the spectrum of Comet I., 1864*. The positions of the three bands seen by him appear to agree with those which the bright bands of this comet occupy.

This comet differs remarkably from the two small comets which I examined in the much smaller relative proportion of the light which forms a continuous spectrum. In Brorsen's comet, as it now appears, the bright middle part of the nebulosity seems to have a constitution analogous at least to that of the nucleus, and to be self-luminous; in the other comets the coma, which surrounded a distinctly marked nucleus, gave a continuous spectrum. The three comets resemble each other in the circumstance that the light of the bright central part was emitted by the cometary matter, while the surrounding nebulosity reflected solar light.

The telescopic observations of the heads of Donati's comet and of other large comets have shown that the luminous material is not at once driven off into the outer portions of the coma and the tail, but usually forms in front of the nucleus a dense luminous cloud, which for a time seems to be identical in the character of its light with that of the nucleus. It is, I believe, the outer portions only of the coma, which are frequently separated by dark spaces from the nucleus, and the tail, which the polariscope has shown to shine by reflected light.

The positions of the bands in this comet would seem to indicate a chemical constitution different from that of the nebulæ, which give a spectrum of bright lines. It will be seen in the diagram that, though the brightest of the bands in the spectrum of the comet differs but little in position from the brightest line of the nebulæ, the other bands are found in parts of the spectrum widely removed from those in which the other lines of the nebulæ occur. The suggestion presents itself whether the broad, nebulous bands may not indicate conditions of temperature and molecular state different from those which occur in the gaseous nebulæ. Plücker has shown that nitrogen and some other substances give totally different spectra, under different conditions of temperature and tension. The spectrum of this comet, however, does not resemble the other spectrum of nitrogen, which Plücker distinguishes as the spectrum of the first order†.

## IX. "Memoir on 'Undevelopable Uniquadric Homographics.'" By MARTIN GARDINER, C.E. Communicated by the Rev. R. TOWNSEND, F.R.S. Received April 13, 1868.

(Abstract.)

In this paper the author's method of investigation is purely geometrical throughout, its arrangement of details is systematic and natural, and it is divided into eight chapters, the first seven of which are preparatory to the consideration of the interesting problem discussed at length in the eighth,

* Philosophical Transactions, 1864, p. 158.

† Ibid. 1865, p. 9.

the direct, general, and complete solution of which is claimed to be given in it for the first time.

Chapter I., after some preliminary general properties, resulting immediately from the known properties of homographic systems of points on plane conics, treats more particularly of the simplest case of such systems on undevelopable quadrics, viz. of the case of systems in perspective, the several pairs of whose corresponding constituents possess manifestly the property of interchangeability; shows that systems having three double points not in the same tangent plane to the quadric on which they lie are necessarily of that class, except only when they have a fourth double point not in the plane of the other three, in which case they altogether coincide; and gives simple instances in which the problem, whose solution is the principal object of the memoir, is manifestly either "wholly or partially porismatic," as he terms it.

Chapter II. treats of systems whose several pairs of corresponding points are interchangeable but which are not in perspective; shows that their several chords of connexion intersect the same two reciprocal lines with respect to the quadric on which they lie; shows the relation between either system and the perspective of the other to any point on either of those lines, also the relation between either of two systems in perspective and the perspective of either to any point conjugate to their centre of perspective with respect to the surface; and shows how to construct the two reciprocal lines from two pairs of corresponding points of the systems.

Chapter III. treats of systems whose several pairs of corresponding points connect through a single common line, which have therefore an infinite number of double planes passing through that line; shows that their several chords of connexion, besides intersecting the line, all touch a second quadric having double contact with the original, both at its two points of intersection with the line, and also at its two points of intersection with the reciprocal line; proves a property of the cone enveloping either surface from any vertex taken arbitrarily on either line, shows the relation between either system and the perspective of the other to any point on either line; and shows how to construct the two reciprocal lines from two pairs of corresponding points of the systems.

