The chief specimens described in the memoir are eight in number, and were found in the lower divisions of the Lancashire and Yorkshire coal-measures imbedded in calcareous nodules occurring in seams of coal.

No. 1, *Diploxylon cycadoideum*, was from the first-named district, and the same locality as the *Trigonocarpon*, described by Dr. J. D. Hooker, F.R.S., and the author, in a memoir on the structure of certain limestone nodules inclosed in seams of bituminous coal, with a description of some *Trigonocarpons* contained therein*, and the other seven (*Sigillaria vascularis*) were from the same seam of coal in the lower coal-measures in which the specimens described in a paper entitled "On some Fossil Plants showing structure from the Lower Coal-measures of Lancashire"†, were met with, but from a different locality in Yorkshire.

III. "On Symbolical Expansions." By W. H. L. Russell, Esq., A.B. Communicated by Prof. Stokes, Sec. R.S. Received May 13, 1865.

Among the papers on symbolical algebra by the lamented Professor Boole, there is one on the Theory of Development, published in the fourth volume of the 'Cambridge Mathematical Journal.' The expansion of $f\left(x+\frac{d}{dx}\right)$ is there given in a very elegant form. I am desirous to terminate my own investigations on the Calculus of Symbols by pointing out the connexion of the binomial theorems given in my first paper on this subject with the expansions due to Professor Boole, and propose with that view to expand $f\left(x+x\frac{d}{dx}\right)$ in terms of $\frac{d}{dx}$, which will be sufficient to indicate the general method. When the term of the expansion which does not contain $\frac{d}{dx}$ is known, the other terms are easily found by a method given by Professor Boole in the paper I have just mentioned. The main object of the present paper, therefore, will be to ascertain that part of the expansion of $f\left(x+x\frac{d}{dx}\right)$ which does not contain $\frac{d}{dx}$.

Putting, as usual, ρ for (x) and π for $x\frac{d}{dx}$, the expression becomes $f(\rho+\pi)$. Our first object must be to ascertain that part of the expansion of $(\rho+\pi)^n$ which is independent of (π), from whence we may easily deduce the corresponding portion of $f(\rho+\pi)$. Now by a former paper the part of $(\rho+\pi)^n$, independent of π, will be

$$\rho^n+\Sigma n\,.\,\rho^{n-1}+\Sigma(n-1)\Sigma n\rho^{\ -2}+\Sigma(n-2)\Sigma(n-1)\Sigma n\rho^{n-3}$$
$$+\ \&\text{c.}+\Sigma(n-r+1)\Sigma(n-r+2)\ \ldots\ldots\ \Sigma n\rho^{n-r}+\ \ldots\ldots$$

* Philosophical Transactions, 1855, p. 149.

† Quarterly Journal of the Geological Society of London for May 1862.

And we must first endeavour to find a suitable expression for

$$\Sigma(n-r+1)\Sigma(n-r+2)\Sigma(n-r+3)\ldots\ldots\Sigma n.$$

With this purpose let us assume

$$\begin{aligned}&\Sigma(n-r+1)\Sigma(n-r+2)\ldots\ldots\Sigma n=\\&A_r^{(1)}(n-2r+1)(n-2r+2)\ldots\ldots n\\&+A_r^{(2)}(n-2r+2)(n-2r+3)\ldots\ldots n\\&+A_r^{(3)}(n-2r+3)(n-2r+4)\ldots\ldots n+\&c.\end{aligned}$$

Whence

$$\begin{aligned}&\Sigma(n-r)\Sigma(n-r+1)\Sigma(n-r+2)\ldots\ldots\Sigma n=\\&\frac{A_r^{(1)}}{2r+2}(n-2r-1)(n-2r)(n-2r+1)\ldots\ldots n\\&+\frac{rA_r^{(1)}}{2r+1}(n-2r)(n-2r+1)\ldots\ldots n\\&+\frac{A_r^{(2)}}{2r+1}(n-2r)(n-2r+1)\ldots\ldots n\\&+\frac{(r-1)A_r^{(2)}}{2r}(n-2r+1)(n-2r+2)\ldots\ldots n\\&+\frac{A_r^{(3)}}{2r}(n-2r+1)(n-2r+2)\ldots\ldots n\\&+\frac{(r-2)A_r^{(3)}}{2r-1}(n-2r+2)(n-2r+3)\ldots\ldots n\\&+\ \&c.\\&=A_{r+1}^{(1)}(n-2r-1)(n-2r)\ldots\ldots n\\&+A_{r+1}^{(2)}(n-2r)(n-2r+1)\ldots\ldots n\\&+A_{r+1}^{(3)}(n-2r+1)(n-2r+2)\ldots\ldots n+\ \&c.\end{aligned}$$

Hence

$$A_{r+1}^{(1)}=\frac{A_r^{(1)}}{2r+2},\quad A_{r+1}^{(2)}=\frac{A_r^{(2)}}{2r+1}+\frac{rA_r^{(3)}}{2r+1}$$

$$A_{r+1}^{(3)}=\frac{A_r^{(3)}}{2r}+\frac{(r-1)A_r^{2}}{2r}\ldots\&c.$$

This will give us

$$A_r^{(1)}=\frac{1}{2r(2r-2)(2r-4)\ldots\ldots 2}$$

$$A_r^{(2)}=\frac{1}{(2r-1)(2r-3)\ldots\ldots 1}\Sigma\frac{r(2r-1)(2r-3)\ldots\ldots 1}{2r(2r-2)\ldots\ldots 2}$$

&c. = &c.

