ b\flat\flat,\ f\flat,\ c\flat,\ g\flat,\ d\flat,\ a\flat,\ e\flat,\ b\flat,$$
$$f,\ c,\ g,\ d,\ a,\ e,\ b,$$
$$f\sharp,\ c\sharp,\ g\sharp,\ d\sharp,\ a\sharp,\ e\sharp,\ b\sharp,\ f\times,\ c\times,\ g\times.$$

It will be obvious from Table V. (Proceedings, vol. xiii. facing p. 108), when the signs † ‡ are omitted, that these 27 tones suffice for all keys from $C\flat$ to $C\sharp$. This also appears from observing that the complete key of C requires 7 naturals, 3 flats and 3 sharps, or 13 tones, and that one flat or sharp is introduced for each additional flat or sharp in the signature of the key. Hence for 7 flats and 7 sharps in the signature 14 additional tones are required, making 27 in all. The rarity of the modulations into $d\flat$, $g\flat$ or $c\flat$ minor enables us generally to dispense with the three tones $a\flat\flat$, $e\flat\flat$, $b\flat\flat$, and thus to reduce all music to 24 tones. The system of writing music usually adopted is only suitable to such a tempered scale, and therefore requires the addition of the acute and grave signs († ‡) to adapt it for a representation of the just scale founded on the numbers 1, 2, 3, 5.

To calculate the value which must be assigned to v so as to fulfil the conditions supposed to produce the least disagreeable system of temperament, it will be most convenient to use logarithms, and to put log $v=\log\frac{3}{2}-x=\cdot1760913-x$. The above arrangement of the requisite 27 tones in order of Vths, therefore, enables us to calculate the logarithms of the ratios of the pitches of all the tones to the pitch of c in terms of x, by continual additions and subtractions of log v, rejecting or adding log 2 $=\cdot3010300$, when necessary, to keep all the tones in the same VIIIve. The result is tabulated in Table XII., column T. From this we immediately deduce

$$\begin{aligned}
\log m &= \log d - \log c = \cdot0511526 - 2x \\
\log l &= \log f - \log e = \cdot0226335 + 5x \\
\log \sharp &= \log f\sharp - \log f = \cdot0285191 - 7x \\
\log \delta &= \log g\flat - \log f\sharp = -\cdot0058851 + 12x.
\end{aligned}$$

To find the interval errors, the just intervals must be taken for the commonest modulations into the subdominant and dominant keys, as explained in my paper on a Perfect Musical Scale (Proceedings, vol. xiii. p. 97). As the method of determining temperament here supposed makes the errors the same for the same intervals in all keys, that is, makes the temperament *equal*, it is sufficient to determine the interval errors for a single key. Hence the just intervals are calculated in Table XII., column J, for the key of C, and the interval error is given in column ϵ, in terms of x and $k=\log\frac{81}{80}$, the interval of a comma. From these interval errors the beat meters for the six concordant dyads are calculated in column β. To these are added the values of $\Sigma\,\epsilon^2$ and $\Sigma\,\beta^2$, also in terms

of x and k. If for k we put its value ·0053950, these last expressions become

$$\Sigma\epsilon^2 = \cdot 0009314 - 1\cdot 1437400x + 420x^2$$
$$\Sigma\beta^2 = \cdot 00043659 - 5\cdot 8158100x + 1998x^2.$$

Hence Table XII. suffices to give complete information respecting the effect of any system of temperament when x is known. The following are some of the principal conditions on which it has been proposed to found a system of temperament. I shall first determine the value of x and $\log v$ on these conditions, and then compare the results.

A. Harmonic Systems of Equal Temperament.

I. *Systems with two concords perfect.*

No. 1 (45)*. System of perfect 4ths and Vths.

Here $x=0$, $\log v = \cdot 1760913$.

This is the old Greek or Pythagorean system of musical tones, more developed in the modern Arabic scale of 17 tones. No nation using it has shown any appreciation of harmony.

No. 2 (2). System of perfect IIIrds and 6ths.

Here ϵ for III, or $k-4x=0$, $x=\frac{1}{4}k=\cdot 00134875$, $\log v = \cdot 17474255$. Hence $\log m = \log d = \cdot 0484551 = \frac{1}{2}\log\frac{5}{4} = \frac{1}{2}(\log\frac{9}{8} + \log\frac{10}{9})$, so that the tempered mean tone is an exact mean between the just major and minor tones. Hence this is known as the System of Mean Tones, or the *Mesotonic System*, as it will be here termed. It was the earliest system of temperament, and is claimed by Zarlino and Salinas. See also Nos. 13 and 19.

No. 3 (23). System of perfect 3rds and VIths.

Here ϵ for 3, or $-k+3x=0$, $x=\frac{1}{3}k=\cdot 0017983$, $\log v = \cdot 1742930$.

II. *Systems in which the harmony of two concords is equal.*

No. 4 (20). The IIIrd and Vth to the same bass; beat equally and in opposite directions†.

Here β for III $+\beta$ for V $=0$, or $(5k-20x)-3x=0$, $x=\frac{5}{23}k=\cdot 0011725$, $\log v = 1749188$.

No. 5 (15). The 6th and Vth beat equally, and in the same direction‡.

Here β for $6=\beta$ for V, or $-8k+32x=-3x$, $x=\frac{8}{35}k=\cdot 0012331$, $\log v = \cdot 1748582$.

* The number preceded by No. points out the order of the system in the present classification. The number in a parenthesis shows the position of the system in the comparative Table XV., which is explained hereafter (p. 418).

† That is, one interval is too great, or "beats sharp," and the other too small, or "beats flat."

‡ That is, both "beat sharp" or both "beat flat."

No. 6 (21). The IIIrd and 4th beat equally, and in the same direction.

Here β for III$=\beta$ for 4, or $5k-20x=4x$, $x=\frac{5}{24}k=\cdot0011239$, log $v\doteq\cdot1749674$.

No. 7 (18). The 6th and 4th beat equally, and in opposite directions.

Here β for $6+\beta$ for $4=0$, or $(-8k+32x)+4x=0$, $x=\frac{2}{9}k=\cdot0011989$, log $v=1748924$.

No. 8 (16). The 3rd and Vth beat equally, and in the same direction.

Here β for $3=\beta$ for V, or $-6k+18x=-3x$, $x=\frac{2}{7}x=\cdot0015414$, log $v=\cdot1745199$. See No. 20.

No. 9 (13). The VIth and Vth beat equally, and in opposite directions.

Here β for VI$+\beta$ for V$=0$, or $(5k-15x)-3x=0$, $x=\frac{5}{18}k=\cdot0014986$, log $v=\cdot1745927$.

This coincides with Dr. Smith's system of equal harmony, as contained in the Table facing p. 224 of his 'Harmonics,' 2nd ed.

No. 10 (9). The 3rd and 4th beat equally, and in opposite directions.

Here β for $3+\beta$ for $4=0$, or $(-6k+18x)+4x=0$, $x=\frac{3}{11}k=\cdot0014713$, log $v=\cdot1746200$.

No. 11 (2). The VIth and 4th beat equally, and in the same direction.

Here β for VI$=\beta$ for 4, or $5k-15x=4x$, $x=\frac{5}{19}k=\cdot0014197$, log $v=\cdot1746716$.

III. *Systems in which the harmony of two concords is in a given ratio.*

No. 12 (24). The beats of the IIIrd and Vth are as 5 : 3, but in opposite directions.

Here β for III : β for V$=-5:3$, or $15k-60x=15x$, $x=\frac{1}{5}k=\cdot0010790$, log $v=\cdot1750123$.

M. Romieu gives this temperament under the title of "système tempéré de $\frac{1}{5}$ comma," Mém. de l'Acad. 1758. See No. 18.

No. 13 (2). The beats of the 3rd and Vth are as 2 : 1, and in the same direction.

Here β for 3 : β for V$=2$, or $-6k+18x=-6x$, $x=\frac{1}{4}k$, as in No. 2.

No. 14 (12). The beats of the 3rd and Vth are as 5 : 2, and in the same direction.

Here β for 3 : β for V$=5:2$, or $-12k+36x=-15x$, $x=\frac{4}{17}k=\cdot0012694$, log $v=\cdot1748219$. See No. 29.

IV. *Systems of least harmonic errors.*

No. 15 (7). The harmonic errors of all the harmonic intervals conjointly are a minimum.

This is determined by putting the sum of the squares of the beat meters,

or (by Table XII.) $150k^2-1078kx+1998x^2$=a minimum, which gives

$$x=\tfrac{539}{1998}\ k=\cdot 0014554,\ \log v=\cdot 1746359.$$

If we had used the sum of the squares of the beat *factors*, we should have obtained an equation of 16 dimensions in v, which gives $\log v=\cdot 1746387$. The difference between the two values of $\log v$ is not appreciable to the ear.

No. 16 (14). The harmonic errors of the 3rd, IIIrd, and Vth conjointly are a minimum.

Here $(\beta$ for $3)^2+(\beta$ for $\mathrm{III})^2+(\beta$ for $\mathrm{V})^2$, or
$(6k-18x)^2+(5k-20x)^2+9x^2$=a minimum, which gives

$$x=\tfrac{208}{733}k=\cdot 0015309,\ \log v=\cdot 1744404.$$

No. 17 (6). The harmonic errors of the Vth and IIIrd conjointly are a minimum.

Here $(\beta$ for $\mathrm{V})^2+(\beta$ for $\mathrm{III})^2$, or $9x^2+(5k-20x)^2$=a minimum, $x=\tfrac{100}{409}k=\cdot 0013190$, $\log v=\cdot 1747723$.

