The remainder is as follows:—

17550 06500 84834 29272 80492 56652 30056 77179 51985 96380
06400 15769 36917 76044 90943 15598 00090 70477 96549 03362
30614 39569 71063 83855 24053 35869 04219 87709 24604 49236
79071 67965 18350 79865 61803 71534 89641 16619 31638 84825
79008 0

Aug. 28th, 1871.

II. Second Paper "On the Numerical Value of Euler's Constant, and on the Summation of the Harmonic Series employed in obtaining such Value." By WILLIAM SHANKS, Houghton-le-Spring, Durham. Communicated by Prof. G. G. STOKES, Sec. R.S. Received August 30, 1871.

Three cases and sources of inaccuracy in finding the value of E in the former paper (Proc. Roy. Soc. vol. xv. p. 429) having been pointed out by Mr. Glaisher, and some other minor errors not noticed by him having since been detected by the author, and these having vitiated the results, but only in a slight degree the inferences drawn from them (for in the former paper the *last and leading conclusion* as to the value of E, though *limited*, was certainly *correct*), the author has been led, from a deep sense of obligation to the Royal Society, to revise, correct, and extend what he had previously done. And it will be seen, from comparing Mr. Glaisher's remarks and results with what follows in this paper, that the supplementary matter herein given, including the extension of E &c. to 110 places of decimals, can scarcely be without interest to mathematicians, and especially as regards the summation of the harmonic series in the formula for finding the value of E.

Not having seen M. Oettinger's article in Crelle's 'Journal,' "On Computing the value of E," the author is unable to state what artifices he used in summing the harmonic series. Mr. Glaisher gives a very simple and obvious one from M. Oettinger, which the author could not but see and employ for calculating the values of the reciprocals of the even numbers.

In summing the harmonic series, the author found the reciprocals of all numbers up to 200, as far as 200 places of decimals; next the reciprocals from 200 to 500, to only 105 decimals; and afterwards the reciprocals of the odd composite numbers up to 5000, to the same extent. In passing from S_{2000} to S_{5000} some extra calculation was necessary, which need not be stated here. It is, however, necessary to calculate, *in extenso*, the reciprocals of the odd composite numbers only to half the number of terms which it is proposed to sum. The reciprocals of all the prime numbers must of course be calculated separately.

The leading artifices the author employed to shorten calculation may be best stated and explained by supposing that the reciprocals of all the odd numbers below 5000 have been computed and retained separately, also

that the sum of the series to 5000 terms has been found, and that the series is required to be summed to 10,000 terms.

To obtain the sum of the reciprocals of the even numbers from 5000 to 10,000, we have:—

$$\left(\tfrac{1}{2501}+\tfrac{1}{2503}+\ldots\tfrac{1}{4999}\right)\div 2. \qquad \left(\tfrac{1}{41}+\ldots\ldots\ldots\tfrac{1}{77}\right)\div 2^7.$$
$$\left(\tfrac{1}{1251}+\tfrac{1}{1253}+\ldots\tfrac{1}{2499}\right)\div 2^2. \qquad \left(\tfrac{1}{21}+\ldots\ldots\ldots\tfrac{1}{39}\right)\div 2^8.$$
$$\left(\tfrac{1}{627}+\ldots\ldots\ldots\tfrac{1}{1249}\right)\div 2^3. \qquad \left(\tfrac{1}{11}+\ldots\ldots\ldots\tfrac{1}{19}\right)\div 2^9.$$
$$\left(\tfrac{1}{313}+\ldots\ldots\ldots\tfrac{1}{625}\right)\div 2^4. \qquad \left(\tfrac{1}{5}+\ldots\ldots\ldots\tfrac{1}{9}\right)\div 2^{10}.$$
$$\left(\tfrac{1}{157}+\ldots\ldots\ldots\tfrac{1}{311}\right)\div 2^5. \qquad \tfrac{1}{3}\div 2^{11}.$$
$$\left(\tfrac{1}{79}+\ldots\ldots\ldots\tfrac{1}{155}\right)\div 2^6. \qquad 1\div 2^{13}.$$

These twelve quotients, when added together, give the valne of the reciprocals of the even numbers between 5000 and 10,000, including the latter number.