Chapter IV. treats of systems having two of their four double planes non-tangential to the quadric on which they lie; shows that the chords connecting their several pairs of corresponding points touch two cones enveloping the quadric along two planes colinear with and harmonically conjugate to each other with respect to the two non-tangential double planes; proves that those touching along a plane section of either cone generate a skew quadric; shows that the two homographic systems determined by the two correspondents in the two systems of a variable point on the quadric are of the class considered in the preceding chapter, having an infinite number of double planes passing through the line of intersection of the two non-tangential double planes of the original systems; and shows the

relation between either system and the perspective of the other to any point on that line of intersection.

Chapter V. treats of systems having their four double planes all tangential to the quadric on which they lie, shows that the chords connecting their several pairs of corresponding points touch two other quadrics having quadruple contact with the original at the four double points of the systems, and gives various constructions for the determination of the four double points when the law connecting the several pairs of corresponding points of the systems is given or known.

Chapter VI. gives various criteria for determining in certain cases to which of the preceding classes two homographic systems belong, where, as in the problem whose solution forms the principal object of the memoir, the law connecting the same pairs of corresponding points of the systems is given or known.

Chapter VII. contains numerous theorems, several of much interest and originality, respecting open and closed polygons inscribed in undevelopable quadrics, whose sides pass in the same order of sequence through a common system of points in space, all deduced from the principles established in the preceding chapters, and several having direct reference to the interesting problem to be considered in the next and closing chapter.

Chapter VIII.—Given an undevelopable quadric and $n$ fixed points in to find the space; first extremities of inscribable closed $n$'gons, or the locus of the first extremities when the inscription of the closed $n$'gons is porismatic.

*When the number* n *of given points is* odd.

Assume any three points $a_1$, $b_1$, $c_1$ in the surface, no two of which are on one generator, as first extremities, and proceed to inscribe $2n$'gons.

(1) If the three points be found to be first extremities of closed $n$'gons, then will the trace of their plane be the locus of first extremities of closed $n$'gons, the problem in such case being partially porismatic.

(2) If the points are first extremities of closed $2n$'gons, or if two of them be first extremities of closed $2n$'gons and the third one a first extremity of a closed $n$'gon, or if one of the points be the first extremity of a closed $2n$'gon, and the other two points first extremities of closed $n$'gons, then the line or lines forming the closing chords of the open $n$'gons composing the $2n$'gon or $2n$'gons (as may be) and the tangent plane or planes at the first extremity or extremities of the closed $n$'gon or $n$'gons (as may be) meet in one point $\rho$, the trace of whose polar plane R is the locus of first extremities of inscribable closed $n$'gons, the problem in such case being partially porismatic.

(3) If two of the points be first extremities of closed $n$'gons and the third point the first extremity of an open $2n$'gon, then the problem is non-porismatic, and the two closed $n$'gons are the only inscribable closed $n$'gons. Moreover the reciprocal of the line joining the first extremities of the two closed $n$'gons will pierce the quadric in points which are the first extremi-

ties of inscribable closed $2n$'gons (real or imaginary according as the quadric S is ruled or convex).

(4) If one of the three chosen points be the first extremity of an open $2n$'gon (no matter as to the other two points) the problem is non-porismatic, and we can find the first extremities of the closed $n$'gons by either of the four following methods:—

*First method.*—Continue the $2n$'gon until a $4n$'gon be formed, and draw the plane P which contains the extremities of this $4n$'gon and the point of junction of the two open $2n$'gons composing it. Assume another point in the surface, not in the trace of the plane P, and, making it a first extremity, inscribe another $4n$'gon; and through the extremities of this $4n$'gon and the point of junction of the two open $2n$'gons composing it draw a plane Q. Then with the line $xx$ of intersection of the planes P and Q pierce the quadric in the only points (real or imaginary as may be) which are first extremities of closed $n$'gons; and the line $ii$ reciprocal to $xx$ will pierce the quadric in points (real or imaginary as may be) which are first extremities of closed $2n$'gons.