$$A_r^{(3)}=\frac{1}{(2r-2)(2r-4)\ldots\ldots 2}\Sigma\frac{(r-1)(2r-2)(2r-4)\ldots\ldots 2}{(2r-1)(2r-3)\ldots\ldots 1}$$

$$\Sigma\frac{r(2r-1)(2r-3)\ldots\ldots 1}{2r(2r-2)(2r-4)\ldots\ldots 2}.$$

Hence we have generally, using Π as a symbol for a continued product,

$$A_r^{(m)}=\frac{1}{\Pi(2r-m+1)}\Sigma(r-m+2)\frac{\Pi(2r-m+1)}{\Pi(2r-m+2)}$$

$$\Sigma(r-m+3)\frac{\Pi(2r-m+2)}{\Pi(2r-m+3)}\Sigma\ldots\ldots\Sigma r\frac{\Pi(2r-1)}{\Pi 2r};$$

whence the portion of $(\rho+\pi)^n$ which does not contain (π) may be written

$$\rho^n+A_1^{(1)}n(n-1)\rho^{n-1}$$
$$+\{A_2^{(1)}(n-3)(n-2)(n-1)n+A_2^{(1)}(n-2)(n-1)n\}\rho^{n-2}$$
$$+\{A_3^{(1)}(n-5)(n-4)(n-3)(n-2)(n-1)n+$$
$$\{A_3^{(2)}(n-4)(n-3)(n-2)(n-1)n+A_3^{(3)}(n-3)(n-2)(n-1)n\}\rho^{n-3}$$
$$+\ \&c.$$
$$=\rho^n+A_1^{(1)}\rho\frac{d^2}{d\rho^2}\rho^n+\left\{A_2^{(1)}\rho^2\frac{d^4}{d\rho^4}+A_2^{(2)}\rho\frac{d^3}{d\rho^3}\right\}\rho^n$$
$$+\left\{A_3^{(1)}\rho^3\frac{d^6}{d\rho^6}+A_3^{(2)}\rho^2\frac{d^5}{d\rho^5}+A_3^{(3)}\rho\frac{d^4}{d\rho^4}\right\}\rho^n+\ldots,$$

the general term being $A_r^{(m)}\rho^{r-m+1}\dfrac{d^{2r-m+1}}{d\rho^{2r-m+1}}\rho^n$;

whence the part of the expansion of $f(\rho+\pi)$, which does not contain π, is

$$f(\rho)+A_1^{(1)}\rho\frac{d^2}{d\rho^2}f(\rho)+\left\{A_2^{(1)}\rho^2\frac{d^4f\rho}{d\rho^4}+A_2^{(2)}\rho\frac{d^3f(\rho)}{d\rho^3}\right\}$$
$$+\ \&c.$$

If, then, we put

$$f\left(x+x\frac{d}{dx}\right)=f_0(x)+f_1(x)\frac{d}{dx}+f_2(x)\frac{d^2}{dx^2}+\ldots,$$

we have

$$f_0(x)=f(x)+A_1^{(1)}x\frac{d^2f(x)}{dx^2}+\left\{A_2^{(1)}x^2\frac{d^4f(x)}{dx^4}+A_2^{(2)}x\frac{d^3f(x)}{dx^3}\right\}$$
$$+\left\{A_3^{(1)}x^3\frac{d^6f(x)}{dx^6}+A_3^{(2)}x^2\frac{d^5f(x)}{dx^5}+A_3^{(3)}x\frac{d^4f(x)}{dx^4}\right\}$$
$$+\ \&c.,$$

the general term being

$$A_r^{(m)}x^{r-m+1}\frac{d^{2r-m+1}f(x)}{dx^{2r-m+1}},$$

where $A_r^{(m)}$ has the value given above; and $f_1(x)$, $f_2(x)$, $f_3(x)$, &c. are given by the following formula, which, as I have before said, can be immediately

deduced from one in the paper of Professor Boole on the Theory of Development:

$$f_{n+1}(x)=\frac{x}{n+1}f'_n(x)-\frac{n}{n+1}f_n(x).$$

The method of the present paper is of course of far more general application; but I have said enough in it to explain the principle on which such expansions must be conducted.

IV. "On the Summation of Series." By W. H. L. Russell, Esq., A.B. Communicated by Professor Stokes, Sec. R.S. Received May 13, 1865.

In a Memoir published in the Philosophical Transactions for the year 1855, I applied the Theory of Definite Integrals to the summation of many intricate series. I have thought my researches on this subject might well be terminated by the following paper, in which I have pointed out methods for the summation of series of a far more complicated nature.

I commence with some remarks intended to give clear conceptions of the general method of calculation.

In any series,

$$u_0+\alpha u_1+\alpha^2 u_2+\alpha^3 u_3+\&c.+\alpha^x u_x+\&c.$$

Where α is less than unity, it is evident that we can sum the series by a definite integral when $u_x=\int du\,U_1\,U^x$, U_1 and U being functions of u, and the integral being taken between certain assigned limits. For it is manifest that the quantity under the integral sign then becomes a geometrical progression.

Again, for a similar reason we can express by a definite integral the sum of the series

$$u_0v_0w_0\ldots+\alpha u_1v_1w_1\ldots+\alpha^2u_2v_2w_2\ldots+\&c.$$
$$+\alpha^x u_xv_xw_x\ldots+\&c.,$$

where

$$u_x=\int du\,U_1U^x,\quad v_x=\int dv V_1V^x,$$

$$w_x=\int dw W_1W^x,\ \&c.$$

Lastly, we can sum the series

$$u_0v_0w_0\ldots+\alpha\,u_1v_1w_1\ldots+\alpha^2u_2v_2w_2\ldots+\&c.$$
$$+\alpha^x u_xv_xw_x\ldots+\&c.$$

by a definite integral when