we shall find the above equations satisfied; and consequently the last investigation gives the law of the formation of the remainders. Each remainder will of course be subject to the three conditions already exhibited.

These results point out the foundations on which symbolical division, as applied to non-linear functions, must rest. We have confined our attention to external division, as more particularly applicable to these functions. When a non-linear equation is proposed for reduction, we must ascertain whether it admits of an external factor by employing the method of division as already explained.

"On the Calculus of Symbols.—Fifth Memoir. With Application to Linear Partial Differential Equations, and the Calculus of Functions." By W. H. L. Russell, Esq., A.B. Communicated by Professor Stokes, Sec. R.S. Received April 7, 1864*.

In applying the calculus of symbols to partial differential equations, we find an extensive class with coefficients involving the independent variables which may in fact, like differential equations with constant coefficients, be solved by the rules which apply to ordinary algebraical equations; for there are certain functions of the symbols of partial differentiation which combine with certain functions of the independent variables according to the laws of combination of common algebraical quantities. In the first part of this memoir I have investigated the nature of these symbols, and applied them to the solution of partial differential equations. In the second part I have applied the calculus of symbols to the solution of functional equations. For this purpose I have given some cases of symbolical division on a modified type, so that the symbols may embrace a greater range. I have then shown how certain functional equations may be expressed in a symbolical form, and have solved them by methods analogous to those already explained.

Since
$$\left(x\frac{d}{dy}-y\frac{d}{dx}\right)(x^2+y^2)=0,$$
we shall have
$$\left(x\frac{d}{dy}-y\frac{d}{dx}\right)(x^2+y^2)u=(x^2+y^2)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)u,$$
or, omitting the subject,
$$\left(x\frac{d}{dy}-y\frac{d}{dx}\right)(x^2+y^2)=(x^2+y^2)\left(x\frac{d}{dy}-y\frac{d}{dx}\right),$$
also
$$x\frac{d}{dy}-y\frac{d}{dx}+x^2+y^2=x^2+y^2+x\frac{d}{dy}-y\frac{d}{dx};$$
therefore the symbols $x\dfrac{d}{dy}-y\dfrac{d}{dx}$ and x^2+y^2 combine according to the laws of ordinary algebraical symbols, and consequently partial differential

* Read April 28, 1864. See Abstract, vol. xiii. p. 227.

equations, which can be put in a form involving these functions exclusively, can be solved like algebraical equations. We shall give some instances of this.

Consider, first, the equation

$$x^2\frac{d^2u}{dx^2}-2xy\frac{d^2u}{dx\,dy}+y^2\frac{d^2u}{dy^2}-x\frac{du}{dx}-y\frac{du}{dy}-(x^2+y^2)^2u=f(x,y).$$

This may be put in the form

$$\left\{x\frac{d}{dy}-y\frac{d}{dx}-(x^2+y^2)\right\}\left\{x\frac{d}{dy}-y\frac{d}{dx}+(x^2+y^2)\right\}u=f(x,y);$$

or if $w=\left\{x\frac{d}{dy}-y\frac{d}{dx}+(x^2+y^2)\right\}u$,

we shall have

$$x\frac{dw}{dy}-y\frac{dw}{dx}-(x^2+y^2)w=f(x,y).$$

Lagrange's method will give in this case the equations

$$\frac{dy}{x}=-\frac{dx}{y}=\frac{dw}{f(x,y)+(x^2+y^2)w}.$$

Hence we shall have

$$x^2+y^2=c^2,$$

$$w=-\epsilon^{-c^2\sin^{-1}\frac{x}{c}}\int\frac{dx f(x,\sqrt{c^2-x^2})}{\sqrt{c^2-x^2}}+\beta\epsilon^{-c^2\sin^{-1}\frac{x}{c}}.$$

Let $\int\frac{dx f(x,\sqrt{c^2-x^2})}{\sqrt{c-{}^2x^2}}=\mathrm{F}(x,c)$;

$$\therefore\ w=-\epsilon^{-(x^2+y^2)\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}}\mathrm{F}(x,\sqrt{x^2+y^2})$$

$$+\epsilon^{-(x^2+y^2)\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}}\phi(x^2+y^2),$$

where ϕ is an arbitrary function. We shall denote this expression by $\chi(x,y)$, whence we have for the determination of (u),

$$x\frac{du}{dy}-y\frac{du}{dx}+(x^2+y^2)u=\chi(x,y),$$

which gives

$$u=\epsilon^{(x^2+y^2)\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}}\mathrm{F}_1(x,\sqrt{x^2+y^2})+\epsilon^{(x^2+y^2)\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}}\phi_1(x^2+y^2),$$

where

$$\mathrm{F}_1(x,r)=\int\frac{dx\chi(x,\sqrt{r^2-x^2})}{\sqrt{r^2-x^2}},$$

which completes the solution.