*Second method.*—By the additional inscription of another open $n$'gon convert the open $2n$'gon into an open $3n$'gon, and put A, B, C to represent the three open $n$'gons composing the open $3n$'gon. Find the point of puncture of the line through the first extremity of A and the final extremity of B with the tangent-plane at the junction of A and B; find the point of puncture of the line through the first extremity of B and the final extremity of C with the tangent-plane at the junction of B and C. Then will the line $xx$ through the two points of puncture pierce the quadric in two points (real or imaginary as may be) which are the first extremities of the only inscribable closed $n$'gons; and the line $ii$, which is reciprocal to $xx$, will pierce in first extremities of closed $2n$'gons.

N.B. When S is a hyperboloid of one sheet, and that the first extremities of the closed $n$'gons are *real*, then the first extremities of the closed $2n$'gons are also *real*; and it is evident there are two pairs of generators which are corresponding interchangeable lines in the homographic figures in which the extremities of the inscribable $n$'gons are pairs of corresponding points. It is moreover evident that when all the corresponding points of such figures are not interchangeable, these are the only pairs of interchangeable generators; and we must not assume the first extremities of the $4n$'gons or $3n$'gons in these lines.

*Third method.*—Let $o_1, o_2, o_3, \ldots o_n$ be the $n$ fixed points through which the sides must pass in order.

Assume the constant homological ratio $-1$ for homological systems, and making $o_1$ vertex and its polar plane axis, find the point $\alpha_1$ homological to the centre $\alpha_0$ of the quadric; assume $o_2$ as vertex and its polar plane as axis, and find the point $\alpha_2$ homological to $\alpha_1$; and proceed thus directly in order through the $n$ points until arrived at the point $\alpha_n$. Assume $o_n$ as vertex and its polar plane as axis, and find the point $\alpha_{-1}$ homological to the

centre $\alpha_0$ of the quadric; assume $o_{n-1}$ as vertex and its polar plane as axis, and find the point $\alpha_{-2}$ homological to $\alpha_{-1}$; and proceed thus in reverse order through the rest of the $n$ given points until arrived at the point $\alpha_{-n}$ (the points $\alpha_n$, $\alpha_{-n}$ will be distinct). Draw the diametral plane $A_0$ which bisects the chords parallel to the line $\alpha_n\alpha_{-n}$ (it will also bisect $\alpha_n\alpha_{-n}$); assume any two points in the trace of this plane as first extremities, and inscribe two $n$'gons in the quadric; through the point $\alpha_n$ and the final extremities of these $n$'gons draw the plane $A_n$; find the line of intersection $ii$ of the planes $A_0$, $A_n$, and its reciprocal $xx$. Then will $xx$ always pierce in the two points (real or imaginary as may be) which are the first extremities of the only inscribable closed $n$'gons; and the line $ii$ will pierce in first extremities of closed $2n$'gons.

*Fourth method.*—Find the points $\alpha_{-n}$ and $\alpha_n$ as in last method; assume any point $a_0$ in the surface as first extremity, and inscribe a $n$'gon whose last extremity we may represent by $a_n$; draw the plane $D_1$ which contains the line $\alpha_{-n}\alpha_0$ and the point $a_0$; draw the plane $D_2$ which contains the $\alpha_n\alpha_0$ and the point $a_n$; in the lines $\alpha_0\alpha_{-n}$, $\alpha_0\alpha_n$ find the points $m_1$, $m_2$ such that

$$\frac{\alpha_{-n}m_1}{\alpha_0 m_1} = \frac{\alpha_n m_2}{\alpha_0 m_2} = +\sqrt{\frac{\alpha_{-n}D_1 \cdot \alpha_n D_2}{\alpha_0 D_1 \cdot \alpha_0 D_2}};$$

and in the same lines find the points $h_1$, $h_2$ such that

$$\frac{\alpha_{-n}h_1}{\alpha_0 h_1} = \frac{\alpha_n h_2}{\alpha_0 h_2} = -\sqrt{\frac{\alpha_{-n}D_1 \cdot \alpha_n D_2}{\alpha_0 D_1 \cdot \alpha_0 D_2}}.$$

