"On the Calculus of Symbols.—Fourth Memoir. With Applications to the Theory of Non-Linear Differential Equations." By W. H. L. Russell, Esq., A.B. Received July 31, 1863*.

In the preceding memoirs on the Calculus of Symbols, systems have been constructed for the multiplication and division of non-commutative symbols subject to certain laws of combination; and these systems suffice for linear differential equations. But when we enter upon the consideration of non-linear equations, we see at once that these methods do not apply. It becomes necessary to invent some fresh mode of calculation, and a new notation, in order to bring non-linear functions into a condition which admits of treatment by symbolical algebra. This is the object of the following memoir. Professor Boole has given, in his 'Treatise on Differential Equations,' a method due to M. Sarrus, by which we ascertain whether a given non-linear function is a complete differential. This method, as will be seen by anyone who will refer to Professor Boole's treatise, is equivalent to finding the conditions that a non-linear function may be externally divisible by the symbol of differentiation. In the following paper I have given a notation by which I obtain the actual expressions for these conditions, and for the symbolical remainders arising in the course of the division, and have extended my investigations to ascertaining the results of the symbolical division of non-linear functions by linear functions of the symbol of differentiation.

Let F $(x, y, y_1, y_2, y_3 \ldots . y_n)$ be any non-linear function, in which $y_1, y_2, y_3, \ldots . y_n$ denote respectively the first, second, third, ... nth differential of y with respect to (x).

Let U_r denote $\int dy_r$, *i.e.* the integral of a function involving $x, y, y_1, y_2 \ldots.$ with reference to y_r alone.

Let V_r in like manner denote $\frac{d}{dy_r}$ when the differentiation is supposed effected with reference to y_r alone, so that $V_r U_r F = F$.

The next definition is the most important, as it is that on which all our subsequent calculations will depend. We may suppose F differentiated (m) times with reference to y_n, y_{n-1}, or y_{n-2}, &c., and y_n, y_{n-1}, or y_{n-2}, &c., as the case may be, afterward equated to zero. We shall denote this entire process by $Z_n^{(m)}$, $Z_{n-1}^{(m)}$, $Z_{n-2}^{(m)}$, &c.

The following definition is also of importance: we shall denote the expression

$$\frac{d}{dx}+y_1\frac{d}{dy}+y_2\frac{d}{dy_1}+y_3\frac{d}{dy_3}+\ldots+y_r\frac{d}{dy_{r-1}}$$

by the symbol Y_r.

* Read Feb. 11, 1864; see Abstract, vol. xiii. p. 126.

Having thus explained the notation I propose to make use of, I proceed to determine the conditions that F may be externally divisible by $\frac{d}{dx}$, or, in other words, that F may be a perfect differential with respect to (x). It will be seen that the above notation will enable us to obtain expressions for the conditions indicated by the process of M. Sarrus.

It is obvious that if we expand F in terms of y_n, in order that the symbolical division with reference $\frac{d}{dx}$ may be possible, the terms involving y_n^2, y_n^3, &c. must vanish.

Hence $V_n^2 F=0$, and consequently

$$F=Z_n^0F+y_nZ'_nF,$$

where, of course, Z_n^0F, Z'_nF do not contain y_n.

Hence we have

$$\frac{d}{dx}(U_{n-1}Z'_nF)=Y_{n-1}U_{n-1}Z'_nF+y_nZ'_nF,$$

and therefore F becomes

$$\frac{d}{dx}(U_{n-1}Z'_nF)+Z_n^0F-Y_{n-1}U_{n-1}Z'_nF;$$

and if R_1 be the first remainder,

$$R_1=Z_n^0F-Y_{n-1}U_{n-1}Z'_nF.$$

The condition that this may be divisible by $\frac{d}{dx}$ will be, as before, $V_{n-1}^2R_1=0$; hence R_1 becomes

$$Z_{n-1}^0Z_n^0F-Z_{n-1}^0Y_{n-1}U_{n-1}Z'_nF+y_{n-1}(Z'_{1-n}Z_u^0F-Z'_{n-1}Y_{n-1}U_{n-1}Z'_nF).$$