Next let us take the equation

$$x^2\frac{d^2u}{dx^2}-2xy\frac{d^2u}{dx\,dy}+y^2\frac{d^2u}{dy^2}-(2x^3+2xy^2+y)\frac{du}{dy}$$

$$+(2y^3+2x^2y-x)\frac{du}{dx}+(x^2+y^2)^2u=f(x,y).$$

This equation may be written

$$\left(x\frac{d}{dy}-y\frac{d}{dx}\right)^2 u-2(x^2+y^2)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)u+(x^2+y^2)^2u=f(x,y),$$

$$\left\{x\frac{d}{dy}-y\frac{d}{dx}-(x^2+y^2)\right\}\left\{x\frac{d}{dy}-y\frac{d}{dx}+(x^2+y^2)\right\}u=f(x,y),$$

which may be treated as before.

In order to find the most general form of equation to which the symbols $x\frac{d}{dy}-y\frac{d}{dx}$ and x^2+y^2 give rise, we must determine the expansion of $\left(x\frac{d}{dy}-y\frac{d}{dx}\right)^n$. As $x\frac{d}{dy}+y\frac{d}{dx}$ and x^2-y^2 likewise combine according to the laws of algebraical symbols, we shall take $\left(x\frac{d}{dy}+y\frac{d}{dx}\right)^n$ to avoid the negative sign.

Now the expansion of $\left(x\frac{d}{dy}+y\frac{d}{dx}\right)^n$ will consist of all the terms of the form

$$\ldots\left(x\frac{d}{dy}\right)^\beta\left(y\frac{d}{dx}\right)^b\left(x\frac{d}{dy}\right)^\alpha\left(y\frac{d}{dx}\right)^a,$$

in which

$$a+b+\ldots+\alpha+\beta+\ldots=n.$$

We shall write δ_x for $\frac{d}{dx}$, and δ_y for $\frac{d}{dy}$, where it is to be understood that δ_x and δ_y do not apply to the subject.

Moreover we shall use, as in the third memoir,

$$a_r \text{ for } a\,.\frac{a-1}{2}\,.\frac{a-2}{3}\ldots\frac{a-r+1}{r}.$$

Then we shall have, if $a+\alpha=r$,

$$\left(x\frac{d}{dy}\right)^\alpha\left(y\frac{d}{dx}\right)^a=x^\alpha y^a\frac{d^n}{dy^\alpha\,dx^a}+\alpha x^\alpha\delta_y y^a\frac{d^{n-1}}{dy^{\alpha-1}\,dx^a}+\alpha_2 x^\alpha\delta^2{}_y y^a\frac{d^{n-2}}{dy^{\alpha-2}\,dx^a}$$
$$+\alpha_3 x^\alpha\delta^3{}_y y^a\frac{d^{n-3}}{dy^{\alpha-3}dx^a}+\ldots$$

Again, $a+\alpha+b=n$,

$$\left(y\frac{d}{dx}\right)^b\left(x\frac{d}{dy}\right)^\alpha\left(y\frac{d}{dx}\right)^a$$

$$=y^{a+b}x^\alpha\frac{d^n}{dy^\alpha\,dx^{a+b}}+b_1y^{a+b}\delta_x x^\alpha\frac{d^{n-1}}{dy^\alpha\,dx^{a+b-1}}+b_2y^{a+b}\delta^2{}_x x^\alpha\frac{d^{n-2}}{dy^\alpha\,dx^{a+b-2}}$$
$$+b_3y^{a+b}\delta^3{}_x x^\alpha\frac{d^{n-3}}{dy^\alpha\,dx^{a+b-3}}\ldots$$

$$+\alpha_1 x^\alpha y^b \delta_y y^a \frac{d^{n-1}}{dy^{\alpha-1}\,dx^{a+b}} + \alpha_1 b_1 y^b \delta x^\alpha \delta_y y^a \frac{d^{n-2}}{dy^{\alpha-1} dx^{a+b-1}}$$