Put $\Sigma_{,}$, $\Sigma_{,,}$ to represent the homographic figures in which the first and final extremities of inscribable $n$'gons are corresponding points. Regard $m_2$ and $h_2$ as points in $\Sigma_{,}$, and find their correspondents $m_3$, $h_3$ in $\Sigma_{,,}$; draw the planes $m_1m_2m_3$, $h_1h_2h_3$, and find their line of intersection $xx$ and its reciprocal $ii$. Then will the line $xx$ pierce the quadric in the only points (real or imaginary as may be) which are first extremities of inscribable closed $n$'gons, and the line $ii$ will pierce in first extremities of closed $2n$'gons.

Moreover the planes $m_1m_2m_3$, $h_1h_2h_3$ are the only two double planes of the figures $\Sigma_{,}$, $\Sigma_{,,}$ which are non-tangential to the quadric.

N.B. When the points $\alpha_{-n}$, $\alpha_n$ are coincident, the inscription of the closed $n$'gons is partially porismatic, and one of the two points which divide $\alpha_{-n}\alpha_n$ and the diameter coincident with it harmonically is the point of concurrence of the closing chords of all inscribable open $n$'gons, and the polar plane of which passes through the other, &c.

When the centre of the quadric is a double point, then according as all the closing chords are parallels or pass through the centre, so will the locus of the first extremities of the closed $n$'gons be the trace of a diametral plane or of a plane at infinity.

*When the number* n *of given points is* even.

Assume three points $a_1$, $b_1$, $c_1$ on the surface, no two of which are on

the same generator; and, making them first extremities, proceed to inscribe $2n$'gons.

(1) If these assumed points be found to be first extremities of closed $n$'gons, the problem is fully porismatic, and every point in the surface will be the first extremity of an inscribable closed $n$'gon.

(2) If two of the points be found to be first extremities of closed $n$'gons and the other not, then the line $xx$ through these two points, and the line $ii$ reciprocal to $xx$, pierce the quadric in four points (the punctures made by $ii$ being real or imaginary as may be) which constitute the first extremities of all the inscribable closed $n$'gons.

(3) If one of the points be the first extremity of a closed $n$'gon, and another of them be first extremity of a closed $2n$'gon. Draw the closing chord of the two open $n$'gons composing the closed $2n$'gon to pierce the tangent-plane at the first extremity of the closed $n$'gon in the point $\rho$; in the closing chord find the point $\mu$ which is conjugate to the point of puncture $\rho$; through $\mu$ and the first extremity of the closed $n$'gon draw the line $xx$, and find the line $ii$ reciprocal to $xx$. Then will the lines $xx$ and $ii$ pierce the quadric in the four points which constitute the first extremities of all the inscribable closed $n$'gons.

(4) If two of the assumed points be found to be first extremities of closed $2n$'gons, we may find the first extremities of the inscribable closed $n$'gons by either of the four following methods:—

*First method.*—Draw the two closing chords of the open $n$'gons composing the $2n$'gons; and, if these chords intersect in a point $\rho$, draw the line $xx$ which is polar of $\rho$ in respect to the trace of the plane of the two chords; find the line $ii$ reciprocal to $xx$. Then will $xx$ and $ii$ pierce the quadric in the first extremities of the closed $n$'gons. But if the chords do not intersect, draw tangent-planes at their extremities; find the two pairs of points (one pair in each chord) which divide these closing chords and the segments intercepted by the tangent-planes harmonically; draw the line $xx$ through the two of these points which divide the chords internally; draw the line $ii$ through the two points which divide the chords externally. Then will $xx$ and $ii$ be reciprocal lines piercing the quadric in four points which are first extremities of the inscribable closed $n$'gons.