B. Melodic Systems of Equal Temperament.

V. *Systems of equal or equal and opposite interval errors.*

No. 18 (24). The interval errors of the IIIrd and Vth are equal and opposite.

Here ϵ for III+ ϵ for V=0, or $k-4x=x$, $x=\tfrac{1}{5}k$, as in No. 12.

No. 19 (2). The interval errors of the 3rd and Vth are equal.
Here ϵ for 3 $=\epsilon$ for V, or $-k+3x=-x$, $x=\tfrac{1}{4}k$, as in No. 2,

No. 20 (16). The interval errors of the IIIrd and 3rd are equal.
Here ϵ for III$=\epsilon$ for 3, or $k-4x=-k+3x$, $x=\tfrac{2}{7}k$, as in No. 8.

VI. *Systems in which the interval errors of two intervals are in a given ratio.*

No. 21 (17). The errors of the IIIrd and Vth are as 5 : 3, but in opposite directions.

Here ϵ for III : ϵ for V $=-5:3$, or $3k-12x=5x$, $x=\tfrac{3}{17}k=\cdot 0015750$, $\log v=\cdot 1745163$.

This is the theoretical determination of M. Romieu's anacratic temperament (Mém. de l'Acad. 1758, p. 510), to which, however, he has in practice preferred No. 22.

No. 22 (29). The errors of the IIIrd and Vth are as 2 : 1, but in opposite directions.

Here ϵ for III : ϵ for V$=-2$, or $k-4x=2x$, $x=\tfrac{1}{6}k=\cdot 0008975$, $\log v=\cdot 1751938$.

This is M. Romieu's anacratic temperament. See No. 21.

No. 23 (26). The errors of the IIIrd and Vth are as 1·94 : 1, and in opposite directions.

Here ϵ for III : ϵ for V $=-1{\cdot}94$, or $k-4x=1{\cdot}94x$, $x=\frac{100}{594}k={\cdot}0009683$, log $v={\cdot}1751830$.

This is the temperament calculated by Drobisch (Nachträge, § 10) from Delezenne's conclusion (Rec. Soc. Lille, 1826–27, pp. 9 and 10), that the ear can detect an error of $\cdot 284k$ in the IIIrd, and $\cdot 146k$ in the Vth, which gives the comparative sensibility as $\cdot 284 : \cdot 146 = 1{\cdot}94$.

No. 24 (20). The errors of the IIIrd and 3rd are as 2 : 5, but in opposite directions.

Here ϵ for III : for ϵ 3 $=-2:5$, or $5k-20x=2k-6x$, $x=\frac{3}{14}k$ $={\cdot}0011561$, log $v={\cdot}1749352$. See No. 27.

No. 25 (46). The errors of the 3rd and IIIrd are as 2 : 1, but in opposite directions, or the errors of the Vth and 3rd are equal and opposite.

Here ϵ for 3 : ϵ for III $=-2$, or $2k-6x=k-4x$, or else $x=-k+3x$; both give $x=\frac{1}{2}k={\cdot}0026975$, log $v={\cdot}1733938$.

Here the error of the Vth reaches the utmost limit of endurance.

VII. *Systems of least melodic errors.*

No. 26 (1). The interval errors of all the melodic intervals conjointly are a minimum.

Here the sum of the squares of the 23 interval errors in Table XII., or $32k^2-212kx+420x^2=$ a minimum, $x=\frac{53}{210}k={\cdot}0013616$, log $v={\cdot}1747297$.

No. 27 (20). The melodic errors of the IInd, IIIrd, 4th, Vth, VIth, and VIIth conjointly, are a minimum.

Here (ϵ for II)^2+(ϵ for III)^2+(ϵ for 4)^2+(ϵ for V)^2+(ϵ for VI)^2+ (ϵ for VII)^2, or $4x^2+(k-4x)^2+2x^2+(k-3x)^2+(k-5x)^2=$ a minimum, $x=\frac{3}{14}k$, as in No. 24.

This is Drobisch's "most perfect possible" (*möglich reinste*) temperament (Poggendorff's Annalen, vol. xc. p. 353, as corrected in Nachträge, § 7). It is only the "most perfect possible" for the major scale.

No. 28 (5). The melodic errors of the 3rd, IIIrd, and 4th conjointly are a minimum.

Here (ϵ for 3)^2+(ϵ for III)^2+(ϵ for 4)^2, or $(-k+3x)^2+(k-4x^2)$ $+x^2=$ a minimum, $x=\frac{7}{26}k={\cdot}0014525$, log $v={\cdot}1746388$.

This is Woolhouse's Equal Harmony (Essay on Musical Intervals, p. 45).

No. 29 (12). The melodic errors of the IIIrd and Vth conjointly are a minimum.

Here (ϵ for III)^2+(ϵ for V)^2, or $x^2+(k-4x)^2=$ a minimum, $x=\frac{4}{17}k$, as in No. 14.

This is given by Drobisch (Nachträge, § 8) as "the simplest solution of the problem."

C. Combined Systems of Equal Temperament.

No. 30 (4). The combined harmonic and melodic errors are a minimum.

By combining the equations of No. 15 and No. 26, we have $(539+106)\,k = (1998+420)x$, or $x=\frac{645}{2418}\,k = \cdot 001444$, $\log v = \cdot 1746439$.

No. 31 (32). The tones are a mean between those of No. 1 and No. 2.

Here $x=\frac{1}{2}$ (sum of the two values of x in No. 1 and No. 2) $= \cdot 0006744$, $\log v = \cdot 1754169$.

This is proposed by Drobisch (Nachträge, § 9).

No. 32 (42). The errors occasioned by using the tempered *c, d, f ,f, g, b♭♭, c²♭, c²* for the just *c, d, e, f, g, a, b, c²* are a minimum.

Using s for $\cdot 0004901$, and forming the values of these errors by Table XII., we have $4x^2+(s-8x)^2+x^2+(s-9x)^2+(s-7x)^2 =$ a minimum, $x=\frac{24}{199}\,s = \cdot 000059084$, $\log v = \cdot 1760322$.

This is proposed by Drobisch as a system of temperament adapted to bowed instruments (Mus. Tonbestim. § 57), allowing them to use a system of perfect fifths, and yet play the perfect scale very nearly by substitution. Such a system would be more complicated than the just scale for any instrument, and would require many more than 27 tones. It is, therefore, unnecessary for the violin, and impossible on instruments with fixed tones.

D. Cyclic Systems of Equal Temperament.

When it was supposed that the number of just tones required would be infinite, importance was attached to cycles of tones which by a limited number expressed all possible tones. Hence Huyghens's celebrated Cyclus Harmonicus, which he proposed to employ for an instrument with 31 strings, struck by levers and acted upon by a moveable finger-board (*abacus mobilis*), acting like a shifting piano or harmonium. The condition of forming a cycle is not properly harmonic or melodic; it is rather arithmetic. If $\log v : \log 2$ be converted into a continued fraction for any of the preceding values of $\log v$, and $y : z$ be any of the convergents, then, putting $\log 2 = z\,.\,h$, we shall have $\log v = yh$, which is commensurable with log 2, and consequently the logarithms of all the intervals will be multiples of h, and therefore commensurable with log 2. A cycle of z tones to the octave will thus be formed. If z is less than 27, the number of tones otherwise necessary, the cycle may be useful, otherwise it can only be judged by its merits as an equal temperament. As an historical interest attaches to several of these cycles, I subjoin a new method for deducing them all, without reference to previous calculations of $\log v$.

Since $\log v = y\,.\,h$, and $\log \delta = 7 \log 2 - 12 \log v = (7z-12y)\,.\,h$, we have only to put $7z-12y = \ldots -2, -1, 0, 1, 2, \ldots$ and find all the positive integral solutions of the resulting equations. This gives for

$$7z-12y=-2, \qquad \frac{y}{z}=\frac{13}{22}, \frac{27}{46}, \frac{41}{70}, \frac{55}{94}, \frac{69}{118}, \ldots$$

$$7z-12y=-1, \quad \frac{y}{z}=\frac{3}{5}, \frac{10}{17}, \frac{17}{29}, \frac{24}{41}, \frac{31}{53}, \frac{38}{65}, \frac{45}{77}, \frac{52}{89}, \ldots$$

$$7z-12y=\ \ 0, \quad \frac{y}{z}=\frac{7}{12}.$$

$$7z-12y=\ \ 1, \quad \frac{y}{z}=\frac{4}{7}, \frac{11}{19}, \frac{18}{31}, \frac{25}{43}, \frac{32}{55}, \frac{39}{67}, \frac{46}{79}, \frac{53}{91}, \ldots$$

$$7z-12y=\ \ 2, \quad \frac{y}{z}=\frac{1}{2}, \frac{15}{26}, \frac{29}{50}, \frac{43}{74}, \frac{57}{98}, \ldots$$

$$7z-12y=\ \ 3, \quad \frac{y}{z}=\frac{5}{9}, \frac{12}{21}, \frac{19}{33}, \frac{26}{45}, \frac{33}{57}, \frac{40}{69}, \frac{47}{81}, \frac{54}{93}, \ldots$$

Many of these cycles are quite useless. The following selection is arranged in order of magnitude, from the greatest to the smallest cycle.

No. 33 (38). Cycle of 118; $h=\cdot 0025511$, $\log v=69h=\cdot 1760259$.
This is Drobisch's cycle (Mus. Ton. § 58) representing No. 32.

No. 34 (8). Cycle of 93; $h=\cdot 0032368$, $\log v=54h=\cdot 1747872$.
This may represent No. 2.

No. 35 (3). Cycle of 81; $h=\cdot 0037164$, $\log v=47h=\cdot 1746708$.
This may represent No. 11 (2).