$$+\alpha_1 b_2 y^b \delta^2{}_x x^\alpha \delta_y y^a \frac{d^{n-3}}{dy^{\alpha-1}\,dx^{a+b-2}}$$

$$+\alpha_2 y^b x^\alpha \delta^2{}_y y^a \frac{d^{n-2}}{dy^{\alpha-2} dx^{a+b}} + \alpha_2 b_1 y^b \delta_x x^\alpha \delta^2{}_y y^a \frac{d^{n-3}}{dy^{\alpha-2} dx^{a+b-1}}$$

$$+\alpha_3 y^b x^\alpha \delta^3{}_y y^a \frac{d^{n-3}}{dy^{\alpha-3}\,dx^{a+b}} +$$

$$+ \&c.$$

And again, if $a+\alpha+b+\beta=n$, we shall have

$$\left(x\frac{d}{dy}\right)^\beta \left(y\frac{d}{dx}\right)^b \left(x\frac{d}{dy}\right)^\alpha \left(y\frac{d}{dx}\right)^a$$

$$=y^{a+b}\,x^{\alpha+\beta} \frac{d^n}{dy^{\alpha+\beta}\,dx^{a+b}} + b_1 y^{a+b}\,x^\beta \delta_x x^\alpha \frac{d^{n-1}}{dy^{\alpha+\beta}\,dx^{a+b-1}}$$

$$+b_2 y^{a+b} x^\beta \delta^2 x^\alpha \frac{d^{n-2}}{dy^{\alpha+\beta}\,dx^{a+b-2}} + b_3 y^{a+b}\,x^\beta\,\delta^3 x^\alpha \frac{d^{n-3}}{dy^{\alpha+\beta}\,dx^{a+b-3}} \cdots$$

$$+\alpha_1 x^{\alpha+\beta} y^b \delta_y y^a \frac{d^{n-1}}{dy^{\alpha+\beta-1} dx^{a+b}} + \alpha_1 b_1 x^\beta y^b \delta_x x^\alpha \delta_y y^a \frac{d^{n-2}}{dy^{\alpha+\beta-1}\,dx^{a+b-1}}$$

$$+\alpha_1 b_2 x^\beta y^b \delta^2{}_x x^\alpha \delta_y y^a \frac{d^{n-3}}{dy^{\alpha+\beta-1}\,dx^{a+b-2}} + \cdots$$

$$+\alpha_2 y^b x^{\alpha+\beta} \delta^2{}_y y^a \frac{d^{n-2}}{dy^{\alpha+\beta-2}\,dx^{a+b}} + \alpha_2 b_1 y^b x^\beta \delta_x x^\alpha \delta^2{}_y y^a \frac{d^{n-3}}{dy^{\alpha+\beta-2}\,dx^{a+b-1}} + \cdot\cdot$$

$$+\alpha_3 y^b x^{\alpha+\beta} \delta^3{}_y y^a \frac{d^{n-3}}{dy^{\alpha+\beta-3}\,dx^{a+b}}$$

$$+\cdots$$

$$+\beta_1 x^{\alpha+\beta} \delta_y y^{a+b} \frac{d^{n-1}}{dy^{\alpha+\beta-1}\,dx^{a+b}} + \beta_1 b_1 x^\beta \delta_y y^{a+b} \delta_x x^\alpha \frac{d^{n-2}}{dy^{\alpha+\beta-1}\,dx^{a+b-1}}$$

$$+\beta_1 b_2 x^\beta \delta_y y^{a+b} \delta^2{}_x x^\alpha \frac{d^{n-3}}{dy^{\alpha+\beta-1}\,dx^{a+b-2}}$$

$$+\beta_1 \alpha_1 x^{\alpha+\beta} \delta_y y^b \delta_y y^a \frac{d^{n-2}}{dy^{\alpha+\beta-2}\,dx^{a+b}} + \alpha_1 b_1 \beta_1 \delta_y y^b \delta_x x^\alpha \delta_y y^a \frac{d^{n-3}}{dy^{\alpha+\beta-2} dx^{a+b-1}}$$

$$+\alpha_2 \beta_1\, x^{\alpha+\beta} \delta_y y^b \delta^2{}_y y^a \frac{d^{n-3}}{dy^{\alpha+\beta-3}\,dx^{a+b-1}}$$