*Second method.*—Find $a_1a_2$, $b_1b_2$, $c_1c_2$ the closing chords of three open $n$'gons. If any two of these intersect, proceed as in the last method; but if not, proceed as follows:—In the chord $a_1a_2$ find the point $m$ which corresponds to infinity (on the same line) in one of the homographic figures in which the extremities of all inscribable $n$'gons are corresponding interchangeable points; in the same chord $a_1a_2$ find the two points $x$ and $i$ such that $mx = mi = \sqrt{ma_1 \,.\, ma_2}$; through $x$ draw the line $xx$ which cuts the two non-planar chords $b_1b_2$, $c_1c_2$; through $i$ draw the line $ii$ which cuts the same two non-planar chords $b_1b_2$, $c_1c_2$. Then will $xx$ and $ii$ be reciprocal lines piercing the quadric in the four points which are the first extremities of closed $n$'gons.

*Third method.*—Draw $a_1a_2$, $b_1b_2$ the closing chords of the open $n$'gons composing the closed $2n$'gons. If these two lines intersect in a point $\rho$, draw the line $xx$ through the points in $a_1a_2$, $b_1b_2$ which are harmonic conjugates to $\rho$ in respect to the segments $a_1a_2$, $b_1b_2$; find the line $ii$ reciprocal to $xx$. Then will $xx$ and $ii$ pierce the quadric in the four points which are first extremities of closed $n$'gons. But if $a_1a_2$, $b_1b_2$ do not intersect, then find the final extremity $c_2$ of the $n$'gon having the point $c_1$ as first extremity; and if the line $c_1c_2$ cuts either $a_1a_2$ or $b_1b_2$, the lines $xx$ and $ii$ can be found in the manner just indicated. If $c_1c_2$ do not cut either of the lines $a_1a_2$, $b_1b_2$, then find the points $\alpha_1$, $\alpha_2$ in which $a_1a_2$ pierces the planes $b_1b_2c_1$, $b_1b_2c_2$; find the points $\beta_1$, $\beta_2$ in which $b_1b_2$ pierces the planes $a_1a_2c_1$, $a_1a_2c_2$; find the points $h_1$, $h_2$ which divide the segments $a_1a_2$, $\alpha_1\alpha_2$ harmonically; find the points $k_1$, $k_2$ which divide the segments $b_1b_2$, $\beta_1\beta_2$ harmonically; through the two points $h_1$, $k_1$, cutting $a_1a_2$, $b_1b_2$ internally, draw the line $xx$; through the two points $h_2$, $k_2$, cutting $a_1a_2$, $b_1b_2$ externally, draw the line $ii$. Then will $xx$ and $ii$ be reciprocal lines piercing the quadric in the four points which are first extremities of closed $n$'gons.

*Fourth method.*—Put $\Sigma_{\prime}$ and $\Sigma_{\prime\prime}$ to represent the homographic figures in which the first and final extremities of all inscribable $n$'gons are corresponding points; find the point $\alpha_1$ which corresponds in either of the figures $\Sigma$, $\Sigma_{\prime}$ to the centre $\alpha_0$ of the quadric regarded as belonging to the other figure; find the diameter $d_1d_2$ which contains the points $\alpha_0$, $\alpha_1$; find the points $p$, $q$ which divide the segments $d_1d_2$ and $\alpha_0\alpha_1$ harmonically; draw the plane P which is polar to the point $p$ which lies outside the quadric; find the point $a_2$ which is final extremity of a $n$'gon whose first extremity is in the trace of the plane P; draw $xx$ the diameter of the trace of P which bisects $a_1a_2$ (the point $a_2$ will be in the trace of P); find the line $ii$ reciprocal to $xx$. Then will $xx$ and $ii$ pierce in first extremities of the four inscribable closed $n$'gons. But if the centre $\alpha_0$ of the quadric be a double point of the figures $\Sigma_{\prime}$, $\Sigma_{\prime\prime}$, proceed as follows:—Inscribe any $n$'gon in the quadric, and draw the diameter $xx$ which bisects its closing chord. Then will the diameter $xx$ and its reciprocal at infinity pierce the quadric in the four points which are first extremities of closed $n$'gons.