No. 36 (39). Cycle of 77; $h=\cdot 0039095$, $\log v=45h=\cdot 1759275$.
This is the same as No. 52.

No. 37 (19). Cycle of 74; $h=\cdot 004068$, $\log v=43h=\cdot 1749200$.
This is another of Drobisch's cycles (Nachträge, § 7) representing No. 27.

No. 38 (22). Cycle of 69; $h=\cdot 004363$, $\log v=40h=\cdot 1745200$.

No. 39 (28). Cycle of 67; $h=\cdot 004493$, $\log v=39h=\cdot 1752270$.

No. 40 (40). Cycle of 65; $h=\cdot 0046123$, $\log v=38h=\cdot 17598674$.

No. 41 (27). Cycle of 57; $h=\cdot 0052812$, $\log v=33h=\cdot 1742796$.

No. 42 (30). Cycle of 55; $h=\cdot 0054733$, $\log v=32h=\cdot 1751456$.
This is mentioned by Sauveur (Mém. de l'Acad. 1707) as the commonly received cycle in his time. Estève (*loc. cit.* p. 135) calls it the Musicians' Cycle.

No. 43 (11). Cycle of 50; $h=\cdot 0060206$, $\log v=29h=\cdot 1745974$.
This is Henfling's cycle (*loc. cit.* p. 281), and is used by Dr. Smith to represent No. 9.

No. 44 (43). Cycle of 53; $h=\cdot 0056798$, $\log v=31h=\cdot 1760800$.
This is the cycle employed by Nicholas Mercator (as reported by Holder, 'Treatise on Harmony,' p. 79) to represent approximately the just scale. He did not propose it as a system of temperament as has been recently done by Drobisch (Musik. Tonbestim. Einleit.). It was the foundation of

the division into degrees and sixteenths adopted in my previous paper Proceedings, vol. xiii. p. 96.

No. 45 (37). Cycle of 45; $h=\cdot 006689$, $\log v=26h=\cdot 1738940$.

No. 46 (25). Cycle of 43; $h=\cdot 0070007$, $\log v=25h=\cdot 1750175$.

This is Sauveur's cycle, defended in Mém. de l'Acad. for 1701, 1702, 1707, and 1711.

No. 47 (10). Cycle of 31; $h=\cdot 009711$, $\log v=\bar{1}8h=\cdot 1747900$.

This is Huyghens's Cyclus Harmonicus, which nearly represents No. 2 (2). It was adopted, apparently without acknowledgment, by Galin (Delezenne, *loc. cit.* p. 19).

No. 48 (44). Cycle of 26; $h=\cdot 011578$, $\log v=15h=\cdot\bar{1}736700$.

No. 49 (25). Cycle of 19; $h=\cdot 0158437$, $\log v=11h=\cdot 1742807$.

This is the cycle adopted by Mr. Woolhouse (Essay on Beats, p. 50) as most convenient for organs and pianos. It may therefore go by his name, although it is frequently mentioned by older writers. It is almost exactly the same as No. 3 (23).

No. 50 (35). Cycle of 12; $h=\cdot 0250858$, $\log v=7h=\cdot 1756008$.

As this is a cycle of twelve equal semitones, it may be termed the *Hemitonic* temperament. It is the one most advocated at the present day, and generally spoken of as "equal temperament" without any qualification, as if there were no other. It was consequently referred to by that name only in my former paper (Proceedings, vol. xiii. p. 95). For its harmonic character see No. 53.

E. Defective Systems of Equal Temperament.

It has been from the earliest times customary to have only twelve fixed tones to the octave, on the organ, harpsichord, piano, &c., and to play the other fifteen by substitution, as shown below, where the tones tuned, arranged in dominative order, occupy the middle line, and the tones for which they are used as substitutes are placed in the outer lines, and are bracketed.

$$
\begin{array}{cccccccccccc}
 & & & & [A\flat\flat, & E\flat\flat, & B\flat\flat, & F\flat, & C\flat, & G\flat, & D\flat, & A\flat] \\
E\flat, & B\flat, & F, & C, & G, & D, & A, & E, & B, & F\sharp, & C\sharp, & G\sharp \\
[D\sharp, & A\sharp, & E\sharp, & B\sharp, & F\times, & C\times, & G\times]. & & & & &
\end{array}
$$

The consequence was, that while the Vths in the middle line were uniform, the Vths and 4ths produced in passing from one line to the other (as $G\sharp E\flat$ for $A\flat E\flat$ or $G\sharp D\sharp$) were strikingly different. Similar errors arose in the other concordant intervals. It is evident that the interval error thus produced must be the usual interval error of the system increased or diminished by the logarithm of the diesis, where $\log \delta=\log g\flat-\log f\sharp=$

$-\cdot0058851+12x=-k-s+12x$, where $s=\cdot0004901$ *. Such interval errors are termed *wolves*, from their howling discordance. In Table XIII. will be found an enumeration of all the wolves, with a notation for them, and an expression of their interval errors and beat meters in terms of k, s, and x.

No. 51 (33). System of least wolf melodic errors.

The sum of the squares of the wolf interval errors, or

$$2k^2+2ks+6s^2-4\,(11k+28s)\,x+266x^2,$$

is a minimum. Hence $22k+56s=266x$, or $x=\cdot0005495$, $\log v=\cdot1755418$.

No. 52 (39). System of least wolf harmonic errors.

The sum of the squares of the wolf beat meters, or

$$25k^2+175s^2+50ks-\,(550k+3072s)\,x+13662x^2,$$

is a minimum. Hence $275k+1536s=13662x$, or $x=\cdot0001638$, $\log v=\cdot1759275$, as in No. 36 (39).

No. 53 (35). The wolf interval errors are equal to the usual interval errors, that is, there are no wolves, or there are none but wolves.

In this case $\log\delta=0$, or, since $\delta=g\flat : f\sharp=2^7 : v^{12}$, $7\log 2=12\log v$. Hence this system is the cycle of 12, No. 50. When δ is greater than 1, $g\flat$ is *sharper* than $f\sharp$, and $\log v$ is less than $\frac{7}{12}\log 2$, or $\cdot1756008$. But if δ is less than 1, $g\flat$ is *flatter* than $f\sharp$, and $\log v$ is less than $\frac{7}{12}\log 2$, or $\cdot1756008$. The latter case is, according to Drobisch, indispensable for musical theory and violin practice (Musik. Tonbestim. Einleit.). Since this temperament thus forms the boundary of the two other classes, distinguished by $g\flat$ being flatter or sharper than $f\sharp$, Drobisch terms it the "mean" temperament (*ibid.* § 51). It is this property of making $g\flat=f\sharp$ which renders this temperament so popular, as the ear is never distressed by the occurrence of intervals different from those expected, and the whole number of tones is reduced to 12.

No. 54 (31). The wolf interval error of the IIIrd is to its usual interval error as 14 : 5.

This gives $-s+8x : k-4x=14 : 5$, or $96x=14k+5s$, $x=\cdot0008123$, $\log v=\cdot1752790$. This is Marsh's system of temperament; see Phil. Mag. vol. xxxvi. p. 437, and p. 39 *seqq.* Schol. 8.

No. 55 (36). The wolf errors of the IIIrd and Vth conjointly are a minimum.

Here $(-s+8x)^2+(-k-s+11x)^2$ is a minimum, whence $11k+19s=185x$, $x=\cdot0003712$, $\log v=\cdot1757201$.

No. 56 (37). The wolf errors of the Vth and IIIrd are equal and opposite.

Here $-k-s+11x=s-8x$, $19x=k+2s$, $x=\cdot0003356$, $\log v=\cdot1757557$.

No. 57 (34). There is no Vth wolf.

Here $-k-s+11x=0$, $x=\cdot0005351$, $\log v=\cdot1755562$.

---

* It appears from Proceedings, vol. xiii. p. 95, that s must be nearly the logarithm of the schisma or log ¶. Actual calculation shows that s and log ¶ agree to 14 places of decimals.

No. 58 (41). There is no IIIrd wolf.

Here $-s+8x=0$, $x=\cdot 0000613$, $\log v=\cdot 1760300$. This is almost exactly No. 32 (42).

No. 59 (42). There is no 3rd wolf.

Here $s-9x=0$, $x=\cdot 0000545$, $\log v=\cdot 1760368$.

F. Systems of Unequal Temperament.

In a defective equal temperament the same just concordance is represented by two different discordances. As performers limited themselves to twelve tones to the octave, those who found the Hemitonic temperament No. 50 (35) too rough, accepted this variety of representatives of the same concordance as the basis of a temperament, hoping to have better IIIrds in the usual chords, without the wolves of the defective temperament. Others conceived that an advantage would be gained by altering the character of the different keys. Thus arose *unequal temperament*, properly so called, which must be carefully distinguished from any defective equal temperament with which it is popularly confused.

