And he submits for consideration whether it may not be desirable to try two shorter wires, the two ends of each wire making connexion with the earth on opposite sides of the Observatory, and the register of each being made, at the Observatory, near the middle of its length.

February 13, 1868.

Lieut.-General SABINE, President, in the Chair.

The following communications were read:—

I. "On the Mysteries of Numbers alluded to by Fermat."—Second Communication. By the Right Hon. Sir Frederick Pollock, Bart., F.R.S. Received January 14, 1868.

(Abstract.)

This paper is not adapted to be read *in extenso*; so much of it is connected with mere calculation, so much more of it requires continual reference to diagrams, that no adequate knowledge of its contents would be acquired by merely hearing it read aloud; but a statement has been prepared of what it contains which will give a general view of the result.

The properties ascribed to all odd numbers, in addition to those contained in Fermat's theorem, are these:—1st. The algebraic sum of the roots in some form of the 4 squares which compose the number will equal 1, 3, 5, 7, &c. (every odd number which it is large enough to produce); 2ndly, the difference between some 2 of the roots will be any odd or even number whatever, subject to the same limitation.

1 3 5 7 9
The series 1, 3, 7, 13, &c. ($n, n, n, n+l$) will give 1, 3, 5, &c. as the sum
2 4 6 8
of the roots of its terms; and each term is the smallest that will give that
1 3 5 7
amount. So 1, 5, 13, 25, &c. is the series whose terms are the smallest that
4 8 12 16
give the odd numbers as a difference of the roots, and 1, 3, 9, 19, &c. that
2 6 10
give the even numbers. And these are the three series that compose *The Square* (the subject of the *last* paper) when the 1st term is 1; and they are the cause of its properties. A portion of the paper is devoted to an investigation of the change effected in the sum of the squares, by a change in the roots. If 2 roots differ by n, they may be represented by a and $a+n$; and if the smaller be diminished by 1, and the larger increased by 1, the sum of the squares is increased by $2n+2$; if $n=0$, the difference is 2; and it becomes 4, 6, 8, &c. as n becomes 1, 2, 3, 4, &c. On the other hand, if the smaller root be increased and the larger diminished by 1, the sum of the squares becomes less by $2n-2$.

A similar property belongs to all polygonal numbers; in the trigonal

the increase is.............. $n+1$,
in the square it is .... $2n+2$,
in the pentagonal .... $3n+3$,
and in the m-gonal.......... $mn+m$.

When the reverse operation takes place and the sum of the squares is diminished, the $+$ (plus) in the above expressions becomes $-$ (minus). There are some other modes mentioned also in dealing with the roots so as to increase the sum of the squares by 2, although there be not two of the roots which are equal. A proof is offered, by means of a supplemental square decreasing as the other increases, that if every number up to $2n+1$ has the properties of odd numbers above enumerated, then the number $2n+3$ will also possess them; and if this be so, then every subsequent odd number will likewise possess them. This is a mode of proof not unfrequent in mathematical investigations: it cannot be abbreviated; but it may be useful to state that the proof chiefly arises from this, that if one term of a series corresponds with the law of it, then every term will do so, and in all the series but two there will be one term obedient to the law which renders all the rest so; the other two series are treated differently.

It is shown that if a term, in the series 1, 3, 7, &c., whose terms (represented in the roots of the 4 squares of which they are the sum) will be $n, n, n\ (n \pm 1)$ be increased by 2, the roots being altered in the manner described above, the operation may be carried on till one of the terms becomes zero (0); but the next term in the series will be reached before that occurs. Then the next term may be taken as the beginning of another similar operation, and may go on till *another* term is reached, and so on without end. In this way the 4 squares into which any odd numbers may be divided will be obtained; and if every odd number is divisible into 4 squares, every even number will be so likewise.

The next subject is considered the most material and important in the paper, because it connects Lagrange's proof of the square numbers with *The Square* (the subject of the last paper). Euler thought that no assistance could be derived from the proof of Lagrange as to the other branches of the theorem (see Euler, 'Opuscula Analytica,' vol. ii. p. 4). But if every odd number is composed of 4 squares or less, then a number of the form $4n+2$ must be composed of 2, 3, or 4 squares, and in any of these cases n (any number) will be equal to 4 trigonal numbers, *which is shown in the paper*. The expression a^2+a+b^2 has been proved in a former paper of the author to be a general expression for any 2 trigonal numbers; and if any number is composed of 4 trigonal numbers or less, $a^2+a+b^2+m^2+m+n^2$ will represent any number whatever, odd or even, and $2a^2+2a+2b^2+2m^2+2m+2n^2$ will represent any even number. This connects Lagrange's proof of the squares with *The Square*, which is the subject of the last paper; and if a series be composed of squares and double trigonal numbers beginning with nothing, and having differences 2, 2, 4, 4, 6, 6,

