from specimens of portions of the skull in the British Museum, and from a cast and photographs of the entire cranium in the Australian Museum at Sydney, New South Wales. The descriptions of the mandible, and of the dentition in both upper and lower jaws, are taken from actual specimens in the British Museum, in the Museum of Natural History at Worcester, and in the Museum at Adelaide, S. Australia, all of which have been confided to the author for this purpose. The results of comparisons of these fossils of *Nototherium* with the answerable parts in *Diprotodon, Macropus, Phascolarctos,* and *Phascolomys* are detailed.

Characters of three species, *Nototherium Mitchelli, N. inerme,* and *N. Victoriæ,* are defined chiefly from modifications of the mandible and mandibular molars. A table of the localities where fossils of *Nototherium* have been found, with the dates of discovery and names of the finders or donors, is appended. The paper is illustrated by subjects for nine quarto Plates.

II. "On Cyclides and Sphero-Quartics." By John Casey, LL.D., M.R.I.A. Communicated by Prof. Cayley, F.R.S. Received May 11, 1871.

(Abstract.)

The curves and surfaces considered in this paper are, I believe, some of the most fertile in properties in the whole range of geometry. For the purpose of giving a full and comprehensive discussion, I have divided the paper into several chapters. The following is an outline of the method of investigation pursued, together with a statement of some of the results arrived at.

If we take the most general equation of the second degree in $(\alpha, \beta, \gamma, \delta)$, where these variables denote spheres instead of planes,

$$(a\ b\ c\ d\ l\ m\ n\ p\ q\ r \between \alpha, \beta, \gamma, \delta,)^2=0,$$

we get the most general form in which the equation of a quartic cyclide can be written. Setting out with this equation, I have proved that a quartic cyclide is the envelope of a variable sphere, whose centre moves on a given quadric, and which cuts orthogonally the Jacobian of the spheres of reference $(\alpha, \beta, \gamma, \delta)$.

The Jacobian of $(\alpha, \beta, \gamma, \delta)$ can be written in a form identical with that of the imaginary circle at infinity in the system of quadriplanar coordinates. The square of the Jacobian can be expressed by an equation of the second degree in $\alpha, \beta, \gamma, \delta$. This equation assumes a very simple form when $\alpha, \beta, \gamma, \delta$ are mutually orthogonal. By means of it I have shown that every quartic cyclide can be written in the canonical form,

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\epsilon^2=0,$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are five spheres mutually orthogonal. These are spheres of inversion of the cyclide, and by incorporating constants their equations are connected by an identical relation, $\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2=0$.

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From these equations I have shown that in general a quartic can be generated in five different ways as the envelope of a variable sphere which cuts a given sphere orthogonally, and whose centre moves on a given quadric, which, on account of one of its most important properties, I have named the *focal quadric* of the cyclide. Every cyclide has, in general, five focal quadrics; these focal quadrics are confocal; their focal conics are double, or "nodo-foci" of the cyclide.

I have shown that the locus of the single or ordinary foci of cyclides are sphero-quartics (curves of intersection of a sphere and a quadric). In general a cyclide has five focal sphero-quartics. If we call confocal two cyclides having in common one focal sphero-quartic, through any point can be described three cyclides confocal with a given cyclide. These confocals are mutually orthogonal. Other methods of generating cyclides are also given; thus three circles in space being given, whose planes are diametral planes of a given sphere, and which are orthogonal to the sphere, a cyclide will be generated by a variable circle in space which rests on these three circles. This method is analogous to that for describing ruled quadrics by the motion of a line. The equation of a cyclide may be interpreted in three different ways:—1, so as to denote a cyclide; 2, a sphero-quartic; 3, a tangent cone to the cyclide. Hence it follows that sphero-quartics, both in their modes of generation and in many of their properties, bear a striking analogy to cyclides. Thus the canonical form of the equation of a sphero-quartic is $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, where α, β, γ, δ are circles on a given sphere U; the poles of the planes of α, β, γ, δ with respect to U are the vertices of the four cones which can be described through the sphero-quartic. The equations of α, β, γ, δ are, by incorporating constants, connected by an identical relation, $\alpha^2+\beta^2+\gamma^2+\delta^2=0$. By means of this relation, which holds also for bicircular quartics, I have got the equations of the four focal sphero-conics of the sphero-quartic. These sphero-conics are constructed geometrically as the intersections with U of perpendiculars from its centre on the tangent-planes to the four cones which can be drawn through the sphero-quartic. The focal sphero-conics are confocal, their foci being the double or nodo-foci of the sphero-quartic.

