result to those cases in which the ground whereon we stand is the plane of the Horopter, it follows that, looking straight forward to the horizon we can distinguish the inequalities and the distances of different parts of the ground better than other objects of the same kind and distance.

This is actually true. We can observe it very conspicuously when we look to a plain and open country with very distant hills, at first in the natural position, and afterwards with the head inclined or inverted, looking under the arm or between our legs, as painters sometimes do in order to distinguish the colours of the landscape better. Comparing the aspect of the distant parts of the ground, you will find that we perceive very well that they are level and stretched out into a great distance in the natural position of your head, but that they seem to ascend to the horizon and to be much shorter and narrower when we look at them with the head inverted: we get the same appearance also when our head remains in its natural position, and we look to the distant objects through two rectangular prisms, the hypothenuses of which are fastened on a horizontal piece of wood, and which show inverted images of the objects. But when we invert our head, and invert at the same time also the landscape by the prisms, we have again the natural view and the accurate perception of distances as in the natural position of our head, because then the apparent situation of the ground is again the plane of the Horopter of our eyes.

The alteration of colour in the distant parts of a landscape when viewed with inverted head, or in an inverted optical image, can be explained, I think, by the defective perception of distance. The alterations of the colour of really distant objects produced by the opacity of the air, are well known to us, and appear as a natural sign of distance; but if the same alterations are found on objects apparently less distant, the alteration of colour appears unusual, and is more easily perceived.

It is evident that this very accurate perception of the form and the distances of the ground, even when viewed indirectly, is a great advantage, because by means of this arrangement of our eyes we are able to look at distant objects, without turning our eyes to the ground, when we walk.

April 21, 1864.

Major-General SABINE, President, in the Chair.

The following communications were read:—

I. "On the Orders and Genera of Quadratic Forms containing more than three Indeterminates." By H. T. STEPHEN SMITH, M.A., F.R.S., Savilian Professor of Geometry in the University of Oxford. Received March 22, 1864.

Let us represent by f_1 a homogeneous form or quantic of any order containing n indeterminates; by $(\alpha^{(1)})$, a square matrix of order n; by

$(\alpha^{(\ )})$, its ith derived matrix, *i. e.* the matrix of order $\frac{\lfloor n}{\lfloor i \lfloor n-i} = \mathrm{I}$, the constituents of which are the minor determinants of order i of the matrix $(\alpha^{(1)})$; and lastly, by f_i, a form of any order containing I indeterminates, the coefficients of which depend on the coefficients of f_1. When f_1 is transformed by $(\alpha^{(1)})$, let f_i be transformed by $(\alpha^{(i)})$; if, after division or multiplication by a power of the modulus of transformation, the metamorphic of f_i depends on the metamorphic of f_1, in the same way in which f_i depends on f_1, f_i is said to be a concomitant of the ith species of f_1. Thus: a concomitant of the 1st species is a covariant; a concomitant of the $(n-1)$th species is a contravariant; if $n=2$ there are only covariants; if $n=3$ there are only covariants and contravariants; but if $n>3$, there will exist in general concomitants of the intermediate species.

There is an obvious difference between covariants and contravariants on the one hand, and the intermediate concomitants on the other. The number of indeterminates in a covariant or contravariant is the same as in its primitive; in an intermediate concomitant, the number of indeterminates is always greater than in its primitive. Again, to every metamorphic of a covariant or contravariant, there corresponds a metamorphic of its primitive; whereas, in the case of a concomitant of the intermediate order i, a metamorphic of the primitive will correspond, not to every metamorphic of the concomitant, but only to such metamorphics as result from transformations the matrices of which are the ith derived matrices of matrices of order n.

It is also obvious that, besides the $n-1$ species of concomitance here defined, there are, when n is >3, an infinite number of other species of concomitance of the same general nature. For from any derived matrix we may form another derived matrix, and so on continually; and to every such process of derivation a distinct species of concomitance will correspond.

The notion of intermediate concomitance appears likely to be of use in many researches; in what follows, it is employed to obtain a definition of the ordinal and generic characters of quadratic forms containing more than 3 indeterminates. (The case of quadratic forms containing 3 indeterminates has been considered by Eisenstein in his memoir, "Neue Theoreme des höheren Arithmetik," Crelle, vol. xxxv. pp. 121 and 125.) Let

$$f_1 = \sum_{p=1}^{p=n} \sum_{q=1}^{q=n} \mathrm{A}^{(1)}_{p,q}\, x_p\, x_q$$

represent a quadratic form of n indeterminates; let $(\mathrm{A}^{(1)})$ be the symmetrical matrix of this form, and $(\mathrm{A}^{(i)})$ the ith derived matrix of $(\mathrm{A}^{(1)})$; $(\mathrm{A}^{(i)})$ will also be a symmetrical matrix, and the quadratic form

$$f_i = \sum_{p=1}^{p=\mathrm{I}} \sum_{q=1}^{q=\mathrm{I}} \mathrm{A}^{(i)}_{p,q}\, \mathrm{X}_p\, \mathrm{X}_q \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A})$$

will be a concomitant of the ith species of f_1. It is immaterial what

principle of arrangement is adopted in writing the quadratic matrix $(A^{(i)})$, and the transforming matrix $(\alpha^{(i)})$; provided only that the arrangement be the same in the two matrices, and that in each matrix it be the same in height and in depth.