$$+\cdots$$

$$+\beta_2 x^{\alpha+\beta}\delta^2_y y^{a+b}\frac{d^{n-2}}{dy^{\alpha+\beta-2}dx^{a+b}}+b_1\beta_2 x^\beta\delta_x x^\alpha\delta^2_y y^{a+b}\frac{d^{n-3}}{dy^{\alpha+\beta-2}dx^{a+b-1}}$$

$$+\alpha_1\beta_2\delta^2_y y^b\delta_y y^a\frac{d^{n-3}}{dy^{\alpha+\beta-3}dx^{a+b}}$$

$$+\ldots$$

$$+\beta_3 x^{\alpha+\beta}\delta^3_y y^{a+b}\frac{d^{n-3}}{dy^{\alpha+\beta-3}dx^{a+b}}$$

$$+\ldots$$

We are consequently able to see that the general term of

$$\ldots\left(x\frac{d}{dy}\right)^\beta\left(y\frac{d}{dx}\right)^b\left(x\frac{d}{dy}\right)^\alpha\left(y\frac{d}{dx}\right)^a$$

is

$$\Sigma_s\Sigma_r\ldots c_{q'}b_{p'}\ldots\beta_q\alpha_p\ldots\delta^q_y y^b\,\delta^p_y y^a\ldots\delta^{q'}_x x^\beta\,\delta^{p'}_x x^\alpha.\frac{d^{n-m}}{dy^{\alpha+\beta+\ldots-r}dx^{a+b+\ldots-s}}$$

where

$$p+q+\ldots=r,\quad\ldots\ldots\quad(1)$$

$$p'+q'+\ldots=s,\quad\ldots\ldots\quad(2)$$

and

$$r+s\quad=m.\quad\ldots\ldots\quad(3)$$

Hence the general term of $\left(x\frac{d}{dy}+y\frac{d}{dx}\right)^n$ will be

$$\Sigma_n\Sigma_s\Sigma_r\ldots c_{q'}b_{p'}\ldots\beta_q\alpha_p\ldots\delta^q_y y^b\,\delta^p_y y^a\ldots\delta^{q'}_x x^\beta\,\delta^{p'}_x x^\alpha\frac{d^{n-m}}{dy^{\alpha+\beta+\ldots-r}dx^{a+b\ldots-s}}$$

under the conditions (1), (2), (3), and also

$$a+b+\ldots+\alpha+\beta+\ldots=n.$$

Calling the expanded form of $\left(x\frac{d}{dy}+y\frac{d}{dx}\right)^n$, Δ_n, it is easily seen that we can resolve all linear partial differential equations of the form

$$f(x^2-y^2)\Delta_n u+f_1(x^2-y^2)\Delta_{n-1}u+f_2(x^2-y^2)\Delta_{n-1}u+\&c.=F(x,y).$$

The same property is possessed by a great number of other symbols. Let us examine the condition that

$$(ax+by+c)\frac{d}{dx}-(a'x+b'y+c')\frac{d}{dy},$$

and

$$Ax^2+2Bxy+Cy^2+2Ex+2Fy+H$$

may combine according to the algebraical law. The condition is easily seen to be

$$(Ax+By+E)(ax+by+c)-(Cy+Bx+F)(a'x+b'y+c')=0,$$

from whence we have

$$Aa-Ba'=0,\quad Bb-Cb'=0,$$
$$Ab+Ba=Ca'+Bb',$$
$$Bc+Eb=Cc'+Fb',$$
$$Ea+Ac=Fa'+Bc',$$

and

$$Ec=Fc'.$$

We may consider $B=1$, which gives the following conditions: $a'b=ab'$, $a=b'$. Also

$$A=\frac{a'}{a},\quad C=\frac{b}{b'},\quad E=\frac{c'}{a},\quad F=\frac{c}{a}.$$

And the symbols may be written

$$b(ax+by+c)\frac{d}{dx}-a(ax+by+c')\frac{d}{dy},$$

and

$$a^2x^2+2ab\,xy+b^2y^2+2ac'x+2cby+H.$$

It follows hence that in order to find the form of the differential equations to which these symbols give rise, we must know the expansion of $\left(X\frac{d}{dx}+Y\frac{d}{dy}\right)^n$, where X and Y are functions of x and y.

The expanded form will be a series of terms like

$$\ldots.\left(Y\frac{d}{dy}\right)^\beta\left(X\frac{d}{dx}\right)^b\left(Y\frac{d}{dy}\right)^\alpha\left(X\frac{d}{dx}\right)^a.$$

We must consequently find an expression for $\left(X\frac{d}{dx}\right)^a$ in powers of $\frac{d}{dx}$. It must be remembered that X is a function of x and y, in which (y) during the present process is considered as constant, and therefore X may be looked upon as a function of (x) only.