Arrange the twelve unequally tempered chords as follows, where the identical numbers indicate identical chords with different names:—

1. *C E G.*	7. *F♯ A♯ c♯.*	7. *G♭ B♭ d♭.*
2. *G B d.*	8. *C♯ E♯ G♯.*	8. *D♭ F A♭.*
3. *D F♯ A.*	9. *G♯ B♯ d♯.*	9. *A♭ c e♭.*
4. *A c♯ e.*	10. *D♯ F× A♯.*	10. *E♭ G B♭.*
5. *E G♯ B.*	11. *A♯ c× e♯.*	11. *B♭ d f.*
6. *B d♯ f♯.*	12. *E♯ G× B♯.*	12. *F A c.*

Let T_n, t_n, v_n be the ratios of the IIIrd, 3rd, and Vth in the nth chord, so that, for example, in the 6th chord $d\sharp=T_6 \,.\, B$, $f\sharp=t_6 \,.\, d\sharp$, $f\sharp=v_6 \,.\, B$. Then it is evident from the above scheme that there exist 12 pairs of equations between these 36 ratios, of the form

$$T_n \,.\, t_n=v_n \text{ and } 4T_n=v_n \,.\, v_{n+1} \,.\, v_{n+2} \,.\, v_{n+3}$$

(where, when the subscript numbers exceed 12, they must be diminished by 12), and one condition,

$$v_1 \,.\, v_2 \,.\, v_3 \,.\, v_4 \,.\, v_5 \,.\, v_6 \,.\, v_7 \,.\, v_8 \,.\, v_9 \,.\, v_{10} \,.\, v_{11} \,.\, v_{12}=2^7.$$

Put $\log T_n=\log \frac{5}{4}+y_n$, $\log t_n=\log \frac{6}{5}-z_n$, $\log v_n=\log \frac{3}{2}-x_n$, then the above equations become

$$z_n=x_n+y_n,$$
$$y_n=k-(x_n+x_{n+1}+x_{n+2}+x_{n+3}),$$
$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}+x_{12}=\cdot 0058851,$$

which represent 25 equations, where the second set of 12 may be replaced by the following, which are readily deduced from them and the last condition:—

$$y_1+y_5+y_9=y_2+y_6+y_{10}=y_3+y_7+y_{11}=y_4+y_8+y_{12}=\cdot 0103000\ldots\ldots (a)$$

$$x_5=x_1+y_1-y_2, \qquad x_9=x_1+y_{10}-y_9,$$
$$x_6=x_2+y_2-y_3, \qquad x_{10}=x_2+y_{11}-y_{10},$$
$$x_7=x_3+y_3-y_4, \qquad x_{11}=x_3+y_{12}-y_{11},$$
$$x_8=x_4+y_4-y_5, \qquad x_{12}=x_4+y_1-y_{12}.$$

A system of unequal temperament may therefore be determined by arbitrarily selecting eleven different Vths, or else eight different IIIrds and three Vths. The equations (a) show that if the temperament is not equal (in which case all the y's are equal, and the interval error of the IIIrd is $\frac{1}{3}\times\cdot 0103000=\cdot 0034333$, as in the Hemitonic temperament), at least four IIIrds must have their interval errors greater than $\frac{1}{3}\times\cdot 0103000$, that is, there must be at least four IIIrds in every unequal temperament which are inferior to the very bad IIIrds of the Hemitonic system. Kirnberger, Dr. T. Young, and Lord Stanhope*, in the unequal systems they propose, have each seven IIIrds sharper, and therefore worse than the Hemitonic IIIrds. In one of Prof. de Morgan's unequal temperaments, six IIIrds are sharper than the Hemitonic; in another four are sharper and four the same; in a third all are the same, but the Vths differ †. Hence nothing is gained over the Hemitonic system in the way of harmony, while the uniformity in the representation of the uniformity of just intonation is entirely lost.

In selecting a temperament, therefore, we may dismiss all unequal temperaments, as they must be inferior to the Hemitonic in both harmony and melody, and will have no advantage over it in the relations of chords or the number of tones required. Also, if it is considered necessary to play in all keys with only twelve tones, any system of defective equal temperament will be inferior to the Hemitonic, on account of the various and distressing wolf intervals which occur when the music is not confined to the six major scales of *B♭, F, C, G, D, A*, and the three minor scales of *g, d, a*. Hence the two conditions of having only twelve tones (exclusive of octaves) and of playing in all keys, at once exclude all temperaments but the Hemitonic. As, however, organs, harpsichords, and pianofortes with 14, 16, 17, 19, 21, 22, and 24 tones to the octave have been actually constructed and used ‡, as Mr. Liston used 59 §, Mr. Poole used 50, and Gen. T. Perronet Thomp-

* *Kirnberger*, Kunst des reinen Satzes in der Musik. *Dr. T. Young*, loc. cit. *Charles Earl Stanhope*, Principles of the Science of Tuning, 1806.

† *De Morgan*, loc. cit. p. 129, temperaments *Q, R, S.*

‡ Mr. Farey (Phil. Mag. vol. xxxix. p. 416) gives the particulars of their scales, builders, and localities.

§ The following account of Mr. Liston's organ is deduced from the data of Mr. Farey (Phil. Mag. vol. xxxix. p. 418). Scale: *c †c ‡c♯ c♯ d♭ †c♯ †d♭ ‡c× ‡d c× d †d ‡d♯ e♭ d♯ †e♭ ††e♭ e f♭ †e †f♭ ††f♭ ‡e♯ e♯ f †e♯ †f ‡f♯ f♯ g♭ †f♯ †g♭ ‡f× ‡g f× g †g ‡g♯ a♭ g♯ †a♭ ††a♭ a †a b♭♭ †b♭♭ ‡a♯ a♯ b♭ †a♯ †b♭ ‡b b c♭ †b †c♭ ‡b♯ ‡c b♯.* Chords: Table V. col. III., lines 4 to 13;

son now uses 40 tones to the octave on their justly intoned organs, the condition of having twelve tones and no more, does not seem to be inevitable. It will therefore be necessary to determine what would be the best system of temperament for the complete equally tempered scale of 27 tones, and how great a sacrifice of musical effect is required by the use of the Hemitonic system.

In Table XV. I have calculated for each of the 59 (reducing to 51) systems of equal temperament already named, the interval errors of the Vths, IIIrds, and 3rds, and the sums of the squares of the 23 interval errors and the 6 beat meters of Table XII. I have then arranged the temperaments in order according to each of these five results, and numbered the order. Finally, I have added the five order numbers together and arranged the whole in the order of these sums. The smallest number would therefore clearly give the best temperament, supposing that all the five points of comparison were of equal value. Now the first and second temperament on the list, or No. 26 and No. 2, only differ from each other in the fifth, sixth, or seventh place of decimals with respect to these five results, a difference which no human ear, however finely constituted by nature or assisted by art, could be taught to detect. As No. 2, or the Mesotonic system, is determined in the simplest manner, I consider it as the real head of the list. There is, however, little to choose between it and any one of the ten or twelve systems which follow, except in simplicity of construction and comparative ease in realization. The Hemitonic system, however, comes 35th in the list, and the old Pythagorean, recently defended by Drobisch and Naumann (*op. cit.*), and asserted to be the system actually used by violinists, is the 45th. No one who has heard any harmonies played on the Pythagorean system will dispute the correctness of the position here assigned to it, which fully explains the absence of all feeling for harmony among the nations which use it—the ancient and modern Greeks, the old Chinese, the Gaels, the Arabs, Persians, and Turks. No modern quartett players could be listened to who adopted it.

The contest lies, therefore, between the Mesotonic and the Hemitonic systems. The Mesotonic is that known as "the old organ-tuning," or, since it was generally used as a defective twelve-toned system, as the "unequal temperament." Within the limits of the nine scales already named, the superiority of the Mesotonic to the Hemitonic system has long been practically acknowledged. But the extremely disagreeable effect of the wolves (more especially to the performer himself) has finally expelled the system from Germany altogether, and from England in great measure. On the pianoforte

---

IV., 5 to 14; V., 5 to 16; VI., 6 to 15; VII., 7 to 16; VIII., 7 to 17; IX., 9 to 13; X., 9 to 13. Tones not forming part of any chord and required chiefly by the system of tuning: †*d*♭ ††*e*♭ †*f*♭ ††*f*♭ †*g*♭ ††*a*♭ *b*♭♭ †*b*♭♭ †*c*♭. Complete keys: *F*, *C*, *G*, *D*, †*A*; *E*, *B*, *F*♯. The keys of *E*♭, *B*♭ had their synonymous, and †*E*, †*B* their relative minors perfect.

Table XIII.

Wolves of Defective Equal Temperaments. (See p. 415.)

Wolves.	Interval error, ε.	Beat meter, β.
Vth wolf=Vw. G♯ e♭ = A♭ e♭ = G♯ d♯........	−k−s+11x	−3k−3s+33x
4th wolf=4w. E♭ G♯ = E♭ A♭ = D♯ G♯......	k+s−11x	4k+4s−44x
IIIrd wolf=IIIw. G♯ c = A♭ c B e♭ = B d♯ F♯ B♭ = F♯ A♯ C♯ F = C♯ E♯	−s+8x	−5s+40x
6th wolf=6w. C G♯ = C A♭ E♭ B = D♯ B B♭ f♯ = A♯ f♯ F c♯ = E♯ c♯	s−8x	8s−64x
3rd wolf=3w. E♭ F♯ = D♯ F♯ B♭ c♯ = A♯ c♯ F G♯ = E♯ G♯	s−9x	6s−54x
VIth wolf=VIw. F♯ e♭ = F♯ d♯ C♯ B♭ = C♯ A♯ G♯ f = G♯ e♯	−s+9x	−5s+45x

Where $k=\cdot 0053950$, $s=\cdot 0004901$, and x is arbitrary.

Table XIV. (See p. 419.)

Comparative Table of the Mesotonic and Hemitonic Temperaments.

J. Just Intonation in the keys of *B♭, F, C, G, D,* or System of *C*; 33 tones.
M. Mesotonic Temperament in all keys. No. 2 (2); 27 tones.
H. Hemitonic Temperament in all keys. No. 50 (35); 12 tones.

