III. On Fermat's Theorem of the Polygonal Numbers, with Supplement. By the Right Hon. Sir Frederick Pollock, F.R.S. Received December 5, 1864.

(Abstract.)

This paper (with its Supplement) proposes a proof of the first two theorems of Fermat, relating to the polygonal numbers, viz. that every number is composed of not exceeding three triangular numbers, and not exceeding four square numbers. And this is done by a method entirely new, founded on the properties of the triangular numbers and the square numbers, and the relation they bear to each other, and on the expansion of an algebraical expression of three members into a *line*, a *square*, and a *cube*, so as to obtain every possible value of the whole expression; and throughout the proof every number or term in a series (except in the Table) is expressed by the roots of the squares that compose it, and the *roots only* are dealt with, and *not* the *numbers* or the *squares* that compose them; a Table is constructed from the triangular numbers, *thus* (see opposite page).

Mode of constructing the Table.

The series of triangular numbers is in the centre of the Table. Below that series the adjoining terms are united, and they form the square numbers 1, 4, 9, &c.; the next adjoining terms are united, and they form the next row, and so on.

Above the triangular numbers each term is doubled, forming the series above, and then the adjacent terms are added together and form the next row, and so on; the differences above are 1, 3, 5, 7, 9, &c. $(2n+1)$, and below are 2, 4, 6, 8, 10, &c. $(2n)$.

From the examination of which Table it appears that the sum of any two triangular numbers, however remote from each other in the series, is always a square number plus a double triangular number; that is,

$$\frac{a^2+a}{2}+\frac{b^2+b}{2}=c^2+c+d^2;$$

and the difference between the sum of two triangular numbers and the sum of some other two triangular numbers may be any number whatever, odd or even, positive or negative. The first of these propositions is mentioned and proved algebraically in the Philosophical Transactions of the year 1861, p. 412; the result is perhaps more clear when presented in a tabular form as above (but more extensively and at large in the paper); it is obviously capable of strict algebraical proof. These two propositions, and the expansion of an algebraical expression into a line, a square, and a cube (exhausting every possible value of the expression), are the foundation of the whole proof, which, in addition to proving the first and second theorems of Fermat, proves also that every odd number has in some form or other

		27	31	37	45	55	67	81	97	115	135
		9	9	9	9	9	9	9	9	9	9
6. And so on		18	22	28	36	46	58	72	88	106	126
		7	7	7	7	7	7	7	7	7	7
5. Sum of the next		11	15	21	29	39	51	65	81	99	119
		5	5	5	5	5	5	5	5	5	5
4. Sum of those next to the others		6	10	16	24	34	46	60	76	94	114
		3	3	3	3	3	3	3	3	3	3
3. Sum of the next adjoining triangular numbers		3	7	13	21	31	43	57	73	91	111
		1	1	1	1	1	1	1	1	1	1
2. Double of the triangular numbers ..	0	2	6	12	20	30	42	56	72	90	110
1. Series of *Triangular Numbers*	**0**	**1**	**3**	**6**	**10**	**15**	**21**	**28**	**36**	**45**	**55**
2. Square numbers (sum of adjoining triangular numbers)............ ..		1	4	9	16	25	36	49	64	81	100
		2	2	2	2	2	2	2	2	2	2
3. Sum of the next adjoining numbers.. ..		3	6	11	18	27	38	51	66	83	102
		4	4	4	4	4	4	4	4	4	4
4. Sum of the next		7	10	15	22	31	42	55	70	87	106
		6	6	6	6	6	6	6	6	6	6
5. Sum of the next		13	16	21	28	37	48	61	76	93	112
		8	8	8	8	8	8	8	8	8	8
6. And so on........................ ..		21	24	29	36	45	56	69	84	101	120
		10	10	10	10	10	10	10	10	10	10
		31	34	39	46	55	66	79	94	111	130

of the roots, two roots equal, and also in some form two roots differing by 1; also that in some form of the roots the algebraic sum of the roots will be equal to 1.

If $a^2+a+b^2-(m^2+m+n^2)$ be equal to any number whatever, odd or even, it must equal a number represented by $p^2-(c^2+c)$; and as

$$a^2+a+b^2-(m^2+m+n^2)=p^2-(c^2+c),$$
$$\therefore\ a^2+a+b^2+c^2+c=m^2+m+n^2+p^2.$$

These two expressions are equivalent to each other, and any number which is of the one form is also of the other form; and if they be doubled and 1 be added to each, they will become

$$2a^2+2a+1+2b^2+2c^2+2c,\ 2m^2+2m+1+2n^2+2p^2,$$

and either of them will represent any odd number whatever. For $a^2+a+b^2-(m^2+m+n^2)$ not only equals $p^2-(c^2+c)$, but it also equals $p^2-(c^2+c)+q$; and if both be doubled and 1 be added,

$$2m^2+2m+1+2n^2+2p^2+2q=2a^2+2a+1+2b^2+2c^2+2c;$$

if therefore to either form any even number ($2q$) be added, it is still of the form of the other, and therefore still of its own form, that is,

$$2m^2+2m+1+2n^2+2p^2+2q$$

is still of the form $2m^2+2m+1+2n^2+2p^2$, and that form therefore represents any odd number.