Now we shall find after a few differentiations, that

$$\begin{aligned}\left(X\frac{d}{dx}\right)^5 &= X^5\frac{d^5}{dx^5}\\ &+(X\delta X^4+X^2\delta X^3+X^3\delta X^2+X^4\delta X)\frac{d^4}{dx^4}\\ &+(X\delta X\delta X^3+X\delta X^2\delta X^2+X^2\delta X\delta X^2\\ &+X\delta X^3\delta X+X^2\delta X^2\delta X+X^3\delta X\delta X)\frac{d^3}{dx^3}\\ &+(X\delta X\delta X\delta X^2+X\delta X\delta X^2\delta X+X\delta X^2\delta X\delta X)\frac{d^2}{dx^2}\\ &+X\delta X\delta X\delta X\delta X\frac{d}{dx}.\end{aligned}$$

Now let

$$\left(X\frac{d}{dx}\right)^n=X_n\frac{d^n}{dx^n}+X_n^{(1)}\frac{d^{n-1}}{dx^{n-1}}+X_n^{(2)}\frac{d^{n-2}}{dx^{n-2}}$$
$$+\ldots+X_n^{(r)}\frac{d^{n-r}}{dx^{n-r}}+$$

Then

$$X_n^{(r)}=\Sigma_n X^\alpha\delta X^\beta\delta X^\gamma\ldots$$

where there are r δ's, and $\alpha+\beta+\gamma+\ldots.=n$. Hence we shall have

$$\ldots\left(Y\frac{d}{dy}\right)^\beta\left(X\frac{d}{dx}\right)^b\left(Y\frac{d}{dy}\right)^\alpha\left(X\frac{d}{dx}\right)^a$$

$$= \dots \left\{ \mathrm{Y}_\beta \frac{d^\beta}{dy^\beta} + \mathrm{Y}^{(1)}_\beta \frac{d^{\beta-1}}{dy^{\beta-1}} + \mathrm{Y}^{(2)}_\beta \frac{d^{\beta-2}}{dy^{\beta-2}} + \dots \right\}$$

$$\left\{ \mathrm{X}_b \frac{d^b}{dx^b} + \mathrm{X}^{(1)}_b \frac{d^{b-1}}{dx^{b-1}} + \mathrm{X}^{(2)}_b \frac{d^{b-2}}{dx^{b-2}} + \dots \right\}$$

$$\left\{ \mathrm{Y}_\alpha \frac{d^\alpha}{dy^\alpha} + \mathrm{Y}^{(1)}_\alpha \frac{d^{\alpha-1}}{dy^{\alpha-1}} + \mathrm{Y}^{(2)}_\alpha \frac{d^{\alpha-2}}{dy^{\alpha-2}} + \dots \right\}$$

$$\left\{ \mathrm{X}_a \frac{d^a}{dx^a} + \mathrm{X}^{(1)}_a \frac{d^{a-1}}{dx^{a-1}} + \mathrm{X}^{(2)}_a \frac{d^{a-2}}{dx^{a-2}} + \dots \right\}.$$

Whence it is obvious that the general expression for the expansion of $\left(\mathrm{X}\frac{d}{dx}+\mathrm{Y}\frac{d}{dy}\right)^n$ will depend upon principles not materially differing from those already considered.

The symbols we have already considered are only of the first order of differentiation; we shall show that there exist symbols of the second order which combine with certain algebraical quantities as if they were themselves algebraic.

Let us take the symbols

$$a\frac{d^2}{dx^2}+2b\frac{d^2}{dx\,dy}+c\frac{d^2}{dy^2}+2a'\frac{d}{dx}+2b'\frac{d}{dy}+e,$$

and

$$\mathrm{A}x^2+2\mathrm{B}xy+\mathrm{C}y^2+2\mathrm{A}'x+2\mathrm{B}'y+\mathrm{H}.$$