Table XIV. (co

Notes.			Logarithms.			Inte
J	M	H	J	M	H	M−
c †c	c	c	·00000 ·00540	·00000	·00000	·000 −·005
‡c♯ c♯ d♭	c♯ d♭	c♯	·01773 ·02312 ·02803	·01908 ·02938	·02509	+·001 −·004 +·001
‡‡d ‡d d	c× d e♭♭	d	·04036 ·04576 ·05115	·03816 ·04846 ·05876	·05017	+·008 +·002 −·002
‡d♯ †e♭	d♯ e♭	d♯	·06888 ·07918	·06753 ·07783	·07526	−·001 −·001
e †e	e f♭	e	·09691 ·10231	·09691 ·10721	·10034	·000 −·005
‡f f †f	e♯ f	f	·11954 ·12494 ·13033	·11599 ·12629	·12543	+·006 +·001 −·004
‡f♯ f♯ g♭	f♯ g♭	f♯	·14267 ·14806 ·15297	·14537 ·15567	·15051	+·002 −·002 +·002
‡g g †g	f× g a♭♭	g	·17070 ·17609 ·18149	·16444 ·17474 ·18504	·17560	+·004 −·001 −·006
‡g♯ a♭ †a♭	g♯ a♭	g♯	·19382 ·19873 ·20412	·19382 ·20412	·20068	·000 +·005 ·000
‡a a †a	g× a b♭♭	a	·21645 ·22185 ·22724	·21290 ·22320 ·23350	·22577	+·006 +·001 −·004
a♯ b♭ †b♭	a♯ b♭	a♯	·24497 ·24988 ·25527	·24228 ·25258	·25086	−·002 +·002 −·002
‡b b	b c^2♭	b	·26761 ·27300	·27165 ·28195	·27594	+·004 −·001
‡c^2 c^2	b♯ c^2	c^2	·29563 30103	·29073 ·30103	·30103	+·005 ·000
Sum of squares............						00047

XIV. (*continued*).

	Interval Errors.		Beat Factor.		Inter-vals.
	M − J	H − J	M	H	
000	·00000 − ·00540	·00000 − ·00540			I †I
509	+ ·00135 − ·00404 + ·00135	+ ·00736 + ·00197 − ·00294			‡I♯ I♯ 2
017	+ ·00810 + ·00270 − ·00270	+ ·00981 + ·00441 − ·00098			‡‡II ‡II II
526	− ·00135 − ·00135	+ ·00638 − ·00392	 − ·018605	 − ·053965	‡II♯ 3
034	·00000 − ·00540	+ ·00343 − ·00197	·000000	+ ·039684	III †III
543	+ ·00675 + ·00135 − ·00405	+ ·00599 + ·00049 − ·00491	 + ·012440	 + ·004520	‡4 4 †4
051	+ ·00270 − ·00270 + ·00270	+ ·00784 + ·00245 − ·00246			‡IV IV 5
560	+ ·00405 − ·00135 − ·00676	+ ·00490 − ·00049 − ·00589	 − ·009304	 − ·003386	†V V †V
068	·00000 + ·00539 ·00000	+ ·00686 + ·00195 − ·00344	 ·000000	 − ·062995	‡V♯ ‡6 6
577	+ ·00675 + ·00135 − ·00405	+ ·00932 + ·00392 − ·00147	 + ·015553	 + ·045379	‡VI VI †VI
086	− ·00270 + ·00270 − ·00270	+ ·00589 + ·00098 − ·00441			VI♯ 7 †7
594	+ ·00405 − ·00135	+ ·00833 + ·00294			‡VII VII
103	+ ·00540 ·00000	+ ·00540 ·00000			†VIII VIII
. . .	0004735	·0008748	·000829	·010547	

Table XV.—Comparative Table of Equal Temperaments. (See p. 418.)

System-atic No.	Name.	Log *v*.	Error of Vth.		Error of IIIrd.		Error of 3rd.		Melodic Errors.	
			Order.	Error.	Order.	Error.	Order.	Error.	Order.	$\Sigma\epsilon^2$.
26	Least Errors	·1747297	32	−·0013616	2	−·0000514	17	−·0013102	1	·0001527
2, 13, 19	Mesotonic	·1747426	31	−·0013488	1	0	18	−·0013488	2	·0001528
11	Equal beats of VI and 4	·1746716	33	−·0014197	6	−·0002838	15	−·0011359	4	·0001541
35	Cycle of 81	·1746703	34	−·0014205	7	−·0002870	14	−·0011335	6	·0001542
30	Least Errors and Beats	·1746439	35	−·0014474	9	−·0003946	13	−·0010528	8	·0001559
28	*Woolhouse's* Equal Harmony	·1746388	36	−·0014525	10	−·0004150	12	−·0010375	9	·0001562
17	Least Beats of V and III	·1747723	30	−·0013190	3	+·0001190	19	−·0014380	3	·0001535
15	Least Beats	·1746359	37	−·0014554	11	−·0004266	11	−·0010288	11	·0001564
34	Cycle of 93	·1747872	29	−·0013039	4	+·0001794	20	−·0014833	5	·0001541
10	Equal and opp. Beats of 3 and 4	·1746200	38	−·0014713	13	−·0004902	10	−·0009811	12	·0001578
47	*Huyghens's* Cycle of 31	·1747900	28	−·0013013	5	+·0001898	21	−·0014911	7	·0001543
43	*Henfling's* Cycle of 50	·1745974	39	−·0014939	14	−·0005806	9	−·0009133	14	·0001601
14, 29	*Drobisch's* Simplest	·1748219	27	−·0012694	8	−·0003174	22	−·0015868	10	·0001563
9	*Dr. Smith's* Equal Harmony	·1745927	40	−·0014986	15	−·0005994	8	−·0008992	15	·0001606
16	Least Beats of 3, III, V	·1744404	41	−·0015309	18	−·0007286	7	−·0008023	18	·0001648
5	Equal Beats of 6 and V	·1748582	26	−·0012331	12	+·0004626	24	−·0016957	13	·0001597
8, 20	Equal Errors of III and 3	·1745199	42	−·0015414	19	−·0007707	6	−·0007707	19	·0001664
21	*Romieu's* Theoretic	·1745163	44	−·0015750	22	−·0009050	5	−·0006700	16	·0001719
7	Equal and opp. Beats of 6 and 4	·1748924	25	−·0011989	15	+·0005994	25	−·0017983	17	·0001639
37	*Drobisch's* Cycle of 74	·1749200	23	−·0011713	17	+·0007108	28	−·0018811	20	·0001679
24, 27	*Drobisch's* least Errors	·1749352	22	−·0011561	19	+·0007706	29	−·0019267	21	·0001705
4	Equal and opp. Beats of III & V	·1749188	24	−·0011725	16	+·0007050	27	−·0018775	26	·0001983
6	Equal Beats of III and 4	·1749674	21	−·0011239	21	+·0008994	30	−·0020233	23	·0001764
38	Cycle of 22	·1745200	43	−·0015713	20	−·0008902	23	−·0016811	22	·0001706
3	Perfect 3rds and VIths	·1742930	45	−·0017983	26	−·0017982	1	0	28	·0002329
12, 18	Errors of III and V eq, and op.	·1750123	20	−·0010790	23	+·0010790	31	−·0020580	24	·0001863
46	*Sauveur's* Cycle of 43	·1750175	19	−·0010738	24	+·0011008	32	−·0021736	25	·0001875
49	*Woolhouse's* Cycle of 19	·1742807	46	−·0018106	27	−·0018474	2	+·0000368	29	·0002375
23	*Drobisch* after *Delezenne*	·1751830	18	−·0019683	25	+·0015218	33	−·0026901	27	·0002177
41	Cycle of 57	·1742796	47	−·0018117	28	−·0018518	3	+·0000401	30	·0002378
39	Cycle of 67	·1752270	16	−·0008643	29	+·0021378	36	−·0028021	32	·0002563
22	*Romieu's* Anacratic	·1751938	17	−·0008975	32	+·0026050	35	−·0027025	31	·0002432
42	Musicians' Cycle of 55	·1751456	48	−·0019457	31	−·0023878	4	+·0004421	34	·0002961
54	*Marsh's*	·1752790	15	−·0008123	30	+·0021458	37	−·0029581	33	·0002794
31	*Drobisch's* V and III combined	·1754169	14	−·0006744	33	+·0026974	38	−·0033718	35	·0003511
51	Least Wolf Errors	·1755418	13	−·0005495	34	+·0031970	39	−·0037465	36	·0004297
57	No V Wolf	·1755562	12	−·0005351	35	+·0032546	40	−·0037897	37	·0004397
50, 53	Hemitonic	·1756008	11	−·0004905	37	+·0034333	41	−·0039235	39	·0004714
55	Least III and V Wolves	·1757201	10	−·0003712	38	+·0039102	42	−·0042814	40	·0005648
45	Cycle of 45	·1738940	49	−·0021973	36	−·0033942	16	+·0011969	38	·0004461
56	III and V Wolves eq. and opp.	·1757557	9	−·0003356	39	+·0040526	43	−·0043882	41	·0005939
33	*Drobisch's* Cycle of 118	·1760259	6	−·0000654	43	+·0051334	48	−·0051988	51	·0009584
36, 52	Least Wolf Beats	·1759275	8	−·0001638	41	+·0047398	44	−·0049036	43	·0007554
40	Cycle of 65	·1759867	7	−·0001046	42	+·0049766	45	−·0050812	44	·0008164
58	No III Wolf	·1760300	5	−·0000613	44	+·0051498	46	−·0052111	45	·0008628
32	*Drobisch's* Violin	·1760322	4	−·0000591	45	+·0051586	49	−·0052177	46	·0008651
59	No 3rd Wolf	·1760368	3	−·0000545	46	+·0051770	47	−·0052315	47	·0008703
44	*N. Mercator* and *Drobisch*	·1760800	2	−·0000113	47	+·0053498	50	−·0053611	49	·0009185
48	Cycle of 26	·1736700	50	−·0024213	40	−·0042902	26	−·0018689	42	·0006244
1	Pythagorean	·1760913	1	0	48	+·0053950	51	−·0053950	50	·0009314
25	Error of V and 3 eq. and opp.	·1733938	51	−·0026975	48	−·0053950	34	+·0026975	48	·0009028