Now

$$\frac{d}{dx}U_{n-2}(Z'_{n-1}Z_n^0F-Z'_{n-1}Y_{n-1}U_{n-1}Z'_nF)=$$
$$Y_{n-2}U_{n-2}Z'_{n-1}Z_n^0F-Y_{n-2}U_{n-2}Z'_{n-1}Y_{n-1}U_{n-1}Z'_nF$$
$$+y_{n-1}(Z'_{n-1}Z_n^0F-Z'_{n-1}Y_{n-1}U_{n-1}Z'_nF);$$

and if R_2 be the second remainder, we find

$$R_2=Z_{n-1}^0Z_n^0F-Z_{n-1}^0Y_{n-1}U_{n-1}Z'_nF$$
$$-Y_{n-2}U_{n-2}Z'_{n-1}Z_n^0F+Y_{n-2}U_{n-2}Z'_{n-1}Y_{n-1}U_{n-1}Z'_nF:$$

the next condition is $V_{n-2}R_2=0$, and therefore

$$R_2=Z_{n-2}^0Z_{n-1}^0Z_n^0F$$
$$-Z_{n-2}^0Z_{n-1}^0Y_{n-1}U_{n-1}Z'_nF-Z_{n-2}^0Y_{n-2}U_{n-2}Z'_{n-1}Z_n^0F$$
$$+Z_{n-2}^0Y_{n-2}U_{n-2}Z'_{n-1}Y_{n-1}U_{n-1}Z'_nF$$
$$+y_{n-2}(Z'_{n-2}Z_{n-1}^0Z_n^0F-Z'_{n-2}Z_{n-1}^0Y_{n-1}U_{n-1}Z'_nF$$
$$-Z'_{n-2}Y_{n-2}U_{n-2}Z'_{n-1}Z_n^0F+Z'_{n-2}Y_{n-2}U_{n-2}Z'_{n-1}Y_{n-1}U_{n-1}Z'_nF).$$

But

$$\frac{d}{dx}\Big\{\mathrm{U}_{n-3}(\mathrm{Z}'_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Z}^0_n\mathrm{F}-\mathrm{Z}'_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}$$
$$-\mathrm{Z}'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Z}^0_n\mathrm{F}+\mathrm{Z}'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F})\Big\}$$
$$=\mathrm{Y}_{n-3}\mathrm{U}_{n-3}(\mathrm{Z}'_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Z}^0_n\mathrm{F}-\mathrm{Z}'_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}$$
$$-\mathrm{Z}'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Z}^0_n\mathrm{F}+\mathrm{Z}'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F})$$
$$+y_{n-2}(\mathrm{Z}'_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Z}^0_n\mathrm{F}-\mathrm{Z}'_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}$$
$$-\mathrm{Z}'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Z}^0_n\mathrm{F}+\mathrm{Z}'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F});$$

whence we find

$$\mathrm{R}_3=\mathrm{Z}^0_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Z}^0_n\mathrm{F}-\mathrm{Z}^0_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}$$
$$-\mathrm{Z}^0_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Z}^0_n\mathrm{F}+\mathrm{Z}^0_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}$$
$$-\mathrm{Y}_{n-3}\mathrm{U}_{n-3}\mathrm{Z}'_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Z}^0_n\mathrm{F}+\mathrm{Y}_{n-3}\mathrm{U}_{n-3}\mathrm{Z}'_{n-2}\mathrm{Z}^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}$$
$$+\mathrm{Y}_{n-3}\mathrm{U}_{n-3}\mathrm{Z}'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Z}^0_n\mathrm{F}$$
$$-\mathrm{Y}_{n-3}\mathrm{U}_{n-3}\mathrm{Z}'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}.$$

Hence we infer the following rule for the formation of R_r.

Construct the term

$$\mathrm{Y}_{n-r}\mathrm{U}_{n-r}\mathrm{Z}'_{n-r+1}\mathrm{Y}_{n-r+1}\mathrm{U}_{n-r+1}\mathrm{Z}'_{n-r+2}\mathrm{Y}_{n-r+2}\mathrm{U}_{n-r+2}\mathrm{Z}'_{n-r+2}$$
$$\ldots\ldots\mathrm{Y}_{n-2}\mathrm{U}_{n-2}\mathrm{Z}'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}.$$

In any symbol Z'_m the accent may be changed into a zero, *i. e.* we may at pleasure substitute Z^0_m anywhere for Z'_m; but in such case the previous symbolical factor $\mathrm{Y}_{m-1}\mathrm{U}_{m-1}$ must be omitted. This term is positive or negative according as the symbol Z' occurs an even or an odd number of times in it; the aggregate of all the terms thus formed constitute the remainder R_r, and the conditions that F may be externally divisible by $\frac{d}{dx}$ are

$$\mathrm{V}_n\mathrm{F}=0,\ \mathrm{V}_{n-1}\mathrm{R}_1=0,\ \mathrm{V}_{n-2}\mathrm{R}_2=0,\ \mathrm{V}_{n-3}\mathrm{R}_3=0,\ \&\mathrm{c}.$$

We shall now investigate the conditions that $\frac{d}{dx}+\mathrm{P}$ may externally divide F where P is a function of (x) and (y).