Proceeding as before, we arrive at the following conditions:

$$\mathrm{A}b+\mathrm{B}c=0, \qquad (1)$$

$$\mathrm{B}b+\mathrm{C}c=0, \qquad (2)$$

$$\mathrm{A}'b+\mathrm{B}'c=0, \qquad (3)$$

$$\mathrm{A}a'+\mathrm{B}b'=0, \qquad (4)$$

$$\mathrm{B}a'+\mathrm{C}b'=0, \qquad (5)$$

$$\mathrm{A}a+\mathrm{B}b=0, \qquad (6)$$

$$\mathrm{B}a+\mathrm{C}b=0, \qquad (7)$$

$$\mathrm{A}'a+\mathrm{B}'b=0, \qquad (8)$$

$$2a\mathrm{A}+4b\mathrm{B}+2c\mathrm{C}+4a'\mathrm{A}'+4b'\mathrm{B}'+e=0. \qquad (9)$$

Whence we have, putting $\mathrm{B}=1$,

$$\mathrm{A}=-\frac{c}{b}, \quad \mathrm{C}=-\frac{b}{c};$$

and with the following conditions,

$$ac=b^2, \quad a'c-bb'=0$$

$$4a'\mathrm{A}'+4b'\mathrm{B}'+e=0;$$

the condition $a'c-bb'$ may be otherwise written $a'b-ab'=0$, in consequence of the equation $ac=b^2$.

It will be observed that several of the nine above equations are not independent of the rest; so that the result is much simplified.

I now proceed to apply the calculus of symbols to the solution of functional equations.

Let

$$\int \frac{dx}{\psi(x)} = \chi(x).$$

Then the following formulæ are known:

$$\epsilon^{\psi(x)\frac{d}{dx}} f(x) = f\{\chi^{-1}(\chi x + 1)\},$$

$$(\epsilon^{\psi(x)\frac{d}{dx}})^2 f(x) = f\{\chi^{-1}(\chi x + 2)\}.$$

$$\&c. \qquad = \ldots$$

$$(\epsilon^{\psi(x)\frac{d}{dx}})^r f(x) = f\{\chi^{-1}(\chi(x) + r)\}.$$

These formulæ may be thus expressed in the notation of the calculus of symbols: if $\rho = \epsilon^{\psi(x)\frac{d}{dx}}$, $\pi = x$, θ a functional symbol acting on $f(\pi)$ in such a manner as to convert $f(\pi)$ into $f\chi^{-1}(\chi\pi + 1)$; then

$$\rho f(\pi) = \theta f\pi \, . \, \rho,$$

a general law of symbolical combination due to Professor Boole.

We will now consider two cases of internal and external division in which the symbols combine according to this law. The results, as will appear afterward, will be found useful in the solution of functional equations.

And, first, for internal division. We shall determine the condition that $\rho\psi_1(\pi) + \psi_0(\pi)$ may divide $\rho^n\phi_n(\pi) + \rho^{n-1}\phi_{n-1}(\pi) + \ldots\ldots$ The process will be, *mutatis mutandis*, the same as in my former memoir. The symbolical quotient is

$$\rho^{n-1}\theta\frac{\phi_n\pi}{\psi_1\pi} + \rho^{n-2}\left\{\theta\frac{\phi_{n-1}\pi}{\psi_1\pi} - \theta\frac{\psi_0\pi}{\psi_1\pi}\,\theta\frac{\phi_n\pi}{\psi_1\pi}\right\} + \ldots;$$

and the required condition is found by equating the symbolical final remainder to zero, and we have

$$\phi_0\pi - \psi_0\pi\theta\frac{\phi_1\pi}{\psi_1\pi} + \psi_0\pi\theta\frac{\psi_0\pi}{\psi_1\pi}\,\theta\frac{\phi_2\pi}{\psi_1\pi} - \&c. \pm \psi_0\pi\theta\frac{\psi_0\pi}{\psi_1\pi}\,\theta\frac{\psi_0\pi}{\psi_1\pi}\ldots\theta\frac{\phi_n\pi}{\psi_1\pi} = 0,$$

θ affecting every part of the term which succeeds it.