For example, if $f_1 = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + 2b_1 x_1 x_2 + 2b_2 x_1 x_3 + 2b_3 x_1 x_4 + 2b_4 x_2 x_3 + 2b_5 x_2 x_4 + 2b_6 x_3 x_4$ be a quadratic form containing four indeterminates, the form $f_2 =$

$$\begin{aligned}
&(b_1^2 - a_1 a_2)\, X_1^2 + (b_2^2 - a_1 a_3)\, X_2^2 + (b_3^2 - a_1 a_4)\, X_3^2 \\
&+ (b_4^2 - a_2 a_3)\, X_4^2 + (b_5^2 - a_2 a_4)\, X_5^2 + (b_6^2 - a_3 a_4)\, X_6^2 \\
&+ 2(b_1 b_2 - a_1 b_4)\, X_1 X_2 + 2(b_1 b_3 - a_1 b_5)\, X_2 X_3 \\
&- 2(b_1 b_4 - a_2 b_2)\, X_1 X_4 - 2(b_1 b_5 - a_2 b_3)\, X_1 X_5 \\
&- 2(b_2 b_5 - b_3 b_4)\, X_1 X_6 + 2(b_2 b_3 - a_1 b_6)\, X_2 X_3 \\
&+ 2(b_2 b_4 - a_3 b_1)\, X_2 X_4 - 2(b_1 b_6 - b_3 b_4)\, X_2 X_5 \\
&- 2(b_2 b_6 - a_3 b_3)\, X_2 X_6 - 2(b_1 b_6 - b_2 b_5)\, X_3 X_4 \\
&+ 2(b_3 b_5 - a_4 b_1)\, X_3 X_5 + 2(b_3 b_6 - a_4 b_2)\, X_3 X_6 \\
&+ 2(b_4 b_5 - a_2 b_6)\, X_4 X_5 - 2(b_4 b_6 - a_3 b_5)\, X_4 X_6 \\
&\qquad + 2(b_5 b_6 - a_4 b_4)\, X_5 X_6
\end{aligned}$$

is the concomitant of the second species of f.

The $n-1$ forms defined by the formula (A), of which the first is the form f_1 itself, and the last the contravariant of f_1, we shall term *the fundamental concomitants of* f_1; in contradistinction to those other quadratic concomitants (infinite in number) of which the matrices are the symmetrical matrices that may be derived, by a multiplicate derivation, from $(A^{(1)})$ . . . . Passing to the arithmetical theory of quadratic forms—*i. e.* supposing that the constituents of $(A^{(1)})$ are integral numbers, we shall designate by $\nabla_1, \nabla_2, \ldots \nabla_n$ the greatest common divisors (taken positively) of the minors of different orders of the matrix $(A^{(1)})$, so that, in particular, ∇_1 is the greatest common divisor of its constituents, and ∇_n is the absolute value of its determinant, here supposed to be different from zero. By the primary divisor of a quadratic form we shall understand the greatest common divisor of the coefficients of the squares and double rectangles in the quadratic form; by the secondary divisor we shall understand the greatest common divisor of the coefficients of the squares and of the rectangles; so that the primary divisor is equal to, or is half of, the secondary divisor, according as the quadratic form (to use the phraseology of Gauss) is derived from a form properly or improperly primitive. It will be seen that $\nabla_1, \nabla_2, \ldots . \nabla_{n-1}$ are the primary divisors of the forms $f_1, f_2, \ldots . f_{n-1}$ respectively.

We now consider the totality of arithmetical quadratic forms, containing n indeterminates, and having a given index of inertia, and a given determinant.

If a quadratic form be reduced to a sum of squares by any linear transformation, the number of positive and of negative squares is the same,

whatever be the real transformation by which the reduction is effected. For the index of inertia we may take the number of the positive squares; it is equal to the number of continuations of sign in a series of ascending principal minors of the matrix of the quadratic form, the series commencing with unity, *i. e.* with a minor of order 0, and each minor being so taken as to contain that which precedes it in the series (see Professor Sylvester "On Formulæ connected with Sturm's Theorem," Phil. Trans. vol. cxliii. p. 481). The distribution of these forms into Orders depends on the following principle:—

"Two forms belong to the same order when the primary and secondary divisors of their corresponding concomitants are identical."