As before, $\mathrm{V}_n\mathrm{F}=0$, and in consequence

$$\mathrm{F}=\mathrm{Z}^0_n\mathrm{F}+y_n\mathrm{Z}'_n\mathrm{F}.$$

Now

$$\left(\frac{d}{dx}+\mathrm{P}\right)\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}=\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}$$
$$+\mathrm{P}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}+y_n\mathrm{Z}'_n\mathrm{F}.$$

Hence we shall have

$$\mathrm{R}_1=\mathrm{Z}^0_n\mathrm{F}-\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}-\mathrm{P}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}.$$

We have $\mathrm{V}^2_{n-1}\mathrm{R}_1=0$ in order that this remainder may contain only the first power of y_{n-1}, and

$$\mathrm{R}_1=\mathrm{Z}^0_{n-1}\mathrm{Z}^0_n\mathrm{F}-\mathrm{Z}^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}-\mathrm{Z}^0_{n-1}\mathrm{P}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}$$
$$+y_{n-1}(\mathrm{Z}'_{n-1}\mathrm{Z}^0_n\mathrm{F}-\mathrm{Z}'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}-\mathrm{Z}'_{n-1}\mathrm{P}\mathrm{U}_{n-1}\mathrm{Z}'_n\mathrm{F}),$$

since

$$\left(\frac{d}{dx}+\mathrm{P}\right)\mathrm{U}_{n-2}(Z'_{n-1}Z^0_n\mathrm{F}-Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$

$$-Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F})=$$

$$\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-2}Z^0_n\mathrm{F}-\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}-$$
$$\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}+y_{n-1}(Z'_{n-1}Z^0_n\mathrm{F}$$
$$-Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}-Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F})+$$
$$\mathrm{PU}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}-\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$-\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}.$$

Whence we find that $\frac{d}{dx}+\mathrm{P}$ divides R_1 with a remainder,

$$\mathrm{R}_2=Z^0_{n-1}Z^0_n\mathrm{F}-Z^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$-Z^0_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}-\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}$$
$$+\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}+\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}$$
$$-\mathrm{PU}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}+\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$+\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}.$$

Putting $\mathrm{V}^2_{n-2}\mathrm{R}_2=0$, we find in like manner,

$$\mathrm{R}_3=Z^0_{n-2}Z^0_{n-1}Z^0_n\mathrm{F}-Z^0_{n-2}Z^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$-Z^0_{n-2}Z^0_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}-Z^0_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}$$
$$+Z^0_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}+Z^0_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}$$
$$-Z^0_{n-2}\mathrm{PU}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}+Z^0_{n-2}\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$+Z^0_{n-2}\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}-\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}Z^0_{n-1}Z^0_n\mathrm{F}$$
$$+\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}Z^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$+\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}Z^0_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}$$
$$+\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}$$
$$-\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$-\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}$$
$$+\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}\mathrm{P}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}$$
$$-\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$-\mathrm{Y}_{n-3}\mathrm{U}_{n-3}Z'_{n-2}\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}$$
$$-\mathrm{PU}_{n-3}Z'_{n-2}Z^0_{n-1}Z^0_n\mathrm{F}+\mathrm{PU}_{n-3}Z'_{n-2}Z^0_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$+\mathrm{PU}_{n-3}Z'_{n-2}Z^0_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}+\mathrm{PU}_{n-3}Z'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}$$
$$-\mathrm{PU}_{n-3}Z'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$-\mathrm{PU}_{n-3}Z'_{n-2}\mathrm{Y}_{n-2}\mathrm{U}_{n-2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}$$
$$+\mathrm{PU}_{n-3}Z'_{n-2}\mathrm{PU}_{n-2}Z'_{n-1}Z^0_n\mathrm{F}$$
$$-\mathrm{PU}_{n-3}Z'_{n-2}\mathrm{PU}_{n-2}Z'_{n-1}\mathrm{Y}_{n-1}\mathrm{U}_{n-1}Z'_n\mathrm{F}$$
$$-\mathrm{PU}_{n-3}Z'_{n-2}\mathrm{PU}_{\ -2}Z'_{n-1}\mathrm{PU}_{n-1}Z'_n\mathrm{F}.$$

We see at once that the value of R_r in this case can be formed from that calculated in the last example, by writing P at pleasure for any one or

more of the symbols Y, and taking the aggregate of the terms so formed. The conditions of division will be, as before,

$$V_n F=0,\ V_{n-1}R_1=0,\ V_{n-2}R_2=0\ \ldots .$$

Let us now investigate the conditions that F may be externally divisible by $\frac{d^2}{dx^2}$.

