

deduced from one in the paper of Professor Boole on the Theory of Development :

$$f_{n+1}(x) = \frac{x}{n+1} f_n'(x) - \frac{n}{n+1} f_n(x).$$

The method of the present paper is of course of far more general application ; but I have said enough in it to explain the principle on which such expansions must be conducted.

IV. "On the Summation of Series." By W. H. L. RUSSELL, Esq., A.B. Communicated by Professor STOKES, Sec. R.S. Received May 13, 1865.

In a Memoir published in the Philosophical Transactions for the year 1855, I applied the Theory of Definite Integrals to the summation of many intricate series. I have thought my researches on this subject might well be terminated by the following paper, in which I have pointed out methods for the summation of series of a far more complicated nature.

I commence with some remarks intended to give clear conceptions of the general method of calculation.

In any series,

$$u_0 + \alpha u_1 + \alpha^2 u_2 + \alpha^3 u_3 + \&c. + \alpha^x u_x + \&c.$$

Where  $\alpha$  is less than unity, it is evident that we can sum the series by a definite integral when  $u_x = \int du U_1 U^x$ ,  $U_1$  and  $U$  being functions of  $u$ , and the integral being taken between certain assigned limits. For it is manifest that the quantity under the integral sign then becomes a geometrical progression.

Again, for a similar reason we can express by a definite integral the sum of the series

$$u_0 v_0 w_0 \dots + \alpha u_1 v_1 w_1 \dots + \alpha^2 u_2 v_2 w_2 \dots + \&c. \\ + \alpha^x u_x v_x w_x \dots + \&c.,$$

where

$$u_x = \int du U_1 U^x, \quad v_x = \int dv V_1 V^x, \\ w_x = \int dw W_1 W^x, \&c.$$

Lastly, we can sum the series

$$u_0 v_0 w_0 \dots + \alpha u_1 v_1 w_1 \dots + \alpha^2 u_2 v_2 w_2 \dots + \&c. \\ + \alpha^x u_x v_x w_x \dots + \&c.$$

by a definite integral when

$$u_x = \int du U_1 U^x + \int du' U_1 U'^x + \int du'' U_1 U''^x + \dots$$

$$v_x = \int dv V_1 V^x + \int dv' V_1 V'^x + \int dv'' V_1 V''^x + \dots$$

$$w_x = \int dw W_1 W^x + \int dw' W_1 W'^x + \int dw'' W_1 W''^x + \dots$$

&c. = &c.,

the number of each set of quantities  $u_1, u', \&c., v_1, v', \&c., w_1, w', \&c.$  being of course finite.

I shall now consider the series

$$\begin{aligned} &\phi(0)^{\psi(0)} + \alpha\phi(1)^{\psi(1)} + \alpha^2\phi(2)^{\psi(2)} + \&c. \\ &+ \alpha^x\phi(x)^{\psi(x)} + \&c., \end{aligned}$$

where  $\phi(x)$  and  $\psi(x)$  are rational functions of  $(x)$ .

Let

$$\phi(x) = \frac{(\alpha + \beta x)(\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 x) \dots}{(a + bx)(a_1 + b_1 x)(a_2 + b_2 x) \dots},$$

$$\psi(x) = \frac{1}{m + nx} + \frac{1}{m_1 + n_1 x} + \frac{1}{m_2 + n_2 x} + \&c.$$

Hence, by what has been said, the problem is reduced to finding a definite

integral of the form  $\int du U_1 U_x$  equivalent to  $\left\{ \frac{\alpha + \beta x}{a + bx} \right\}^{\frac{1}{m + nx}}$ .