I shall now give the corresponding condition for an external factor. The symbolical quotient is

$$\rho^{n-2}\phi_n\pi\theta^{-(n-1)}\frac{1}{\psi_1\pi} + \rho^{n-2}\left\{\phi_{n-1}\pi\theta^{-(n-2)}\frac{1}{\psi_1\pi} - \phi_n\pi\theta^{-(n-2)}\frac{1}{\psi_1\pi}\theta^{-1}\frac{\psi_0\pi}{\psi_1\pi}\right\} + \ldots$$

The required condition is found by equating the final remainder to zero; we have

$$\phi_0\pi - \phi_1\pi\frac{\psi_0\pi}{\psi_1\pi} + \phi_2\pi\frac{\psi_0\pi}{\psi_1\pi}\theta^{-1}\frac{\psi_0\pi}{\psi_1\pi} - \&c. \pm \phi_n\pi\frac{\psi_0\pi}{\psi_1\pi}\theta^{-1}\frac{\psi_0\pi}{\psi_1\pi}\theta^{-1}\frac{\psi_0\pi}{\psi_1\pi}\ldots\theta^{-1}\frac{\psi_0\pi}{\psi_1\pi} = 0,$$

θ^{-1} in each term affecting everything which comes after it.

I conclude with some examples of functional equations.

Let the functional equation be

$$f(x) - \alpha f\frac{x}{\sqrt{1+2x^2}} = \mathrm{F}(x);$$

this may be written

$$f(x)-\alpha\epsilon^{\frac{d}{dx^{-2}}}f(x)=\mathrm{F}(x),$$

or

$$f(x)=\frac{1}{1-\alpha\epsilon^{\frac{d}{dx^{-2}}}}\mathrm{F}(x),$$

$$=\{1+\alpha\epsilon^{\frac{d}{dx^{-2}}}+\alpha^2(\epsilon^{\frac{d}{dx^{-2}}})^2+\ldots\}\mathrm{F}(x)$$

$$=\mathrm{F}(x)+\alpha\mathrm{F}\frac{x}{\sqrt{1+2x^2}}+\alpha^2\mathrm{F}\frac{x}{\sqrt{1+4x^2}}+\ldots$$

To make this solution complete, we must add a complementary function, and we have

$$u=\frac{\mathrm{C}}{(\sqrt{\alpha})^{\frac{1}{x^2}}}+\frac{\mathrm{C}_1}{(-\sqrt{\alpha})^{\frac{1}{x^2}}}$$

$$+\mathrm{F}(x)+\alpha\mathrm{F}\frac{x}{\sqrt{1+2x^2}}+\alpha^2\mathrm{F}\frac{x}{\sqrt{1+4x^2}}+\ldots\ldots$$

As an example of this, put $\mathrm{F}(x)=x$, and the series becomes

$$x+\frac{\alpha x}{\sqrt{1+2x^2}}+\frac{\alpha^2 x}{\sqrt{1+4x^2}}+\frac{\alpha^3 x}{\sqrt{1+6x^2}}+\ldots$$

$$=\frac{2x}{\sqrt{\pi}}\int_0^\infty dv\,\epsilon^{-v^2}(1+\alpha\epsilon^{-2x^2v^2}+\alpha^2\epsilon^{-4x^2v^2}+\ldots)$$

$$=\frac{2x}{\sqrt{\pi}}\int_0^\infty\frac{\epsilon^{-v^2}dv}{1-\alpha\epsilon^{-2xv^2}}.$$

As a second example, we will take the equation

$$f\left(\frac{x}{x+2}\right)-3f\left(\frac{x}{x+1}\right)+2f(x)=\mathrm{F}(x).$$

This equation may be written

$$(\epsilon^{2.\frac{d}{d\frac{1}{x}}}-3\epsilon^{\frac{d}{d\frac{1}{x}}}+2)f(x)=\mathrm{F}(x),$$

$$(\epsilon^{\frac{d}{d\frac{1}{x}}}-1)(\epsilon^{\frac{d}{d.\frac{1}{x}}}-2)f(x)=\mathrm{F}(x).$$

Now let

$$(\epsilon^{\frac{d}{d\frac{1}{x}}}-2)f(x)=\chi(x),$$

and the functional equation resolves itself into the two,

$$f\left(\frac{x}{x+1}\right)-2f(x)=\chi(x),$$

and

$$\chi\left(\frac{x}{x+1}\right)-\chi(x)=\mathrm{F}(x),$$

which are known forms.

As a last example, we will take the equation

$$f\left(\frac{3x-2}{2x-1}\right)-\frac{x^2+3x-1}{x}f\frac{2x-1}{x}+x(x+1)f(x)=\mathrm{F}(x)\,;$$

or putting

$$\epsilon^{-(x-1)^2\frac{d}{dx}}=\rho,\quad x=\pi,$$

we can write the equation (since $\rho=\epsilon^{\frac{d}{d.\frac{x}{x-1}}}$)

$$\left\{\rho^2-\rho\left(\frac{2\pi^2-\pi-3}{\pi-2}\right)+\pi\ (\pi+1)\right\}u=\mathrm{F}(x).$$

Applying the method of divisors, we see that if the symbolical portion of the first member admit of an internal factor, it must be either $\rho-\pi$ or $\rho-(\pi+1)$.