We see at once that F, as before, must take the form $Z_n^0 F+y_n Z'_n F$, and also that $Z'_n F$ can contain neither y_n nor y_{n-1}. Hence we shall have

$$V_n^2 F=0, \text{ and also } V_{n-1}V_n F=0.$$

Now

$$\frac{d^2}{dx^2}(U_{n-2}Z'_n F)=\frac{d}{dx}\{Y_{n-2}+y_{n-1}V_{n-2}\}U_{n-2}Z'_n F$$
$$=XY_{n-2}U_{n-2}Z'_n F+y_{n-1}XZ'_n F+y_n Z'_n F.$$

Hence we shall have

$$R_1=Z_n^0 F-XY_{n-2}U_{n-2}Z'_n F-y_{n-1}XZ'_n F,$$

when we must introduce the conditions

$$V_{n-1}^2 R_1=0, \text{ and } V_{n-2}V_{n-1}R_1=0;$$

consequently we shall have

$$R_1=Z_{n-1}^0 Z_n^0 F-Z_{n-1}^0 XY_{n-2}U_{n-2}Z'_n F+$$
$$(Z'_{n-1}Z_n^0 F-Z'_{n-1}XY_{n-2}U_{n-2}Z'_n F-Z_{n-1}^0 XZ'_n F)y_{n-1}.$$

Now

$$\frac{d^2}{dx^2}U_{n-3}(Z'_{n-1}Z_n^0 F-Z'_{n-1}XY_{n-2}U_{n-2}Z'_n F$$
$$-Z_{n-1}^0 XZ'_n F)=$$
$$XY_{n-3}U_{n-3}Z'_{n-1}Z_n^0 F-XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_n F$$
$$-XY_{n-3}U_{n-3}Z_{n-1}^0 XZ'_n F+(XZ'_{n-1}Z_n^0 F-$$
$$XZ'_{n-1}XY_{n-2}U_{n-2}Z'_n F-XZ_{n-1}^0 XZ'_n F)y_{n-2}+$$
$$(Z'_{n-1}Z_n^0 F-Z'_{n-1}XY_{n-2}U_{n-2}Z'_n F-Z_{n-1}^0 XZ'_n F)y_{n-1}.$$

Hence

$$R_2=Z_{n-1}^0 Z_n^0 F-Z_{n-1}^0 XY_{n-2}U_{n-2}Z'_n F$$
$$-XY_{n-3}U_{n-3}Z'_{n-1}Z_n^0 F+XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_n F$$
$$+XY_{n-3}U_{n-3}Z_{n-1}^0 XZ'_n F+(XZ'_{n-1}XY_{n-2}U_{n-2}Z'_n F$$
$$+XZ_{n-1}^0 XZ'_n F-XZ'_{n-1}Z_n^0 F)y_{n-2}.$$

Introducing the conditions

$$V_{n-2}^2 R_2=0,\ V_{n-3}V_{n-2}R_2=0,$$

we find

$$R_2=Z_{n-2}^0 Z_{n-1}^0 Z_n^0 F-$$
$$Z_{n-2}^0 Z_{n-1}^0 XY_{n-2}U_{n-2}Z'_n F-Z_{n-2}^0 XY_{n-3}U_{n-3}Z'_{n-1}Z_n^0 F$$
$$+Z_{n-2}^0 XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_n F$$
$$+Z_{n-2}^0 XY_{n-3}U_{n-3}Z_{n-1}^0 XZ'_n F+(Z'_{n-2}Z_{n-1}^0 Z_n^0 F$$
$$-Z'_{n-2}Z_{n-1}^0 XY_{n-2}U_{n-2}Z'_n F-Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}Z_n^0 F$$
$$+Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_n F$$
$$+Z'_{n-2}XY_{n-3}U_{n-3}Z_{n-1}^0 XZ'_n F+Z_{n-2}^0 XZ'_{n-1}XY_{n-2}U_{n-2}Z'_n F$$
$$+Z_{n-2}^0 XZ_{n-1}^0 XZ'_n F-Z_{n-2}^0 XZ'_{n-1}Z_n^0 F)y_{n-2}.$$