Errors.	Harmonic Errors.		Comparison.	
$\Sigma\epsilon^2$.	Order.	$\Sigma\beta^2$.	Sum of Orders.	Order of sums.
01527	11	·0001513	63	1
01528	12	·0001565	64	2
01541	6	·0001363	64	2
01542	5	·0001361	66	3
01559	3	·0001339	68	4
01562	2	·0001338	69	5
01535	15	·0001709	70	6
01564	1	·0001337	71	7
01541	16	·0001718	74	8
01578	4	·0001342	77	9
01543	17	·0001812	78	10
01601	7	·0001367	83	11
01563	18	·0002029	85	12
01606	8	·0001375	86	13
01648	9	·0001451	93	14
01597	19	·0002325	94	15
01664	10	·0001485	96	16
01719	14	·0001623	101	17
01639	20	·0002625	102	18
01679	22	·0002950	110	19
01705	23	·0003127	114	20
01983	21	·0002936	114	20
01764	24	·0003533	119	21
01706	13	·0001589	121	22
02329	25	·0003587	125	23
01863	29	·0004168	127	24
01875	30	·0004247	130	25
02375	26	·0003858	130	25
02177	31	·0006077	134	26
02378	27	·0003873	135	27
02563	34	·0008306	147	28
02432	33	·0007556	148	29
02961	32	·0006140	149	30
02794	35	·0009600	150	31
03511	37	·0013524	157	32
04297	38	·0017734	160	33
04397	39	·0018255	163	34
04714	40	·0019940	168	35
05648	42	·0024829	172	36
04461	36	·0012335	175	37
05939	43	·0026391	175	37
09584	28	·0039960	176	38
07554	45	·0034669	181	39
08164	46	·0037795	184	40
08628	47	·0040169	187	41
08651	48	·0040292	192	42
08703	49	·0040549	192	42
09185	50	·0043004	198	43
06244	41	·0019977	199	44
09314	51	·0043659	201	45
09028	44	·0032183	225	46

the Hemitonic system is universally adopted in intention. It is, however, so difficult to realize by the ordinary methods of tuning, that "equal temperament," as the Hemitonic system is usually called, has probably never been attained in this country, with any approach to mathematical precision.

In Table XIV. I have given a detailed comparison of the Mesotonic and Hemitonic temperaments with each other and with just intonation, for the system of *C* (Proceedings, vol. xiii. p. 98), from which the great superiority of the Mesotonic over the Hemitonic both in melody and harmony becomes apparent. But this comparison rests upon the preceding calculations, which were founded upon the beats that arise from rendering the conjunct harmonics pulsative. It was therefore assumed that the qualities of tone employed were such as to develope these beats. The result will consequently be materially modified when the requisite harmonics either do not exist or are very faint. Now

for the	Vth	the conjunct harmonics	are	2 and 3,
,,	4th	,,	,,	3 and 4,
,,	VIth	,,	,,	3 and 5,
,,	IIIrd	,,	,,	4 and 5,
,,	3rd	,,	,,	5 and 6,
,,	6th	,,	,,	5 and 8.

If then only simple tones are used, as in the wide covered pipes of organs, or such qualities as develope the second harmonic only, such as tuning-forks, to which we may add flutes, which have almost simple tones, no beats will be heard, and *any* system of temperament may be used in which the ear can tolerate the interval errors. Now Delezenne's experiments show (*loc. cit.*) that a good ear distinguishes

in the	unison	an interval	error of	$0{\cdot}2807k$,
,,	VIIIve	,,	,,	$0{\cdot}31k$,
,,	Vth	,,	,,	$0{\cdot}1461k$,
,,	IIIrd	,,	,,	$0{\cdot}284k$,
,,	VIth	,,	,,	$0{\cdot}299k$,

and an indifferent ear perceives an error of $0{\cdot}561k$ in the VIIIve, and $0{\cdot}292k$ in the Vth. We may say, therefore, generally that the ear just perceives an interval error of $\frac{1}{4}k$ in the Vth, and $\frac{1}{3}k$ in the other intervals. Now in the Mesotonic system the interval error of the Vth is $-\frac{1}{4}k$, and therefore just perceptible, but in scarcely any other interval does it exceed $\frac{1}{4}k$. Thus it is $-\frac{1}{4}k$ in the VIIth, 0 in the IIIrd, and $+\frac{1}{4}k$ in the VIth, and it is therefore in those intervals imperceptible. In the Hemitonic system the error of the Vth is $-\frac{1}{11}k$, and hence quite imperceptible, but the errors of the VIIth, IIIrd, and VIth are respectively $\frac{6}{11}k$, $\frac{7}{11}k$, and $\frac{8}{11}k$, and therefore perfectly appreciable. It is only in the VIIth that this error is at all agreeable. The sharpness of the IIIrd and VIth is universally disliked. Hence in those qualities of tone which are most favourable to the Hemitonic system, it is much inferior to the Mesotonic. In Table XV.

the Mesotonic stands 2nd in order of melody, inappreciably different from the 1st, and the Hemitonic 39th.

If the 3rd harmonic only is developed in the qualities of tone combined, the beats of the Vth are heard, but those of the other intervals are not perceived. The beats of the IIIrd and VIth, which are so faulty on the Hemitonic system, will not be perceived at all unless the 5th harmonic be developed, and will not be much perceived unless it be strongly developed. Now the 5th harmonic is comparatively weak on all organ pipes and on pianofortes, and hence the errors are not so violently offensive on these instruments. If, however, the 'mixture stops,' which strengthen the upper harmonics by additional pipes, are employed on the organ, the effect is unmistakeably bad, unless drowned by din or dimmed by distance. On the pianoforte, however, these intervals, and even the still worse 3rd and 6th, depending on the 6th and 8th harmonics, which are undeveloped on pianoforte strings, are quite endurable.

Hence the Hemitonic system, except as regards melody, will not be greatly inferior to the Mesotonic on a pianoforte and on soft stops of organs, but will only become offensive on loud stops. But for harmoniums and concertinas, violins and voices, where harmonics up to the 8th, and even higher, are well developed, the Hemitonic temperament is offensive. The roughness of harmoniums is almost entirely due to this mode of tuning. The beats of the VIth, IIIrd, and 3rd are distinctly heard, and the development of differential tones is so strong as frequently to form an unintelligibly inharmonious accompaniment*. Concertinas having 14 tones to the octave are indeed generally tuned mesotonically (or intentionally so), thus *c c*♯, *d d*♯, *e e*♭, *f f*♯, *g g*♯, *a a*♭, *b b*♭. They are, however, occasionally tuned hemitonically (or intentionally so) to accompany pianofortes, thus *c c*♯, *d d*♯,

* The three recognized forms of the common major triad 4, 5, 6; 5, 6, 8; 3, 4, 5, or *C E G*, *E G c*, *G c e*, have the pitches of their tones as $4n$, $5n$, $6n$; $5n$, $6n$, $8n$, and $3n$, $4n$, $5n$ respectively. They produce, therefore, the differential tones n, n, $2n$; n, $2n$, $3n$, and n, n, $2n$ respectively. If the chords are tempered, the altered unisons n, n become pulsative, and the other tones disjunct. Now if in Table XII. we put $x=\log(1+t)$ and neglect t^2, we shall have very nearly $E=\frac{81}{64}.(1-4t).C$; $G=\frac{3}{2}.(1-t).C$; $c=2C$, $e=\frac{81}{32}.(1-4t).C$; $g=3.(1-t).C$. The pairs of pulsative differential tones are therefore $E-C=(\frac{17}{14}-\frac{81}{16}t).C$, $G-C=(\frac{15}{64}+\frac{57}{16}t).C$, and $c-G=(\frac{1}{2}+\frac{3}{2}t).C$, $e-c=(\frac{17}{7}-\frac{81}{8}t).C$. The numbers of beats are the absolute value of the differences of these pairs of numbers, or of $(-\frac{1}{32}+\frac{69}{8}t).C$, and $(-\frac{1}{32}+\frac{93}{8}t).C$. The squares of these expressions, and the sum of their squares, will be minima respectively for $t=\frac{1}{276}$, $x=\cdot 00157070$, $\log v=\cdot 1745206$, which is nearly No. 38 (22); $t=\frac{1}{372}$, $x=\cdot 0011658$, $\log v=\cdot 1749255$, which is nearly No. 24 (20); and $t=\frac{9}{2980}$, $x=\cdot 0013096$, $\log v=\cdot 1747817$, which is nearly No. 34 (8). These beats, though perfectly distinct in some octaves, do not appear to be sufficiently prominent to serve as a criterion of the relative value of different systems of temperament, or to form the basis of a system, and they have consequently not been introduced into the text. They were noticed and used by H. Scheibler (Der physikalische und musikalische Tonmesser, p. 15).