To obtain the sum of the reciprocals of the odd composite numbers between 5000 and 10,000, we have, using *prime* divisors,

$$\left(\tfrac{1}{1667}+\tfrac{1}{1669}+\ldots\tfrac{1}{3333}\right)\div 3.$$
$$\left(\tfrac{1}{1001}+\tfrac{1}{1003}+\tfrac{1}{1007}+\ldots\tfrac{1}{1999}\right)\div 5.$$

Here it must be observed that *all odd numbers which are multiples of previous prime divisors* must be excluded: *e. g.* $\tfrac{1}{1005}$ must be excluded from division by 5, because 1005 is a multiple of 3.

$$\left(\tfrac{1}{719}+\ldots\ldots\ldots\tfrac{1}{1427}\right)\div 7. \qquad \left(\tfrac{1}{127}+\ldots\ldots\ldots\tfrac{1}{241}\right)\div 41.$$
$$\left(\tfrac{1}{457}+\ldots\ldots\ldots\tfrac{1}{907}\right)\div 11. \qquad \left(\tfrac{1}{127}+\ldots\ldots\ldots\tfrac{1}{229}\right)\div 43.$$
$$\left(\tfrac{1}{389}+\ldots\ldots\ldots\tfrac{1}{769}\right)\div 13. \qquad \left(\tfrac{1}{107}+\ldots\ldots\ldots\tfrac{1}{211}\right)\div 47.$$
$$\left(\tfrac{1}{307}+\ldots\ldots\ldots\tfrac{1}{587}\right)\div 17. \qquad \left(\tfrac{1}{97}+\ldots\ldots\ldots\tfrac{1}{181}\right)\div 53.$$
$$\left(\tfrac{1}{269}+\ldots\ldots\ldots\tfrac{1}{523}\right)\div 19. \qquad \left(\tfrac{1}{89}+\ldots\ldots\ldots\tfrac{1}{167}\right)\div 59.$$
$$\left(\tfrac{1}{223}+\ldots\ldots\ldots\tfrac{1}{433}\right)\div 23. \qquad \left(\tfrac{1}{83}+\ldots\ldots\ldots\tfrac{1}{163}\right)\div 61.$$
$$\left(\tfrac{1}{173}+\ldots\ldots\ldots\tfrac{1}{337}\right)\div 29. \qquad \left(\tfrac{1}{79}+\ldots\ldots\ldots\tfrac{1}{149}\right)\div 67.$$
$$\left(\tfrac{1}{163}+\ldots\ldots\ldots\tfrac{1}{317}\right)\div 31. \qquad \left(\tfrac{1}{71}+\ldots\ldots\ldots\tfrac{1}{139}\right)\div 71.$$
$$\left(\tfrac{1}{137}+\ldots\ldots\ldots\tfrac{1}{269}\right)\div 37.$$

Here it must be noted that all numbers *below* prime divisors must always be excluded.

$$\left(\tfrac{1}{73}+\ldots\ldots\ldots\tfrac{1}{131}\right)\div 73. \qquad \left(\tfrac{1}{59}+\ldots\ldots\ldots\tfrac{1}{109}\right)\div 89.$$
$$\left(\tfrac{1}{79}+\ldots\ldots\ldots\tfrac{1}{113}\right)\div 79. \qquad \left(\tfrac{1}{97}+\ldots\ldots\ldots\tfrac{1}{103}\right)\div 97.$$
$$\left(\tfrac{1}{83}+\ldots\ldots\ldots\tfrac{1}{113}\right)\div 83.$$

These twenty-four quotients, when added together, give the sum of the reciprocals of the odd composite numbers between 5000 and 10,000. To this sum add the sum of the prime reciprocals between 5000 and 10,000; the result is the value of the reciprocals of all the odd numbers between 5000 and 10,000. It need scarcely be stated that the sum of these two

distinct sets of numbers, increased by the sum of the series to 5000 terms, will give the sum of the harmonic series to 10,000 terms.

Before proceeding further it should be stated that, having obtained the correct value of E, from S_{200} &c. to 110 decimals, verifying Mr. Glaisher's to 99 decimals, it was comparatively easy to extend S_{500} and S_{1000} to 110 decimals, and to correct and extend S_{2000}, S_{5000}, and $S_{10,000}$ to the same extent.

When we have S_{100}, the calculations from Bernoulli's 31 numbers will lead to obtaining E only to about 92 decimals. This value may no doubt be extended by finding the ratio between the last and each succeeding Bernoulli's number. Such ratio is, however, only approximative, and can yield correct results of only a limited number of decimals. The excess of the + Bernoulli terms over the — ones, to 110 decinals, when S_{100} is used, is readily obtained when E and $\log_\epsilon 100$ are known to the same extent. Such excess will be found below ; also the separate sums of the + and — terms in which Bernoulli's numbers enter, both when S=100 and when S=200, to 205 decimals.

The values of S_{500}, S_{1000}, S_{2000}, S_{5000}, and $S_{10,000}$ to 110 decimals, also the corresponding + and — results of the Bernoulli terms to the same extent, are likewise given below, as they involve very considerable calculation, and may thus be tested and verified. The values of S_{100} and S_{200} may as well be also written anew, inasmuch as a few slight errors had crept into them before.

E= ·57721 56649 01532 86060 65120 90082 40243 10421 59335 93992
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495
11141 14421......

S_{100} =5·18737 75176 39620 26080 51176 75658 25315 79089 72126 70845
16531 76533 95658 72195 57532 55049 66056 87768 92312 04135
49921 06986 97779 79182 73403 18717 00828 94825 42444 49096
57618 56474 16326 13467 07313 21114 47132 49733 09103 51129
......