Now

$$\begin{aligned} \left\{ \frac{\alpha + \beta x}{a + bx} \right\}^{\frac{1}{m + nx}} &= \frac{1}{\Gamma \left\{ \frac{1}{m + nx} \right\}} \int_0^1 du \cdot u^{\frac{\alpha + \beta x}{\alpha + \beta x} - 1} \left( \log_e \frac{1}{u} \right)^{\frac{1}{m + nx} - 1} \\ &= \frac{1}{\pi} \sin \frac{\pi}{m + nx} \Gamma \left\{ \frac{m + nx - 1}{m + nx} \right\} \int_0^1 du \cdot u^{\frac{\alpha + \beta x}{\alpha + \beta x} - 1} \left( \log_e \frac{1}{u} \right)^{\frac{1}{m + nx} - 1} \\ \sin \frac{\pi}{m + nx} &= \frac{1}{\sqrt{2}} \cos \left\{ \frac{\pi}{4} - \frac{\pi}{m + nx} \right\} - \frac{1}{\sqrt{2}} \sin \left\{ \frac{\pi}{4} - \frac{\pi}{m + nx} \right\} \\ &= \frac{\sqrt{2}}{\pi} \int_0^1 ds \log_e^{-\frac{1}{2}} \frac{1}{s} \cdot s^{\frac{1}{m + nx} - 1} \int_0^\infty dz \cos(m + nx)z^2 \cos 2\sqrt{\pi} \cdot z \\ &- \frac{\sqrt{2}}{\pi} \int_0^1 ds \log_e^{-\frac{1}{2}} \frac{1}{s} \cdot s^{\frac{1}{m + nx} - 1} \int_0^\infty dz \sin(m + nx)z^2 \cos 2\sqrt{\pi} \cdot z \end{aligned}$$

$$\Gamma\left\{\frac{m+nx-1}{m+nx}\right\} = \int_0^\infty \epsilon^{-v} dv \epsilon^{-\frac{1}{m+nx}}$$

$$\int_0^1 du \cdot u^{\frac{a+bx}{\alpha+\beta x}-1} \left(\log_\epsilon \frac{1}{u}\right)^{\frac{1}{m+nx}-1}$$

$$= \int_{\frac{1}{\epsilon}}^1 du \cdot \epsilon^{\log_\epsilon u \left(\frac{a+bx}{\alpha+\beta x}-1\right)} \epsilon^{\log_\epsilon \log_\epsilon \frac{1}{u} \left(\frac{1}{m+nx}-1\right)}$$

(where  $\log_\epsilon \log_\epsilon \frac{1}{u}$  is negative),

$$+ \int_0^{\frac{1}{\epsilon}} du \epsilon^{\log_\epsilon u \left(\frac{a+bx}{\alpha+\beta x}-1\right)} \epsilon^{\log_\epsilon \log_\epsilon \frac{1}{u} \left(\frac{1}{m+nx}-1\right)}$$

(where  $\log_\epsilon \log_\epsilon \frac{1}{u}$  is positive).

By means of these transformations the series is reduced to forms considered in my previous investigations; for the general term of the transformed series included under the signs of definite integration is of the form

$$P \alpha^x \epsilon^{\frac{\phi}{m+nx}} \{\cos h(m+nx) - \sin k(m+nx)\},$$

a form which I have discussed in the memoir in the Philosophical Transactions mentioned at the beginning of this paper.

Next let us investigate the series

$$\sqrt[n]{\psi(0) + \sqrt[m]{\chi(0)}} + \alpha \sqrt[n]{\psi(1) + \sqrt[m]{\chi(1)}} + \dots + \alpha^x \sqrt[n]{\psi(x) + \sqrt[m]{\chi(x)}} + \dots,$$

where  $\psi(x)$  and  $\chi(x)$  are identical functions of  $(x)$ .

We transform as follows:—

$$\sqrt[n]{\psi(x) + \sqrt[m]{\chi(x)}} = \frac{1}{\Gamma\left(\frac{1}{n}\right)} \int_0^1 du u^{\psi(x) + \sqrt[m]{\chi(x)} - 1} \left(\log_\epsilon \frac{1}{u}\right)^{n-1}$$

$$\epsilon^{\psi(x) + \sqrt[m]{\chi(x)}} = \frac{\sqrt{\psi(x) + \sqrt[m]{\chi(x)}}}{2\sqrt{\pi} \log_\epsilon \frac{1}{u}} \int_{-\infty}^\infty d\rho \epsilon^{-\frac{\psi(x) + \sqrt[m]{\chi(x)}}{4 \log_\epsilon \frac{1}{u}} \rho^2} \cos \rho,$$