(5) When we can inscribe an open $2n$'gon the problem is always nonporismatic, and we can find the lines $xx$, $ii$ which pierce the quadric in first extremities of the four closed $n$'gons by either of the four following methods:—

*First method.*—Put $\Sigma_{\prime}$ and $\Sigma_{\prime\prime}$ to represent the homographic figures in which the first and final extremities of all inscribable $n$'gons are corresponding points. In the figures $\Sigma_{\prime}$ and $\Sigma_{\prime\prime}$ find the points $o_1$ and $o_2$ which are the correspondents of the centre $o$ of the quadric regarded as belonging to the figures $\Sigma_{\prime}$ and $\Sigma_{\prime\prime}$; draw the diameter $ror$ which bisects $o_1o_2$; find the line $r_1o_1r_1$ in $\Sigma_{\prime}$ which corresponds to $ror$ in $\Sigma_{\prime\prime}$; bind the line $r'r'$ reciprocal to $r_1r_1$; through the centre $o$ draw the diametral plane K which bisects all chords parallel to $ror$; find the points $\rho_1$, $\rho'$ in which the reciprocal lines $r_1r_1$,

$r'r'$ pierce the plane K; draw any plane P parallel to K, and through the points $\mu_1$, $\mu'$ in which it cuts $r_1r_1$ and $r'r'$ draw lines parallel to the diameter $rr$ to pierce the plane K in points $\phi_1$, $\phi'$; through the centre $o$ draw (see 'Section of Ratio' of Apollonius) the two lines $ox_1x'$, $oi_1i'$ so that

$$\rho_1 x_1 : \rho' x' :: \rho_1 i_1 : \rho' i' :: \rho_1\phi_1, \rho'\phi';$$

through $x_1$ and $x'$ draw lines parallel to $rr$ to cut $r_1r_1$ and $r'r'$ in points $x$ and $x$; through $i_1$ and $i'$ draw lines parallel to $rr$ to cut $r_1r_1$ and $r'r'$ in $i$ and $i$. Then will the lines $xx$ and $ii$ be reciprocals piercing the quadric in the four points which are first extremities of the inscribable closed $n$'gons.

*Second method.*—In the closing chord $c_1c_2$ of any inscribed open $n$'gon assume any point $p$, and for the moment regard it as $n+1$th point of a series having the $n$ fixed points as first $n$ points; choose four points $b_1$, $d_1$, $e_1$, $f_1$ in the surface so that no two of the five $c_1$, $b_1$, $d_1$, $e_1$, $f_1$ lie on one generator; find the final points $b_2$, $d_2$, $e_2$, $f_2$ of inscribed $(n+1)$'gons whose sides pass in order through the $n+1$ points, and which have $b_1$, $d_1$, $e_1$, $f_1$ as first extremities. Then (representing tangent-planes at points by capital letters of like names, and subscript numbers as the small ones representing the points of contact) in the chord $d_1d_2$ find the point $d_3$ such that

$$\frac{d_1d_3}{p_2d_3} = \frac{d_1\mathrm{B}_1}{d_2\mathrm{B}_2} : \frac{c_1\mathrm{B}_1}{c_1\mathrm{B}_2};$$

in the chord $e_1e_2$ find the point $e_3$ such that

$$\frac{e_1e_3}{e_2e_3} = \frac{e_1\mathrm{B}_1}{e_2\mathrm{B}_2} : \frac{c_1\mathrm{B}_1}{c_1\mathrm{B}_2};$$

in the chord $f_1f_2$ find the point $f_3$ such that

$$\frac{f_1f_3}{f_2f_3} = \frac{f_1\mathrm{B}_1}{f_2\mathrm{B}_2} : \frac{c_1\mathrm{B}_1}{c_1\mathrm{B}_2}$$