Now

$$\frac{d^2}{dx^2}U_{n-4}(Z'_{n-2}Z^0_{n-1}Z^0_nF-$$
$$Z'_{n-2}Z^0_{n-1}XY_{n-2}U_{n-2}Z'_nF-Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}Z^0_nF$$
$$+Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+Z'_{n-2}XY_{n-3}U_{n-3}Z^0_{n-1}XZ'_nF+Z^0_{n-2}XZ'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+Z^0_{n-2}XZ^0_{n-1}XZ'_nF-Z^0_{n-2}XZ'_{n-1}Z^0_nF)$$
$$=XY_{n-4}U_{n-4}Z'_{n-2}Z^0_{n-1}Z^0_nF$$
$$-XY_{n-4}U_{n-4}Z'_{n-2}Z^0_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$-XY_{n-4}U_{n-4}Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}Z^0_nF$$
$$+XY_{n-4}U_{n-4}Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+XY_{n-4}U_{n-4}Z'_{n-2}XY_{n-3}U_{n-3}Z^0_{n-1}XZ'_nF$$
$$+XY_{n-4}U_{n-4}Z^0_{n-2}XZ'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+XY_{n-4}U_{n-4}Z^0_{n-2}XZ^0_{n-1}XZ'_nF$$
$$-XY_{n-4}U_{n-4}Z^0_{n-2}XZ'_{n-1}Z^0_nF$$
$$+(XZ'_{n-2}Z^0_{n-1}Z^0_nF-XZ'_{n-2}Z^0_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$-XZ'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}Z^0_nF$$
$$+XZ'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+XZ'_{n-2}XY_{n-3}U_{n-3}Z^0_{n-1}XZ'_nF$$
$$+XZ^0_{n-2}XZ'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+XZ^0_{n-2}XZ^0_{n-1}XZ'_nF-XZ^0_{n-2}XZ'_{n-1}Z^0_nF)y_{n-3}$$
$$+(Z'_{n-2}Z^0_{n-1}Z^0_nF-Z'_{n-2}Z^0_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$-Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}Z^0_nF+Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+Z'_{n-2}XY_{n-3}U_{n-3}Z^0_{n-1}XZ'_nF+Z^0_{n-2}XZ'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+Z^0_{n-2}XZ^0_{n-1}XZ'_nF-Z^0_{n-2}XZ'_{n-1}Z^0_nF)y_{n-2}.$$

We thus find

$$R_3=Z^0_{n-2}Z^0_{n-1}Z^0_nF-Z^0_{n-2}Z^0_{n-1}XY_{n-2}U_{n-2}Z'_nF-$$
$$Z^0_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}Z^0_nF+Z^0_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+Z^0_{n-2}XY_{n-3}U_{n-3}Z^0_{n-1}XZ'_nF-XY_{n-4}U_{n-4}Z'_{n-2}Z^0_{n-1}Z^0_nF$$
$$+XY_{n-4}U_{n-4}Z'_{n-2}Z^0_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+XY_{n-4}U_{n-4}Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}Z^0_nF$$
$$-XY_{n-4}U_{n-4}Z'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$-XY_{n-4}U_{n-4}Z'_{n-2}XY_{n-3}U_{n-3}Z^0_{n-1}XZ'_nF$$
$$-XY_{n-4}U_{n-4}Z^0_{n-2}XZ'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$-XY_{n-4}U_{n-4}Z^0_{n-2}XZ^0_{n-1}XZ'_nF$$
$$+XY_{n-4}U_{n-4}Z^0_{n-2}XZ'_{n-1}Z^0_nF$$
$$+(XZ'_{n-2}Z^0_{n-1}XY_{n-2}U_{n-2}Z'_nF-XZ'_{n-2}Z^0_{n-1}Z^0_nF$$
$$+XZ'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}Z^0_nF$$
$$-XZ'_{n-2}XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$-XZ'_{n-2}XY_{n-3}U_{n-3}Z^0_{n-1}XZ'_nF-XZ^0_{n-2}XZ'_{n-1}XY_{n-2}U_{n-2}Z'_nF$$
$$+XZ^0_{n-2}XZ^0_{n-1}XZ'_nF+XZ^0_{n-2}XZ'_{n-1}Z^0_nFy_{n-3}).$$

Let us now assume

$$R_r = M_r + N_r y_{n-r}.$$

Then M_r is formed according to the following rule:—Form the term

$$XY_{n-r-1}U_{n-r-1}Z'_{n-r+1}XY_{n-r}U_{n-r}Z'_{n-r+2}XY_{n-r+1}U_{n-r+1}Z'_{n-r+3}$$
$$\ldots . XY_{n-3}U_{n-3}Z'_{n-1}XY_{n-2}U_{n-2}Z'_nF.$$

Z'_m may in any place be changed into Z^0_m; but in this case either the preceding $XY_{m-2}U_{m-2}$ must be omitted, or the succeeding $XY_{m-1}U_{m-1}$ changed into X. The signs of the terms follow this law. A term not containing X introduced in place of XYU is positive if Z' occurs in it an even number of times, negative in the contrary case. But every X introduced in place of XYU occasions a change of sign. The aggregate of all the terms thus formed will give M_r.