It is shown in the paper that when $2a^2+2a+1$, $2b^2$, $2c^2+2c$ is expanded, $2a^2+2a+1$ becomes a series (by the addition of 4, 8, 12, &c.) whose terms will be 0, 0, 0, 1; 0, 0, 1, 2; 0, 0, 2, 3, &c., and may be considered as a line whose general expression is 0, 0, a, $(a+1)$.

When $2b^2$ is added to each term by the addition of 2, 6, 10, 14, &c. it becomes a square whose general term is $b, b, a, a+1$ (these being roots whose squares added together form the term in the square). Lastly, when $2c^2+2c$ is added (by decreasing a and increasing $a+1$, 1 each time) and the square becomes a cube, every term has two roots equal, but is composed of not exceeding 4 square numbers; and as on the addition to any term of any even number ($2q$) the term so increased will still be within the cube (extended indefinitely), the cube will contain every odd number; but if $2m^2+2m+1+2n^2$ be formed into a square, and then by the addition of $2p^2$ be raised into a cube (the terms n, n in each term being one increased, and the other diminished by 1), every term in the cube will have two roots differing by 1, and will be composed of not exceeding 4 square numbers; and this cube also will contain every odd number for the same reason that the other will.

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Lastly, $a^2+a+b^2-(m^2+m+n^2)$ will (as it equals any number) equal either $p^2-\frac{c^2+c}{2}$ or $p^2-\frac{c^2+c}{2}+q$, and therefore $2a^2+2a+1,\ +2b^2+c^2+c$ will equal $2m^2+2m+1+2n^2+2p^2$ with or without $2q$.

In raising $a, a+1, b, b$ to a cube by adding c^2+c, it must be by the

addition of 2, 4, 6, 8, 10, &c., which must be added alternately to each ; 2, 6, 10, 14, &c. to b, b, and 4, 8, 12, 16, &c. to $a, a+1$; but the effect of this alternate addition of 2, 6, 10, &c. to b, b, by increasing one of them by 1 and diminishing the other, and of 4, 8, 12, &c. to $a, a+1$ by decreasing each time by 1 and increasing $a+1$ by 1, is to make the algebraic sum of the four roots at all times equal to 1, as is distinctly shown in the paper; and if $2a^2+2a+1+2b^2+c^2+c$ will represent any odd number, then $2a^2+2a+1+2b^2+c^2+c=2n+1$, deducting 1 and dividing by 2.

$a^2+a+b^2+\frac{c^2+c}{2}=n$, and as a^2+a+b^2 equals the sum of 2 triangular numbers and $\left(\frac{c^2+c}{2}\right.$ is a triangular number$\left.\right)$, therefore every number is composed of not exceeding three triangular numbers.

IV. "On the Structure and Affinities of *Eozoon Canadense.*" In a Letter to the President. By W. B. Carpenter, M.D., F.R.S. Received December 14, 1864.

I cannot doubt that your attention has been drawn to the discovery announced by Sir Charles Lyell in his Presidential Address at the late Meeting of the British Association, of large masses of a fossil organism referable to the Foraminiferous type, near the base of the Laurentian series of rocks in Canada. The geological position of this fossil (almost 40,000 feet beneath the base of the Silurian system) is scarcely more remarkable than its zoological relations; for there is found in it the evidence of a most extraordinary development of that Rhizopod type of animal life which at the present time presents itself only in forms of comparative insignificance —a development which enabled it to separate carbonate of lime from the ocean-waters in quantity sufficient to produce masses rivalling in bulk and solidity those of the stony corals of later epochs, and thus to furnish (as there seems good reason to believe) the materials of those calcareous strata which occur in the higher parts of the Laurentian series.

Although a detailed account of this discovery, including the results of the microscopic examinations into the structure of the fossil which have been made by Dr. Dawson and myself, has been already communicated to the Geological Society by Sir William E. Logan, I venture to believe that the Fellows of the Royal Society may be glad to be more directly made acquainted with my view of its relations to the types of Foraminifera which I have already described in the Philosophical Transactions.

The massive skeletons of the Rhizopod to which the name *Eozoon Canadense* has been given, seem to have extended themselves over the surface of submarine rocks, their base frequently reaching a diameter of 12 inches, and their thickness being usually from 4 to 6 inches. A vertical section of one of these masses exhibits a more or less regular alternation of calcareous and