8, 8, &c., the series will be 2, 4, 8, 12, 18, &c., and any even number will be made with some 4 terms of the series. Now *The Square*, the subject of the last paper, has a property not noticed in the former paper, viz. that the first term of *The Square*, supposing it to be of the form $4n+3$, will be increased in descending down the principal diagonal into the sum of the squares of the roots n, $n+1$, $n+1$, $n+1$, into which the number itself may be divided; and if the form of the number be $4n+1$, a term which is the sum of the roots n, n, n, $n+1$ (into which $4n+1$ may be divided) would appear in the diagonal next below the principal diagonal; and as every odd number is of the form of either $4n+3$, or $4n+1$, this applies to every possible odd number, and each of these numbers is a term in the series already mentioned, 1, 3, 7, 13, &c., and which may be increased by any even number by means of the series $2b^2$, $2a^2+2a$, and so on. This, it is shown in the paper, may be so altered as to correspond with the index of some number in the principal diagonal of the square, or the one below it, and will therefore ascend to the first term in *The Square*, and give the sum of its roots equal to 1; and therefore $(4n+3-1)$ divided by 2 will be composed of 3 trigonal numbers, and in the other case $(4n+1-1)$ is divided by 2; that is, every odd and every even number is composed of 3 trigonal numbers. If this be so, Fermat's theorem of the trigonal numbers is proved from the case of the squares, which (it is believed) has not been done before; but this leads to other conclusions, which are shown in the paper. If 1 be the first term of *The Square*, every term in it will have its roots of the squares that compose it of the form $+1a$, a, b, b, and the term itself will be composed of two trigonal numbers; but if each of these be made the first term of a square, every odd number will be found in some of the resulting squares; and it is shown that every odd number not only is of the form $1+2a^2+2a+2b^2+2m^2+2m+2n^2$, but also of the form $1+2a^2+2a+2b^2+2m^2+2m$, or $1+2a^2+2a+2b^2+2n^2$; so that, with re-respect to every odd number, two of the squares that compose it may be equal, and also two may have their roots differing by 1.

There remains one other matter to be mentioned, viz. a certain remarkable relation which all the polygonal numbers bear to each other, and which forms a connexion that runs through them all, from which it would seem to follow that a solution of the theorem as to one would be a solution as to all the rest (except the first).

This relation arises in the square numbers by a property of the gradation series, already in part alluded to, viz., as to the odd numbers, by which the interval between any two terms can be filled up, all the terms having, as to the odd numbers, the sum of the roots of the squares that compose them equal to the sum of the roots of the first term; but the intervals, as to the *even* numbers, may be also filled up by making the sum of the roots one less than that of the roots of the odd numbers (see the Table in Diagram No. 3, which is thus constructed). A term in the gradation series is assumed (in this case 73); its roots are 4, 4, 4, 5; and the roots of all the odd numbers

between that and the next term are found by the processes mentioned in the former part of this paper. The roots of the even numbers are then obtained by an analogous process; and these are used as bases or roots of the polygonal numbers, which are placed in columns, with their sums, as appears in the Table (see Diagram No. 4 for the mode in which the polygonal numbers are formed).

It will be observed that the sum of the roots or bases is 17; but if they be used to form trigonal numbers, the increment of the sum of the resulting trigonal numbers above the sum of the roots or bases is 28, and so on of the rest, each successive column increasing by the same number, viz. 28. If the roots or bases be $n, n, n, n\pm 1$ (that is, a term in the gradation series), the increment of the sums of the successive columns will be $2n^2\mp n$, a trigonal number.

Again, in the trigonal numbers the difference between the sums of the first and second term is 0; in the square numbers it is 1; in the pentagonal numbers 2; in the hexagonal numbers 3; in the heptagonal numbers 4; but in all of them the difference between the second and third terms is 1, and this continues throughout. The difference between the third and fourth, the fifth and sixth, the seventh and eighth, &c., increases by 1 in each column; but the difference between the second and third, the fourth and fifth, the sixth and seventh, &c., is always 1 in each column; and the result is that, by adding 1 in the pentagonal column, by adding 1, or 1.1 in the hexagonal, by adding 1, or 1.1, or 1.1.1 in the heptagonal, every number, odd or even, can be made by not exceeding four square numbers, or five pentagonal numbers, or, &c., as clearly appears by the Table.

This corresponds with what was discovered by Cauchy, published at the end of Legendre's 'Théorie des Nombres,' viz. that four only of each class of numbers is necessary; the rest may be supplied by 1 repeated as often as necessary. But I must not omit to say that, although all the odd numbers are sufficiently obedient, there is one class of even numbers quite refractory, viz. the powers of 2. They may be easily expressed in squares, pentagonal numbers, &c., but they cannot be brought within the rule that otherwise prevails.

II. "Compounds Isomeric with the Sulphocyanic Ethers.—I. On the Mustard Oil of the Ethyl Series." By A. W. Hofmann, LL.D., F.R.S.

The results of my researches on the chloroform-derivatives of the primary monamines, which, as I have shown, are isomeric with the nitriles, could not fail to direct my attention to allied groups of bodies, with the view of discovering similar isomerisms.

In a note communicated to the Royal Society some months ago, I expressed the expectations which even then appeared to be justified in the following manner:—"In conclusion, I may be permitted to announce as