(the points $d_3$, $e_3$, $f_3$ must be so determined as that a *real* tangent-plane can pass through either of them); draw the plane $d_3e_3f_3$ and it will touch the quadric in a point $a_1$; draw the line $c_1a_1$, and find the point $q$ in it which is conjugate to $p$. Now if we regard the $n$ given fixed points and the point $q$ as the $n+1$ points of a series, every point in the surface will be the first extremity of a closed $2(n+1)$'gon. Find (by last case) the two reciprocal lines $yy$, $zz$ which pierce the quadric in first extremities of closed $(n+1)$'gons whose sides pass in order through these $n+1$ points; find the line $p'q'$ reciprocal to $pq$; draw the lines $xx$, $ii$, each of which cuts the four non-planar lines $yy$ $zz$, $pq$, $p'q'$. Then will $xx$ and $ii$ be reciprocal lines piercing in four points which are the first extremities of the inscribable closed $n$'gons.

*Third method.*—On the closing chord $c_1c_2$ of any inscribed open $n$'gon assume any point $p$, and regard it for the moment as the $n+1$th point of a series to which the $n$ fixed points belong. Assume three points $b_1$, $d_1$, $e_1$ in the surface so that no two of the points $c_1$, $b_1$, $d_1$, $e_1$ are on one generator, and find the final extremities $b_2$, $d_2$, $e_2$ of inscribed $n$'gons having $b_1$, $d_1$, $e_1$ as first extremities; draw the tangent-plane $\mathrm{C}_1$ at the point $c_1$; put $\mathrm{D}_1$, $\mathrm{D}_2$,

$E_1$, $E_2$ to represent the four planes $d_1b_1c_1$, $d_2b_2c_1$, $e_1b_1c_1$, $e_2b_2c_1$ respectively; through the line of intersection of the planes $D_1$, $D_2$ draw the plane P the distances of any point in which from $D_1$ and $D_2$ have to each other the ratio of $\frac{b_1D_1}{b_2D_2}$ to $\frac{b_1C_1}{b_2C_1}$; through the line of intersection of the planes $E_1$, $E_2$ draw the plane Q which is such that the ratio of the distances of any point in it from $E_1$ and $E_2$ is the same as that of $\frac{b_1E_1}{b_2E_2}$ to $\frac{b_1C_1}{b_1C_1}$; find the line of intersection $mm$ of the planes P, Q (this line $mm$ will be a tangent to the quadric), and the point $a_1$ in which it touches the quadric; in the line $c_1a_1$ find the point $q$ conjugate to $p$. Then if we regard the $n$ given fixed points and the point $q$ as $n+1$ points of a series, any point in the surface will be first extremity of a closed $2(n+1)$'gon. Find, by the preceding case, the two reciprocal lines $yy$, $zz$ which pierce the quadric in first extremities of closed $(n+1)$'gons whose sides pass in order through these $n+1$ points; find the line $p'q'$ reciprocal to $pq$; draw the lines $xx$ and $ii$ each of which cuts the four non-planar lines $yy$, $zz$, $pq$, $p'q'$. Then will $xx$ and $ii$ be reciprocals piercing in first extremities of the four inscribable closed $n$'gons.