Sphero-quartics may be inverted into bicirculars; they may also be projected into bicirculars, and that in two ways. First, on either plane of circular section of the quadric, whose intersection with the sphere is the sphero-quartic by lines parallel to the greatest or least axis of the quadric; second, by elliptic projection—that is, by lines of curvature of confocal quadrics passing through each point of the sphero-quartic. The developable formed by tangent planes to the sphere U, at every point of the sphero-quartic, possesses many geometrical properties. Thus the cone whose vertex is at the centre of U, and which stands on its cuspidal edge, may be generated by the focal lines of a variable cone osculating one cone of the second degree, and having double contact with another. The cuspidal edge and the nodal lines of the developable may be projected

into the evolute and the focal conics of a bicircular quartic. The developable possesses numerous anharmonic properties; thus all its generators are divided homographically by the nodal lines and the sphere U.

In the chapters on the inversion and classification of cyclides, I have proved that the presence or absence of nodes depends on the relative positions of the focal quadric and sphere of inversion; thus if they touch there will be a conic node, the cyclide being in this case the inverse of a quadric, which is an hyperboloid or ellipsoid according as the node has a real or imaginary cone of contact. If they osculate, the cyclide will be the inverse of a paraboloid; the node will be biplanar if the paraboloid be an elliptic or hyperbolic one, and it will be uniplanar if the paraboloid be cylindrical. If the focal quadric and sphere of inversion have double contact, the cyclide will be the inverse of a cone of the second degree, and will have two nodes, which must be conic nodes. When a cyclide has nodes, the number of focal quadrics suffers diminution. I have given in the same chapters the equations and the singularities of the tangent cones, and shown that in general every cyclide has as many double tangent cones as it has focal quadrics; in fact the double tangent cones are the reciprocals of the asymptotic cones of the focal quadrics. It is also proved that the lines of intersection of a cyclide, with its spheres of inversion, are lines of curvature on the cyclide, and that the imaginary circle at infinity is a flecnodal curve on its surface of centres.

In the chapter on the classification of sphero-quartics I have given Chasles's characteristics for the osculating circles of a sphero-quartic. By inversion we get the characteristics for the osculating circles of bicircular quartics. Thus $V=24$ for these circles. In the same chapter Professor Cayley's equations, giving the singularities of the cuspidal edges of developables, are transformed so as to give the singularities of the evolute of a plane curve, any three of the singularities of the curve being given.

The last two chapters contain an account of the substitutions by which, from properties of quadrics, may be inferred corresponding properties of cyclides. These chapters are in reality an exposition of a new method of geometrical transformation; in fact, since the general equation in $\alpha, \beta, \gamma, \delta$ which I employ is the same in form as the general equation of a quadric, only that in my method the variables denote spheres in place of planes, it will be readily seen that the theories of invariants, reciprocation, &c. in the geometry of surfaces of the second degree have their analogues in the theory of cyclides, and, in fact, the modes of proof employed in one apply also in the other. This method of transformation is very fertile; I have illustrated it by numerous theorems. Thus the locus of the centre of a variable sphere cutting in two sphero-quartics having double contact, two cyclides having a common sphere of inversion is the developable circumscribed about the focal quadrics of these cyclides, which correspond to the common sphere of inversion.