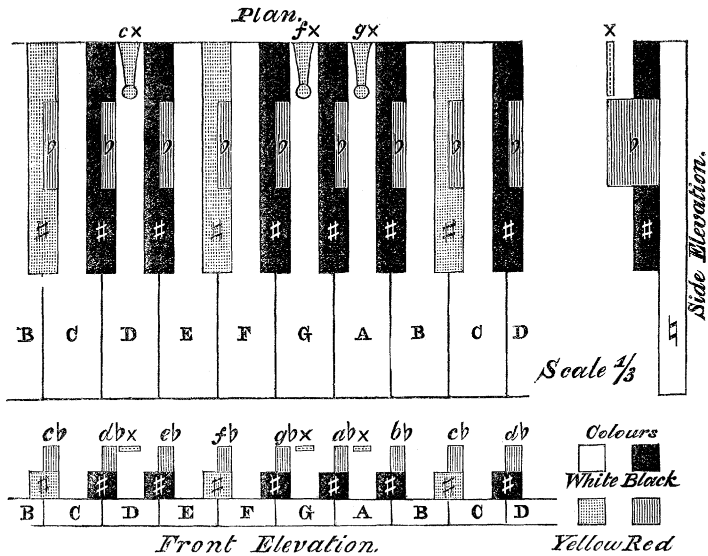
e d♯, *f f♯*, *g g♯*, *a g♯*, *b a♯*. Hence it is easy to compare the different effects of the two systems as applied to the same quality of tone, for harmonies which are common to both. Having two concertinas so tuned, and a third tuned to just intervals, I have been able to make this comparison, and my own feeling is that the Mesotonic is but slightly, though unmistakeably, inferior to the Just, and greatly superior to the Hemitonic.

There are two other points in which the complete Mesotonic system possesses advantages over the Hemitonic. The Mesotonic VIIth is rather flat, but by using the flat VIIIth in its place, when the harmony will allow, the effect of an extremely sharp VIIth is produced, which is sometimes desirable in melodies. Thus log Mesotonic VIII♭ = ·28195, which is sharper even than log Pythagorean VII = ·27840. The ordinary and flatter VIIth can be used when necessary for the harmony. Again, by using the German sharp VIth in place of the dominant 7th, that is, by using the chords *G♭ B♭ D♭ e*, *D♭ F A♭ b*, *A♭ C E♭ f♯*, *E♭ G B♭ c♯*, *B♭ D F g♯*, *F A C d♯*, *C E G a♯*, *G B D e♯*, *D F♯ A b♯*, *A C♯ E f* ×, *E G♯ B c* ×, *B D♯ F♯ g* ×, in place of *G♭ B♭ D♭ f♭*, *D♭ F A♭ c♭*, *A♭ C E♭ g♭*, *E♭ G B♭ d♭*, *B♭ D F a♭*, *F A C e♭*, *C E G b♭*, *G B D f*, *D F♯ A c*, *A C♯ E g*, *E G♯ B d*, *B D♯ F♯ a*, when the progression of parts will allow, an almost perfect natural seventh, better than that obtained by using the corresponding just tones, will result, producing beautiful harmony; for log Mesotonic VI♯ = ·24228, log $\frac{7}{4}$ = ·24304, and log Just VI♯ = ·24497. The ordinary sharper 7th can be used when necessary. Neither of these effective substitutions is possible on the Hemitonic system.

Considering that singers and violinists naturally intone justly (Delezenne, *loc. cit.*), and that the interval errors of the Mesotonic system seldom exceed the natural errors of intonation which may be expected from the inability of the ear to appreciate minute distinctions of pitch, it appears desirable to tune harmoniums at least, and perhaps organs, mesotonically. Except as an instrument for practising singers, however (for which purpose it would be superseded by a Mesotonic harmonium), it would be unnecessary to alter the Hemitonic tuning and arrangement of the piano. But it would be best to teach the Mesotonic intonation on the violin in preference to the Hemitonic, as proposed by Spohr *. As, however, it would be useless to tune mesotonically with only 12 tones to the octave, it is necessary to have some practical arrangement for 27, 24, or 21 tones at least. I propose the following plan for 24 tones, and as these are exactly twice as many as on pianos, &c. of the usual construction, I call my arrangement the

* "Unter reiner Intonation wird natürlich die der gleichschwebenden [Hemitonic] Temperatur verstanden, da es für moderne Musik keine andere giebt. Der angehende Geiger braucht auch nur diese eine zu kennen; es ist deshalb in dieser Schule von einer ungleichschwebenden [defective equal, or unequal] Temperatur eben so wenig die Rede, wie von kleinen und grossen halben Tönen [*c c♯* & *c d♭* = *B c*, that is, ♯ = *l*], weil durch beides die Lehre von der völlig gleichen Grösse aller 12 halben Töne nur in Verwirrung gebracht wird."—Violinschule, p. 3.

DUPLEX FINGER-BOARD.



Let the black and white manuals remain as at present, and let a yellow manual, of the same form as the black, be introduced between *B* and *C*, and *E* and *F*. Cut out about the middle third of each black and yellow manual, up to half its width, on the right side only, and introduce a thin red manual rising as high above the black or yellow as these do above the white. Over *G*, *A*, and *D*, each of which lies between two black manuals, introduce three yellow metal manuals (lacquered or aluminium-bronze) shaped like flute keys, and standing at the height of a red manual above the white one, which can therefore, when necessary, be reached below it. The 7 white manuals are the 7 naturals; the 5 black manuals are the 5 usual sharps, *c*♯ *d*♯ *f*♯ *g*♯ *a*♯; the 2 long yellow manuals are the unusual sharps *e*♯ *b*♯, and the 3 metal yellow manuals are the double sharps *f*× *g*× *c*× ; and the 7 thin red manuals are the 7 flats, *c*♭, *d*♭, *e*♭, *f*♭, *g*♭, *a*♭, *b*♭. The shapes of the red and metal manuals were suggested by those of General T. Perronet Thompson's *quarrils* and *flutals*. The 24 levers opening the valves on the organ or harmonium would lie side by side, being made half the width of those now in use, and metallic, if required for strength. The organ pipes or harmonium reeds would be arranged in two ranks of 12 for each octave, the first rank containing the 7 naturals and 5 usual sharps, and the back rank containing the 7 flats, 2 unusual and 3 double sharps. The use of this finger-board is accurately pointed out by the ordinary musical notation which distinguishes the sharps from the flats, and is therefore in no respect adapted to the Hemitonic fusion of sharps and flats into mean semitones.

Table XIII.

Wolves of Defective Equal Temperaments. (See p. 415.)

Wolves.	Interval error, ε.	Beat meter, β.
Vth wolf = Vw. G♯ e♭ = A♭ e♭ = G♯ d♯	−k − s + 11x	−3k − 3s + 33x
4th wolf = 4w. E♭ G♯ = E♭ A♭ = D♯ G♯	k + s − 11x	4k + 4s − 44x
IIIrd wolf = IIIw. G♯ c = A♭ c B e♭ = B d♯ F♯ B♭ = F♯ A♯ C♯ F = C♯ E♯	−s + 8x	−5s + 40x
6th wolf = 6w. C G♯ = C A♭ E♭ B = D♯ B B♭ f♯ = A♯ f♯ F c♯ = E♯ c♯	s − 8x	8s − 64x
3rd wolf = 3w. E♭ F♯ = D♯ F♯ B♭ c♯ = A♯ c♯ F G♯ = E♯ G♯	s − 9x	6s − 54x
VIth wolf = VIw. F♯ e♭ = F♯ d♯ C♯ B♭ = C♯ A♯ G♯ f = G♯ e♯	−s + 9x	−5s + 45x

Where $k = {\cdot}0053950$, $s = {\cdot}0004901$, and x is arbitrary.

Table XIV. (See p. 419.)

Comparative Table of the Mesotonic and Hemitonic Temperaments.

J. Just Intonation in the keys of *B♭, F, C, G, D*, or System of *C*; 33 tones.
M. Mesotonic Temperament in all keys. No. 2 (2); 27 tones.
H. Hemitonic Temperament in all keys. No. 50 (35); 12 tones.

Table XIV. (*continued*).

Notes.			Logarithms.			Interval Errors.		Beat Factor.		Intervals.
J	M	H	J	M	H	M − J	H − J	M	H	
c †c	c	c	·00000 ·00540	·00000	·00000	·00000 − ·00540	·00000 − ·00540			I †I
‡c♯ c♯ d♭	c♯ d♭	c♯	·01773 ·02312 ·02803	·01908 ·02938	·02509	+ ·00135 − ·00404 + ·00135	+ ·00736 + ·00197 − ·00294			‡I♯ I♯ 2
‡‡d ‡d d	c× d e♭♭	d	·04036 ·04576 ·05115	·03816 ·04846 ·05876	·05017	+ ·00810 + ·00270 − ·00270	+ ·00981 + ·00441 − ·00098			‡‡II ‡II II
‡d♯ †e♭	d♯ e♭	d♯	·06888 ·07918	·06753 ·07783	·07526	− ·00135 − ·00135	+ ·00638 − ·00392	− ·018605	− ·053965	‡II♯ 3
e †e	e f♭	e	·09691 ·10231	·09691 ·10721	·10034	·00000 − ·00540	+ ·00343 − ·00197	·000000	+ ·039084	III †III
‡f f †f	e♯ f	f	·11954 ·12494 ·13033	·11599 ·12629	·12543	+ ·00675 + ·00135 − ·00405	+ ·00589 + ·00049 − ·00491	+ ·012440	+ ·004520	‡4 4 †4
‡f♯ f♯ g♭	f♯ g♭	f♯	·14267 ·14806 ·15297	·14537 ·15567	·15051	+ ·00270 − ·00270 + ·00270	+ ·00784 + ·00245 − ·00246			‡IV IV 5
‡g g †g	f× g a♭♭	g	·17070 ·17609 ·18149	·16444 ·17474 ·18504	·17560	+ ·00405 − ·00135 − ·00676	+ ·00490 − ·00049 − ·00589	− ·008804	− ·003386	‡V V †V
‡g♯ a♭ †a♭	g♯ a♭	g♯	·19382 ·19873 ·20412	·19382 ·20412	·20068	·00000 + ·00539 ·00000	+ ·00686 + ·00195 − ·00344	·000000	− ·062095	‡V♯ ‡6 6
‡a a †a	g× a b♭♭	a	·21645 ·22185 ·22724	·21290 ·22320 ·23350	·22577	+ ·00675 + ·00135 − ·00405	+ ·00932 + ·00392 − ·00147	+ ·015553	+ ·045379	‡VI VI †VI
a♯ b♭ †b♭	a♯ b♭	a♯	·24497 ·24988 ·25527	·24228 ·25258	·25086	− ·00270 + ·00270 − ·00270	+ ·00589 + ·00098 − ·00441			VI♯ 7 †7
‡b b	b c♭	b	·26761 ·27300	·27165 ·28195	·27594	+ ·00405 − ·00135	+ ·00833 + ·00294			‡VII VII
‡c² c²	b♯ c²	c²	·29563 ·30103	·29073 ·30103	·30103	+ ·00540 ·00000	+ ·00540 ·00000			†VIII VIII
Sum of squares............						·0004735	·0008748	·000829	·010547	

TABLE XV.—Comparative Table of Equal Temperaments. (See p. 418.)