Now in this case

$$\rho f(\pi)=f\left(\frac{2\pi-1}{\pi}\right)\rho.$$

Hence

$$\theta f(\pi)=f\left(\frac{2\pi-1}{\pi}\right).$$

Wherefore the equation

$$\phi_0\pi-\psi_0\pi\theta\frac{\phi_1\pi}{\psi_1\pi}+\psi_0\pi\theta\frac{\psi_0\pi}{\psi_1\pi}\theta\frac{\phi_2\pi}{\psi_1\pi}-\&\text{c.}\pm\psi_0\pi\theta\frac{\psi_0\pi}{\psi_1\pi}\theta\frac{\psi_0\pi}{\psi_1\pi}\ldots\ldots\theta\frac{\psi_0\pi}{\psi_1\pi}=0$$

becomes, if we take the factor $\rho-\pi$, and put

$$\psi_1\pi=1,\quad \psi_0\pi=-\pi,$$

$$(\pi+1)-\theta\left(\frac{\pi-3}{\pi-2}\right)-\theta\pi+\theta\pi=0\,;$$

an identical equation if we put for the symbol θ its equivalent as given above.

Hence $\rho-\pi$ is an internal factor of the symbolical portion of the first member. Effecting the internal division, we have

$$(\rho-(\pi+1))(\rho-\pi)f(x)=\mathrm{F}(x).$$

Let

$$(\rho-\pi)f(x)=\chi(x),$$

and the equation resolves itself into the two,

$$(\rho-(\pi+1))\chi(x)=\mathrm{F}(x)$$

and

$$(\rho-\pi)f(x)=\chi(x)\,;$$

or

$$\chi\left(\frac{2x-1}{x}\right)-(x+1)\chi(x)=\mathrm{F}(x)$$

and

$$f\left(\frac{2x-1}{x}\right)-xf(x)=\chi(x)\,;$$

forms which I have considered in my memoir on the Calculus of Functions published in the Philosophical Transactions for 1862, in which the general solution of the equations

$$\phi(x)-\chi(x)\phi\left\{\frac{a+bx}{c+ex}\right\}=\mathrm{F}(x),$$

where ϕ is the unknown function, has been obtained.

Communications received since the end of the Session.

I. "Comparison of Mr. De la Rue's and Padre Secchi's Eclipse Photographs." By Warren De la Rue, F.R.S. Received August 8, 1864.

I have stated, in the Bakerian Lecture read at the Royal Society on April 10, 1862, that the boomerang (prominence E)* was not depicted on Señor Aguilar's photographs. This is true of the prints which came into my hands in England. A visit to Rome in November 1862, however, afforded an opportunity for the examination of the first prints which had been taken in Spain on the day of the eclipse, previous to those printed off for general distribution by Señor Aguilar. I was agreeably surprised to find that the photograph of the first phase of totality showed not only this prominence very distinctly, but also other details, presently to be described, which were quite invisible in Señor Aguilar's copies. I had in fact experienced some difficulty in comparing measurements of my photographs with those of Señor Aguilar's, on account of the indistinctness (woolliness) of the latter, which I have attributed to Padre Secchi's telescope not having followed the sun's motion perfectly. A careful examination of the prints in Padre Secchi's possession has, however, convinced me that this was not the case during the period of exposure of the first negative; for I have been able to identify with a magnifier many minute forms which could only have been depicted by the most perfect following of the sun's apparent motion. For instance, my statement that the prominence H (the fallen tree) was not seen from having been mixed up with the prominence G, is not applicable to Padre Secchi's copy of the first phase of totality, for in it every detail of the fallen tree can be made out.

On expressing to Professor Secchi my surprise at the great discordance between the copy of the first phase of totality sent to me by Señor Aguilar and that of the same phase in his possession, I was informed that after a few positive prints had been taken from the then unvarnished negative, it was strengthened by the usual photographic process with nitrate of silver. This I look upon as an unfortunate mistake, as the images of the prominences were increased and their details hidden, and the beauty of the negative for ever lost.

It occurred to Padre Secchi and myself that although there was no hope

* See Index Map, Plate XV. Phil. Trans. Part I. 1862.