We form N thus: construct the term

$$XZ'_{n-r+r}XY_{n-r}U_{n-r}Z'_{n-r+2}XY_{n-r+1}U_{n-r+1} \ldots\ldots Z'_nF,$$

and a precisely similar rule holds good. R_r is subject to the condition

$$V^2_{n-r}R_r = 0,\ V_{n-r-1}V_{n-r}R_r = 0.$$

Let us now investigate the criterion that F may be divisible by

$$\frac{d^2}{dx^2} + P\frac{d}{dx} + Q,$$

where P and Q are functions of (x) and (y).

Proceeding as before, we have

$$\left(\frac{d^2}{dx^2} + P\frac{d}{dx} + Q\right)U_{n-2}Z'_nF =$$
$$\left(\frac{d}{dx} + P\right)(Y_{n-2} + y_{n-1}V_{n-2})U_{n-2}Z'_nF + QU_{n-2}Z'_nF$$
$$= (XY_{n-2} + PY_{n-2} + Q)U_{n-2}Z_nF + y_{n-1}(X+P)Z'_nF + y_nZ'_nF.$$

The form of this equation gives us the following rule to ascertain the successive remainders. Construct the remainder in the last case as before, and substitute at pleasure Q in any place where XY is found, P in any place where X is found. The aggregate of the term thus formed will give the remainder in this case.

We now investigate the condition that $\frac{d^3}{dx^3}$ may be an external factor of F.

We put, as before, $F = Z^0_nF + y_nZ'_nF$, where Z'_nF must contain neither y_{n-1} nor y_{n-2}, which gives the conditions

$$V^2_nF = 0,\ V_{n-1}V_nF = 0,\ V_{n-2}V_nF = 0.$$

Now we have

$$\frac{d^3}{dx^3}(U_{n-3}Z'_nF) = \frac{d^2}{dx^2}(Y_{n-3} + y_{n-2}V_{n-3})U_{n-3}Z'_nF$$
$$= \frac{d^2}{dx^2}(Y_{n-3}U_{n-3}Z'_nF + y_{n-2}Z'_nF)$$
$$= X^2Y_{n-3}U_{n-3}Z'_nF + y_{n-2}X^2Z'_nF + 2y_{n-1}XZ'_nF + y_nZ'_nF.$$

And we consequently obtain

$$R_1=Z^0_nF-X^2Y_{n-3}U_{n-3}Z'_nF-y_{n-2}X^2Z'_nF-2y_{n-1}XZ'_nF.$$

Introducing the conditions

$$V^2_{n-1}R_1=0,\ V_{n-2}V_{n-1}R_1=0,\ V_{n-3}V_{n-1}R_1=0,$$

and expanding in terms of y_{n-1}, we have

$$\begin{aligned}R_1=&(Z^0_{n-1}Z^0_nF-Z^0_{n-1}X^2Y_{n-3}U_{n-3}Z'_nF)+\\&(Z'_{n-1}Z^0_nF-Z'_{n-1}X^2Y_{n-3}U_{n-3}Z'_nF)y_{n-1}-Z^0_{n-1}X^2Z'_nFy_{n-2}\\&-y_{n-1}y_{n-2}Z'_{n-1}X^2Z'_nF-2y_{n-1}Z^0_{n-1}XZ'_nF.\end{aligned}$$

As the coefficient of y_{n-1} in this cannot contain y_{n-2}, we may write this expression,

$$\begin{aligned}R_1=&(Z^0_{n-1}Z^0_nF-Z^0_{n-1}X^2Y_{n-3}U_{n-3}Z'_nF)-y_{n-2}Z^0_{n-1}X^2Z'_nF\\&+(Z^0_{n-2}Z'_{n-1}Z^0_nF-Z^0_{n-2}Z'_{n-1}X^2Y_{n-3}U_{n-3}Z'_nF-2Z^0_{n-2}Z^0_{n-1}XZ'_nF)y_{n-1}.\end{aligned}$$

Let us now assume

$$R_m=L_m+M_my_{n-m-1}+N_my_{n-m},$$

where R_m is the mth remainder, and N_m does not contain y_{n-m-1} or y_{n-m-2}.