remembering that  $\log_\epsilon u$  is negative,

$$\sqrt[n]{\psi(x) + \sqrt[m]{\chi(x)}} = \frac{\psi(x) + \sqrt[m]{\chi(x)}}{\sqrt{\pi}} \int_{-\infty}^\infty \epsilon^{-v^2(\psi(x) + \sqrt[m]{\chi(x)})} dv,$$

$$\epsilon^{-\left(v^2 + \frac{\rho^2}{4 \log_\epsilon \frac{1}{u}}\right) \sqrt[m]{\chi(x)}} = \frac{1}{\pi} \int_0^\pi \frac{(1 - \sqrt[m]{\chi(x)^2}) F(\theta) d\theta}{1 - 2 \sqrt[m]{\chi(x)} \cos \theta + \sqrt[m]{\chi(x)^2}},$$

where

$$F(\theta) = \epsilon^{-\left(v^2 + \frac{\rho^2}{4 \log_e \frac{1}{u}}\right) \cos \theta} \cos \left\{ \left(v^2 + \frac{\rho^2}{4 \log_e \frac{1}{u}}\right) \sin \theta \right\},$$

and  $\sqrt[m]{\chi(x)}$  is supposed less than unity.

Hence also

$$\begin{aligned} & \epsilon^{-\left(v^2 + \frac{\rho^2}{4 \log_e \frac{1}{u}}\right) \sqrt[m]{\chi(x)}} \\ &= \frac{1}{\pi} \int_0^\pi \frac{(1 - \sqrt[m]{\chi(x)^2}) F_1(\chi(x), \epsilon^{i\theta}) F_1(\chi(x), \epsilon^{-i\theta}) F(\theta) d\theta}{1 - 2\chi(x) \cos m\theta + \chi(x)^2}, \end{aligned}$$

where we render the denominator rational by multiplication, and suppose

$$F_1(\chi(x), \epsilon^{i\theta}) = \chi(x)^{\frac{m-1}{m}} + \epsilon^{i\theta} \chi(x)^{\frac{m-2}{m}} + \epsilon^{2i\theta} \chi(x)^{\frac{m-3}{m}} + \dots,$$

$$F_1(\chi(x), \epsilon^{-i\theta}) = \chi(x)^{\frac{m-1}{m}} + \epsilon^{-i\theta} \chi(x)^{\frac{m-2}{m}} + \epsilon^{-2i\theta} \chi(x)^{\frac{m-3}{m}} + \dots,$$

$$\frac{1}{1 - 2\chi(x) \cos m\theta + \chi(x)^2} = \frac{\chi_1(x)^2}{(\chi_1(x) - \cos m\theta)^2 + \sin^2 m\theta},$$

where

$$\chi_1(x) = \frac{1}{\chi(x)}.$$

$\chi(x)$  is less than unity, hence  $\chi_1(x)$  is greater than unity, and therefore  $\chi_1(x) - \cos m\theta$  is always positive; hence

$$\frac{1}{1 - 2\chi(x) \cos m\theta + \chi(x)^2} = \frac{\chi(x)^2}{\sin m\theta} \int_0^\infty dz \epsilon^{-(\chi_1(x) - \cos m\theta)z} \sin(z \sin m\theta).$$

The general term of the series included under the signs of definite integration is now of the form

$$P\psi(x) \epsilon^{\alpha\psi(x)} \epsilon^{-\frac{\beta}{\chi(x)}} \chi(x)^{\frac{r}{m}} \alpha^x,$$

belonging to a class which I have considered in my former memoir.