Systematic No.	Name.	Log r.	Error of Vth. Order.	Error of Vth. Error.	Error of IIIrd. Order.	Error of IIIrd. Error.	Error of 3rd. Order.	Error of 3rd. Error.	Melodic Errors. Order.	Melodic Errors. Σe^2.	Harmonic Errors. Order.	Harmonic Errors. $\Sigma \beta^2$.	Comparison. Sum of Orders.	Comparison. Order of sums.
26	Least Errors	·1747207	32	−·0013616	2	−·0000514	17	−·0013102	1	·0001527	11	·0001513	63	1
2, 13, 19	MESOTONIC	·1747426	31	−·0013488	1	0	18	−·0013488	2	·0001528	12	·0001565	64	2
11	Equal beats of VI and 4	·1746716	33	−·0014197	6	−·0002838	15	−·0011359	4	·0001541	6	·0001363	64	2
35	Cycle of 81	·1746703	34	−·0014205	7	−·0002870	14	−·0011335	6	·0001542	5	·0001361	66	3
30	Least Errors and Beats	·1746439	35	−·0014474	9	−·0003946	13	−·0010528	8	·0001559	3	·0001339	68	4
28	*Woolhouse's* Equal Harmony	·1746388	36	−·0014525	10	−·0004150	12	−·0010375	9	·0001562	2	·0001338	69	5
17	Least Beats of V and III	·1747723	30	−·0013190	3	+·0001190	19	−·0014380	3	·0001535	15	·0001709	70	6
15	Least Beats	·1746359	37	−·0014554	11	−·0004266	11	−·0010288	11	·0001564	1	·0001337	71	7
34	Cycle of 93	·1747872	29	−·0013039	4	+·0001794	20	−·0014833	5	·0001541	16	·0001718	74	8
10	Equal and opp. Beats of 3 and 4	·1746200	38	−·0014713	13	−·0004902	10	−·0009811	12	·0001578	4	·0001342	77	9
47	*Huyghens's* Cycle of 31	·1747900	28	−·0013013	5	+·0001898	21	−·0014911	7	·0001543	17	·0001812	78	10
43	*Henfling's* Cycle of 50	·1745974	39	−·0014939	14	−·0005806	9	−·0009133	14	·0001601	7	·0001367	83	11
14, 29	*Drobisch's* Simplest	·1748219	27	−·0012694	8	−·0003174	22	−·0015868	10	·0001563	18	·0002029	85	12
9	*Dr. Smith's* Equal Harmony	·1745927	40	−·0014986	15	−·0005994	8	−·0008992	15	·0001606	8	·0001375	86	13
16	Least Beats of 3, III, V	·1744404	41	−·0015309	18	−·0007286	7	−·0008023	18	·0001648	9	·0001451	93	14
5	Equal Beats of 6 and V	·1748582	26	−·0012331	12	+·0004626	24	−·0016957	13	·0001597	19	·0002325	94	15
8, 20	Equal Errors of III and 3	·1745199	42	−·0015414	19	−·0007707	6	−·0007707	19	·0001664	10	·0001485	96	16
21	*Romieu's* Theoretic	·1745163	44	−·0015750	22	−·0009050	5	−·0006700	16	·0001710	14	·0001623	101	17
7	Equal and opp. Beats of 6 and 4	·1748924	25	−·0011989	15	+·0005994	25	−·0017983	17	·0001639	20	·0002625	102	18
37	*Drobisch's* Cycle of 74	·1749200	23	−·0011713	17	+·0007108	28	−·0018811	20	·0001679	22	·0002950	110	19
24, 27	*Drobisch's* least Errors	·1749352	22	−·0011561	19	+·0007706	29	−·0019267	21	·0001705	23	·0003127	114	20
4	Equal and opp. Beats of III & V	·1749188	24	−·0011725	16	+·0007050	27	−·0018775	26	·0001983	21	·0002936	114	20
6	Equal Beats of III and 4	·1749674	21	−·0011239	21	+·0008994	30	−·0020233	23	·0001754	24	·0003533	119	21
38	Cycle of 22	·1745200	43	−·0015713	20	−·0008902	23	−·0016811	22	·0001706	13	·0001589	121	22
3	Perfect 3rds and VIths	·1742930	45	−·0017983	26	−·0017982	1	0	28	·0002329	25	·0003587	125	23
12, 18	Errors of III and V eq. and op.	·1750123	20	−·0010790	23	+·0010790	31	−·0020580	24	·0001863	29	·0004168	127	24
46	*Sauveur's* Cycle of 43	·1750175	19	−·0010738	24	+·0011008	32	−·0021736	25	·0001875	30	·0004247	130	25
49	*Woolhouse's* Cycle of 19	·1742807	46	−·0018106	27	−·0018474	2	+·0000368	29	·0002375	26	·0003858	130	25
23	*Drobisch* after *Delezenne*	·1751830	18	−·0019083	25	+·0015218	33	−·0026301	27	·0002177	31	·0006077	134	26
41	Cycle of 57	·1742796	47	−·0018117	28	−·0018518	3	+·0000401	30	·0002378	27	·0003873	135	27
39	Cycle of 67	·1752270	16	−·0008643	29	+·0021378	36	−·0028021	32	·0002563	34	·0008306	147	28
22	*Romieu's* Anacratic	·1751938	17	−·0008975	32	+·0026050	35	−·0027025	31	·0002432	33	·0007556	148	29
42	Musicians' Cycle of 55	·1751456	48	−·0019457	31	−·0023878	4	+·0004421	34	·0002061	32	·0006140	149	30
54	*Marsh's*	·1752790	15	−·0008123	30	+·0021458	37	−·0029581	33	·0002794	35	·0009300	150	31
31	*Drobisch's* V and III combined	·1754169	14	−·0006744	33	+·0026974	38	−·0033718	35	·0003511	37	·0013524	157	32
51	Least Wolf Errors	·1755418	13	−·0005495	34	+·0031970	39	−·0037465	36	·0004207	38	·0017734	160	33
57	No V Wolf	·1755562	12	−·0005351	35	+·0032546	40	−·0037897	37	·0004397	39	·0018255	163	34
50, 53	HEMITONIC	·1756008	11	−·0004905	37	+·0034333	41	−·0039235	39	·0004714	40	·0019940	168	35
55	Least III and V Wolves	·1757201	10	−·0003712	38	+·0039102	42	−·0042814	40	·0005648	42	·0024829	172	36
45	Cycle of 45	·1738940	49	−·0021973	36	−·0033942	16	+·0011969	38	·0004461	36	·0012335	175	37
56	III and V Wolves eq. and opp.	·1757557	9	−·0003356	39	+·0040526	43	−·0043882	41	·0005939	43	·0026391	175	37
33	*Drobisch's* Cycle of 118	·1760259	6	−·0000654	43	+·0051334	48	−·0051988	51	·0009584	28	·0039060	176	38
36, 52	Least Wolf Beats	·1759275	8	−·0001638	41	+·0047398	44	−·0049036	43	·0007554	45	·0034069	181	39
40	Cycle of 65	·1759867	7	−·0001046	42	+·0049766	45	−·0050812	44	·0008164	46	·0037795	184	40
58	No III Wolf	·1760300	5	−·0000613	44	+·0051498	46	−·0052111	45	·0008628	47	·0040169	187	41
32	*Drobisch's* Violin	·1760322	4	−·0000591	45	+·0051586	49	−·0052177	46	·0008651	48	·0040292	192	42
59	No 3rd Wolf	·1760368	3	−·0000545	46	+·0051770	47	−·0052315	47	·0008703	49	·0040549	192	42
44	*N. Mercator* and *Drobisch*	·1760800	2	−·0000113	47	+·0053498	50	−·0053611	49	·0009185	50	·0043004	198	43
48	Cycle of 26	·1736700	50	−·0024213	40	−·0042902	26	−·0018689	42	·0006244	41	·0019077	199	44
1	PYTHAGOREAN	·1760913	1	0	48	+·0053950	51	−·0053950	50	·0009314	51	·0043659	201	45
25	Error of V and 3 eq. and opp.	·1733938	51	−·0026975	48	−·0053950	34	+·0026975	48	·0009028	44	·0032183	225	46

DUPLEX FINGER-BOARD.