Hence, expanding in terms of y_{n-m}, we have

$$\begin{aligned}R_m=&(Z^0_{n-m}L_m+y_{n-m}Z'_{n-m}L_m)\\&+(Z^0_{n-m}M_m+y_{n-m}Z'_{n-m}M_m)y_{n-m-1}\\&+(Z^0_{n-m}N_m+y_{n-m}Z'_{n-m}N_m)y_{n-m}\\=&Z^0_{n-m}L_m+Z^0_{n-m}M_my_{n-m-1}\\&+(Z^0_{n-m-1}Z'_{n-m}L_m+Z^0_{n-m-1}Z^0_{n-m}N_m)y_{n-m}.\end{aligned}$$

Now

$$\begin{aligned}&\frac{d^3}{dx^3}\{U_{n-m-3}(Z^0_{n-m-1}Z'_{n-m}L_m+Z^0_{n-m-1}Z^0_{n-m}N_m)\}\\&=\frac{d^2}{dx^2}\{Y_{n-m-3}+y_{n-m-2}V_{n-m-3}\}U_{n-m-3}\\&\quad\{Z^0_{n-m-1}Z'_{n-m}L_m+Z^0_{n-m-1}Z^0_{n-m}N_m\}\\&=X^2Y_{n-m-3}U_{n-m-3}(Z^0_{n-m-1}Z'_{n-m}L_m+Z^0_{n-m-1}Z^0_{n-m}N_m)\\&\quad+X^2(Z^0_{n-m-1}Z'_{n-m}L_m+Z^0_{n-m-1}Z^0_{n-m}N_m)\,.\,y_{n-m-2}\\&\quad+2X(Z^0_{n-m-1}Z'_{n-m}L_m+Z^0_{n-m-1}Z^0_{n-m}N_m)y_{n-m-1}\\&\quad+(Z^0_{n-m-1}Z'_{n-m}L_m+Z^0_{n-m-1}Z^0_{n-m}N_m)y_{n-m}.\end{aligned}$$

Hence
$$\begin{aligned}R_{m+1}=&(Z^0_{n-m}-X^2Y_{n-m-3}U_{n-m-3}Z^0_{n-m-1}Z'_{n-m})L_m\\&-X^2Y_{n-m-3}U_{n-m-3}Z^0_{n-m-1}Z^0_{n-m}N_m\\&-(X^2Z^0_{n-m-1}Z'_{n-m}L_m+X^2Z^0_{n-m-1}Z^0_{n-m}N_m)y_{n-m-2}\\&-(2XZ^0_{n-m-1}Z'_{n-m}L_m-Z^0_{n-m}M_m+2XZ^0_{n-m-1}Z^0_{n-m}N_m)y_{n-m-1}.\end{aligned}$$

Now consider for a moment the equations

$$\begin{aligned}L_{m+1}&=G_mL_m+H_mM_m+K_mN_m,\\M_{m+1}&=G'_mL_m+H'_mM_m+K'_mN_m,\\N_{m+1}&=G''_mL_m+H''_mM_m+K''_mN_m,\end{aligned}$$

and suppose that

$$\begin{aligned}L_{m+1}&=\lambda_1 L_{m-1}+\mu_1 M_{m-1}+\nu_1 N_{n-1}\\&=\lambda_2 L_{m-2}+\mu_2 M_{m-2}+\nu_2 N_{n-2}=\&c.\\&=\lambda_r L_{m-r}+\mu_r M_{m-r}+\nu_r N_{n-r}\\&=\&c.\end{aligned}$$

Then we find

$$\lambda_1= \quad G_m G_{m-1}+H_m G'_{m-1}+K_m G''_{m-1},$$

$$\begin{aligned}\lambda_2= \quad & G_m G_{m-1} G_{m-2}+H_m G'_{m-1} G_{m-2}+K_m G''_{m-1} G_{m-2}\\&+G_m H_{m-1} G'_{m-2}+H_m H'_{m-1} G'_{m-2}+K_m H''_{m-2} G'_{m-2}\\&+G_m K_{m-1} G''_{m-2}+H_m K'_{m-1} G''_{m-2}+K_m K''_{m-1} G''_{m-2},\end{aligned}$$