Since, as has been just pointed out, there are, beside the fundamental concomitants, an infinite number of other concomitants, it is important to know whether, in order to obtain the distribution into orders, it is, or is not, necessary to consider those other concomitants. With regard to the primary divisors, it can be shown that it is unnecessary to consider any concomitants other than the fundamental ones; *i. e.* it can be shown that the equality of the primary divisors of the corresponding fundamental concomitants of two quadratic forms, implies the equality of the primary divisors of all their corresponding concomitants. And it is probable (but it seems difficult to prove) that the same thing is true for the secondary divisors also.

Confining our attention (in the next place) to the forms contained in any given order, we proceed to indicate the principle from which the subdivision of that order into genera is deducible.

If F_1 be any quadratic form containing r indeterminates, and F_2 be its concomitant of the second species, we have the identical equation

$$\left.\begin{aligned} &F_1(x_1, x_2, \ldots x_r)\times F_1(y_1, y_2, \ldots y_r)-\tfrac{1}{4}\left[\sum_{k=1}^{k=r} y^k \frac{dF_1}{dx_k}\right]^2 \\ &\quad = F_2\begin{pmatrix} x_1, x_2, \ldots xr \\ y_1, y_2, \ldots y_r \end{pmatrix} \end{aligned}\right\} \quad . \quad . \quad (B)$$

in which the symbol $F_2\begin{pmatrix} x_1, x_2, \ldots x_r \\ y_1, y_2, \ldots yr \end{pmatrix}$ indicates that the determinants $\begin{pmatrix} x_1, x_2, \ldots, xr \\ y_1, y_2, \ldots, y_r \end{pmatrix}$ are to be taken for the indeterminates of F_2, the order in which they are taken being the same as the order in which the determinants of any two horizontal rows of the matrix of F_1 are taken in forming the matrix of F_2. Let $\theta_i=\frac{1}{\nabla}f_1$ for every value of i from 1 to $n-1$; it will be found that, if we form the concomitant of the second species of θ, its primary divisor is the quotient $\frac{\nabla i+1}{\nabla i}\div\frac{\nabla i}{\nabla i-1}$, which, as has been shown elsewhere (see Phil. Trans. vol. cli. p. 317) is always an integral number. Let δ_i be any uneven

prime dividing $\frac{\nabla_{i+1}}{\nabla_i} \div \frac{\nabla_i}{\nabla_{s-1}}$; we infer from the identity (B) that the numbers prime to δ_i, which can be represented by θ_i, are either all quadratic residues of δ^i, or all non-quadratic residues of δ_i. In the former case we attribute to f_i the *particular character* $\left(\frac{\theta_i}{\delta_i}\right) = +1$; in the latter the particular character $\left(\frac{\theta^i}{\delta_i}\right) = -1$. If $\nabla_1 = 1$, *i.e.* if the form f_1 itself do not admit of any primary divisor beside unity (which is the only important case), the product $\left(\frac{\nabla_n}{\nabla_{n-1}} \div \frac{\nabla_{n-1}}{\nabla_{n-2}}\right) \times \left(\frac{\nabla_{n-1}}{\nabla_{n-2}} \div \frac{\nabla_{n-2}}{\nabla_{n-3}}\right) \times \ldots$ is equal to $\frac{\nabla_n}{\nabla_{n-1}}$; whence, inasmuch as every prime that divides ∇_n also divides $\frac{\nabla_n}{\nabla_{n-1}}$, it appears that a primitive quadratic form will always have one particular character, at least with respect to every uneven prime dividing its determinant, and will have more than one if the uneven prime divide more than one of the quotients $\frac{\nabla_{i+1}}{\nabla_i} \div \frac{\nabla_i}{\nabla_{i-1}}$.

The subdivision of an order into genera can now be effected by assigning to the same genus all those forms whose particular characters coincide. But it remains to consider whether the above enumeration of particular characters is complete. It is evident that we might apply the theorem (B) to other concomitants besides those included in the fundamental system; and it might appear as if in this manner we could obtain other particular characters besides those which we have given. But it can be shown that such other particular characters are implicitly contained in ours; *i.e.* it can be shown that two quadratic forms, which coincide in respect of the particular characters deducible from their fundamental concomitants, will also coincide in respect of the particular characters deducible from any other concomitant. Again, it will be found that if the determinant be uneven, there are no particular characters with respect to 4 or 8. For this case, therefore, our enumeration is complete. But when the determinant is even, besides the particular characters arising from its uneven prime divisors, there may also be particular characters with regard to 4 or 8. There is no difficulty in enumerating these particular characters; nevertheless we suppress the enumeration here, not only because it would require a detailed distinction of cases, but also because there appears to be some difficulty in showing that the characters with regard to 4 or 8, which may arise from the excluded concomitants, are virtually included in those which arise from the concomitants of the fundamental set.