Let us now consider the series

$$\begin{aligned} & \sqrt[n]{\phi(0)} \sqrt[m]{\psi(0)} + \alpha \cdot \sqrt[n]{\phi(1)} \sqrt[m]{\psi(1)} + \alpha^2 \cdot \sqrt[n]{\phi(2)} \sqrt[m]{\psi(2)} + \dots \\ & + \alpha^x \cdot \sqrt[n]{\phi(x)} \sqrt[m]{\psi(x)} + \&c., \end{aligned}$$

$\phi(x)$  and  $\psi(x)$  being rational functions of  $(x)$ .

$$\sqrt[n]{\phi(x)} \sqrt[m]{\psi(x)} = \frac{1}{\Gamma \sqrt[m]{\psi(x)}} \int_0^1 u^{\frac{1}{n\sqrt[m]{\phi(x)}} - 1} \left( \log_e \frac{1}{u} \right)^{\sqrt[m]{\psi(x)} - 1} du,$$

where  $\psi(x)$  must be supposed less than unity, in order that the following transformation may hold :—

$$\begin{aligned}\frac{1}{\Gamma\sqrt[m]{\psi(x)}} &= \frac{\sin \pi\sqrt[m]{\psi(x)}}{\pi} \Gamma(1-\sqrt[m]{\psi(x)}) \\ \sin \pi\sqrt[m]{\psi(x)} &= \frac{1}{2\pi} \int_0^\pi \frac{d\theta(1-\sqrt[m]{\psi(x)^2})d\theta(\sin \pi\epsilon^{i\theta} + \sin \pi\epsilon^{-i\theta})}{1-2\cos \theta\sqrt[m]{\psi(x)}+\sqrt[m]{\psi(x)^2}} \\ &= \frac{1}{2\pi} \int_0^\pi \frac{d\theta(1-\sqrt[m]{\psi(x)^2})F(\psi(x),\epsilon^{i\theta})F(\psi(x),\epsilon^{-i\theta})(\sin \pi\epsilon^{i\theta} + \sin \pi\epsilon^{-i\theta})}{1-2\cos m\theta, \psi(x)+\psi(x)^2},\end{aligned}$$

where

$$\begin{aligned}F(\psi(x), \epsilon^{i\theta}) &= \psi(x)^{\frac{m-1}{m}} + \epsilon^{i\theta}\psi(x)^{\frac{m-2}{m}} + \dots, \\ F(\psi(x), \epsilon^{-i\theta}) &= \psi(x)^{\frac{m-1}{m}} + \epsilon^{-i\theta}\psi(x)^{\frac{m-2}{m}} + \dots;\end{aligned}$$

also

$$\Gamma(1-\sqrt[m]{\psi(x)}) = \int_0^\infty e^{-v} v^{-\frac{1}{m}\sqrt[m]{\psi(x)}} dv.$$

The remainder of the process will be evident from the two former examples.

## V. "On a Theorem concerning Discriminants." By J. J. SYLVESTER, F.R.S. Received May 27, 1865.

Let  $F(a, b, c, d) = a^2 d^2 + 4a^3 c + 4d^3 b - 3a^2 b^2 - 6abcd$ , and let  $a, b, c, d$  be four quantities all greater than zero, which make this function vanish.

(1) The cubic equation in  $x$ ,  $F(a, x, c, d)$  will have two positive roots ( $b, b_1$ ); so  $F(a, b_1, x, d)$  will have two such roots ( $c, c_1$ ),  $F(a, x, c_1, d)$  two such ( $b, b_2$ ),  $F(a, b_2, x, d)$  two such ( $c, c_2$ ), and so on *ad infinitum*; we may thus generate the infinite series  $b_1 c_1 b_2 c_2 \dots$ .

Similarly, beginning with the equation  $F(a, b, x, d)$ , and proceeding as above, we shall obtain a similar series,  $c', b', c'', b'' \dots$ ; and combining the two together, and with the initial quantities  $b, c$ , we obtain a series proceeding to infinity in both directions  $\dots b'' c'' b' c' b c b_1 c_1 b_2 c_2 \dots$ .

(2) The four quantities

$$\frac{\partial F}{\partial a}, \frac{\partial F}{\partial b}, \frac{\partial F}{\partial c}, \frac{\partial F}{\partial d},$$

where  $F$  represents  $F(a, b, c, d)$ , will present one or the other of the three following successions of sign,

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ 0 & 0 & 0 & 0 \end{array}$$

(3) When the last is the case, *i.e.* when the differential derivatives all