*Fourth method.*—The following method is applicable in all cases in which $n$ is even. Omit temporarily the $n$th point $o_{n'}$ of the given $n$ points, and find the line U which pierces the quadric in the two first extremities of inscribable closed $(n-1)$'gons whose sides pass in order through the $n-1$ points; find the point $q$ in the line U which is conjugate to the omitted $n$th point $o_n$. Then if we regard the $n-1$ given points and the point $q$ as forming the $n$ points of a new series, any point in the surface will be the first extremity of a closed $2n$'gon. Find the two reciprocal lines $yy$, $zz$ which pierce the quadric in first extremities of closed $n$'gons whose sides pass in order through the new series of $n$ points; find the line $p'q'$ reciprocal to $o_nq$; draw the lines $xx$, $ii$ each of which cuts the four non-planar lines $yy$, $zz$, $o_nq$, $p'q'$. Then will the lines $xx$ and $ii$ be reciprocals piercing the quadric in first extremities of the four inscribable closed $n$'gons whose sides pass in order through the $n$ given points. But if the inscription of the closed $(n-1)$'gons be partially porismatic, and $\rho$ the point of concurrence of the closing chords of the inscribable open $(n-1)$'gons, then will the line $xx$ through $o_n$ and $\rho$, and the line $ii$ reciprocal to $xx$ pierce the quadric in the first extremities of the inscribable closed $n$'gons.

N.B. And if $o_n$ be in such case coincident with $\rho$, then the problem is fully porismatic, and every point in the surface is the first extremity of a closed $n$'gon.

N.B. We may also observe that when the inscription of the closed $(n-1)$'gons is non-porismatic, and that the point $o_n$ is situated in the line U, then, by conceiving $q$ coincident with $o_n$ the lines $yy$, $zz$ will be identical with $xx$ and $ii$.

I may observe that the general problem can be completely solved by

"methods of reduction," amongst which the following is perhaps the most obvious and simple:—

Let S be the quadric, and $o_1$ $o_2$, $o_3$, $o_4$, ... $o_n$ the $n$ given points. Put $xx$ for the line through $o_1$ and $o_2$. Instead of $o_2$ and $o_1$ we can substitute the point $p_2$ in which the line $xx$ is cut by the plane $o_3o_4o_5$, and another point $p_1$ determinable in the same line $xx$. Then instead of the four planar points $p_2$, $o_3$, $o_4$, $o_5$ we can (see theorem 38) substitute two other points $p_3$, $p_4$ in the same plane; and therefore instead of the series of $n$ points, we can substitute the series of $n-2$ points $p_1$, $p_3$, $p_4$, $o_6$, $o_7$, ... $o_n$ and the inscribable $(n-2)$'gons, closed and open as may be, whose sides pass in order through these points will have extremities identical with the extremities of inscribable $n$'gons. And thus step by step we can reduce the number of sides, until at length we find three points or four points, according as $n$ is odd or even, such that the extremities of all inscribed 3'gons or 4'gons whose sides pass in order through such points are identical with extremities of inscribable $n$'gons whose sides pass through the original $n$ points; and therefore to solve the problem all we have to do is to inscribe the closed 3'gons or closed 4'gons as may be.

And in respect to this method we may observe,—

(1) If any four consecutive points of any of the series be colinear and such as to render the inscription of closed 4'gons *real*, we may omit such points altogether from the series.

(2) When $n$ is *odd*, and that we reduce the problem to the inscription of closed 3'gons whose sides pass through three known points, then should such points be colinear or form a conjugate triad, the problem will be partially porismatic.

(3) In the case in which $n$ is *odd*, it is easy to perceive how the problem can be reduced to the drawing of a line through a known point to cut two reciprocal lines (which *point* will be on *one* of the lines when the problem is partially porismatic). And when $n$ is *even*, it is easy to see how the problem can be reduced to the drawing of the two lines which cut two pair of (determinable) reciprocal lines.

(4) The following method of finding the line in the plane of four points which pierces in first extremities of closed 4'gons is obvious:—Let $o_1$, $o_2$, $o_3$, $o_4$ be the four planar points.

Find $p$ the point of intersection of the lines $o_1o_2$, $o_3o_4$; in the line $o_1o_2$ find the point $m$ such that $o_1o_2$, $mp$, and the pair of (real or imaginary) points in which $o_1o_2$ pierces the quadric, will form an involution; in the line $o$ $o_4$ find the point $n$ such that the pairs of points $o_3o_4$, $pn$, and the points in which $o_3o_4$ pieces the quadric, form an involution. Then will the line $mn$ pierce in first extremities of closed 4'gons.