$$\begin{aligned}\lambda_3= \quad & G_m G_{m-1} G_{m-2} G_{m-3}+H_m G'_{m-1} G_{m-2} G_{m-3}\\&+K_m G''_{m-1} G_{m-2} G_{m-3}+G_m H_{m-1} G'_{m-2} G_{m-3}\\&+H_m H'_{m-1} G'_{m-2} G_{m-3}+K_m H''_{m-2} G'_m G_{m-3}\\&+G_m K_{m-1} G''_{m-2} G_{m-3}+H_m K'_{m-1} G''_{m-2} G_{m-3}\\&+K_m K''_{m-1} G''_{m-2} G_{m-3}+G_m G_{m-1} H_{m-2} G'_{m-3}\\&+H_m G'_{m-1} H_{m-2} G'_{m-3}+K_m G''_{m-1} H_m G'_{m-3}\\&+G_m H_{m-1} H'_{m-2} G'_{m-3}+H_m H'_{m-1} H'_{m-2} G'_{m-3}\\&+K_m H''_{m-1} H'_{m-2} G'_{m-3}+K_m H''_{m-1} H'_{m-2} G'_{m-3}\\&+G_m K_{m-1} H''_{m-2} G'_{m-3}+H_m K'_{m-1} H''_{m-2} G'_{m-3}\\&+G_m G_{m-1} K_{m-2} G''_{m-3}+H_m G'_{m-1} K_{m-2} G''_{m-3}\\&+K_m G''_{m-1} K_{m-2} G''_{m-3}+G_m H_{m-1} K'_{m-2} G''_{m-3}\\&+H_m H'_{m-1} K'_{m-2} G''_{m-3}+K_m H''_{m-1} K'_{m-2} G''_{m-3}\\&+G_m K_{m-1} K''_{m-2} G''_{m-3}+H_m K'_{m-1} K''_{m-2} G''_{m-3}\\&+K_m K''_{m-1} K''_{m-2} G''_{m-3}.\end{aligned}$$

Hence we obtain the following rule for the determination of λ_r:—Write down the term $G_m G_{m-1} G_{m-2} \ldots\ldots G_{m-r}$. We may substitute H and K at pleasure for G anywhere except in the last factor, which is always G. Whenever we put H for G, the succeeding letter is to receive a single accent; whenever K for G, the succeeding letter receives a double accent. The aggregate of all the terms thus formed will be λ_r, and we may of course obtain similar expressions for μ_r, &c.

Now if we put

$$\begin{aligned}&G_m=Z^0_{n-m}-X^2 Y_{n-m-3} U_{n-m-3} Z^0_{n-m-1} Z'_{n-m},\\&H_m=0,\\&K_m=-X^2 Y_{n-m-3} U_{n-m-3} Z^0_{n-m-1} Z^0_{n-m},\\&G'_m=-X^2 Z^0_{n-m-1} Z'_{n-m},\\&H'_m=0,\\&K'_m=-X^2 Z^0_{n-m-1} Z^0_{n-m},\\&G''_m=-2X Z^0_{n-m-1} Z'_{n-m},\\&H''_m=+Z^0_{n-m} M_m,\\&K''_m=-2X Z^0_{n-m-1} Z^0_{n-m},\end{aligned}$$

we shall find the above equations satisfied; and consequently the last investigation gives the law of the formation of the remainders. Each remainder will of course be subject to the three conditions already exhibited.

These results point out the foundations on which symbolical division, as applied to non-linear functions, must rest. We have confined our attention to external division, as more particularly applicable to these functions. When a non-linear equation is proposed for reduction, we must ascertain whether it admits of an external factor by employing the method of division as already explained.

"On the Calculus of Symbols.—Fifth Memoir. With Application to Linear Partial Differential Equations, and the Calculus of Functions." By W. H. L. Russell, Esq., A.B. Communicated by Professor Stokes, Sec. R.S. Received April 7, 1864*.

In applying the calculus of symbols to partial differential equations, we find an extensive class with coefficients involving the independent variables which may in fact, like differential equations with constant coefficients, be solved by the rules which apply to ordinary algebraical equations; for there are certain functions of the symbols of partial differentiation which combine with certain functions of the independent variables according to the laws of combination of common algebraical quantities. In the first part of this memoir I have investigated the nature of these symbols, and applied them to the solution of partial differential equations. In the second part I have applied the calculus of symbols to the solution of functional equations. For this purpose I have given some cases of symbolical division on a modified type, so that the symbols may embrace a greater range. I have then shown how certain functional equations may be expressed in a symbolical form, and have solved them by methods analogous to those already explained.

Since
$$\left(x\frac{d}{dy}-y\frac{d}{dx}\right)(x^2+y^2)=0,$$
we shall have
$$\left(x\frac{d}{dy}-y\frac{d}{dx}\right)(x^2+y^2)u=(x^2+y^2)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)u,$$
or, omitting the subject,
$$\left(x\frac{d}{dy}-y\frac{d}{dx}\right)(x^2+y^2)=(x^2+y^2)\left(x\frac{d}{dy}-y\frac{d}{dx}\right),$$
also
$$x\frac{d}{dy}-y\frac{d}{dx}+x^2+y^2=x^2+y^2+x\frac{d}{dy}-y\frac{d}{dx};$$
therefore the symbols $x\frac{d}{dy}-y\frac{d}{dx}$ and x^2+y^2 combine according to the laws of ordinary algebraical symbols, and consequently partial differential

* Read April 28, 1864. See Abstract, vol. xiii. p. 227.