S_{200} =5·87803 09481 21444 47605 73863 97130 86163 68374 00246 53024
30844 64971 94472 28783 30029 84018 15499 64301 86679 89238
37326 83211 85439 05911 76542 77755 27568 86559 30203 06046
25715 75389 22254 75748 47845 75246 64079 54805 61627 08880
......

S_{500} =6·79282 34299 90524 60298 92871 45367 97369 48198 13814 39677
91166 43088 89685 43566 23790 55049 24576 49403 73586 56039
14705 68279......

S_{1000} =7·48547 08605 50344 91265 65182 04333 90017 65216 79169 70880
36657 73626 74995 76993 49165 20244 09599 34437 41184 50813
93907 71134......

S_{2000} =8·17836 81036 10282 40957 76565 71641 69368 79354 66740 91248
77402 20419 74812 15302 80688 34328 60377 35324 29687 02614
20643 33506......

S_{5000} = 9·09450 88529 84436 96726 12455 33393 43939 17829 87811 30381
14506 16283 85209 05328 30500 87619 93914 09299 23691 97409
31969 93538......

$S_{10,000}$ = 9·78760 60360 44382 26417 84779 04851 60533 48592 62945 57769
16183 89460 95668 16020 24943 15950 68001 25127 29008 08825
88669 45713......

When we have S_{100} we have excess of + Bernoulli terms over − Bernoulli terms as follows:—

+·00000 83332 50003 96783 73773 23792 87768 83353 90186 48901
78976 95889 08023 27933 88599 81913 48032 53704 54782 29326
45555 64243......

For S_{100} the sum of the + Bernoulli terms is

+·00000 83333 33337 30158 73773 44885 67821 37321 67823 08773
00082 30639 33761 49846 36254 14224 82920 15089 51978 53129
32622 35457 80834 33466 17877 22649 42919 19943 83311 44605
09910 37086 58144 47970 79756 13813 15081 18650 05862 58830
54265......

For S_{100} the sum of the − Bernoulli terms is

−·00000 00000 83333 33375 00000 21092 80052 53967 77636 59871
21105 34750 25738 21912 47654 32311 34887 61384 93958 76677
70933 50276 12351 60205 99358 40193 39578 24455 06318 69180
82191 70839 75485 79804 74621 35198 75347 46810 82718 00710
69953......

For S_{200} the sum of the + Bernoulli terms is

+·00000 20833 33333 39533 73016 61283 59538 59711 91078 12258
22890 92137 92329 17052 51095 23490 89894 78957 35141 06774
04433 65950 49748 89147 11592 75754 43919 38271 38143 50271
65525 60887 11534 99567 61791 58429 85003 06150 05862 58830
54265......

For S_{200} the sum of the − Bernoulli terms is

−·00000 00000 05208 33333 49609 37505 14960 84887 28877 09510
05685 64006 82170 11770 13023 01074 35246 38458 75084 75675
04222 29829 71026 50819 24197 13084 02015 36765 99447 44527
25466 44760 75767 81146 37524 18401 87847 46810 82718 00710
69953......

For S_{500} the sum of the + Bernoulli terms is

+·00000 03333 33333 33358 73015 87309 34487 73462 42678 21147
87864 83345 53789 61875 16264 59799 20761 22352 44796 75809
19590 05554......

For S_{500} the sum of the − Bernoulli terms is

−·00000 00000 00133 33333 33344 00000 00008 63960 92825 14227 06303 41363 50645 63522 06215 12108 84946 96404 61971 02278 38712 72302......

For S_{1000} the sum of the + Bernoulli terms is

+·00000 00833 33333 33333 73015 87301 59487 73448 77428 21067 82137 32165 00876 22580 99464 52882 37182 41954 63931 48991 92230 67302......

For S_{1000} the sum of the − Bernoulli terms is

−·00000 00000 00008 33333 33333 37500 00000 00210 92796 09323 93526 00040 82093 23718 92820 00915 30332 37713 49061 83360 48493 88803......

For S_{2000} the sum of the + Bernoulli terms is

+·00000 00208 33333 33333 33953 37301 58730 89855 02344 88243 11403 65711 84320 18015 80288 93393 93977 27188 21088 95868 23862 44594......

For S_{2000} the sum of the − Bernoulli terms is

−·00000 00000 00000 52083 33333 33349 60937 50000 05149 60842 10995 59700 64404 23526 83197 86039 81846 60506 95079 95161 64799 80365......

For S_{5000} the sum of the + Bernoulli terms is

+·00000 00033 33333 33333 33335 87301 58730 15880 77344 87734 48909 98210 67821 14787 86482 16512 65337 39125 53345 19304 37661 22499......

For S_{5000} the sum of the − Bernoulli terms is

−·00000 00000 00000 01333 33333 33333 34400 00000 00000 08639 60927 96095 70104 02470 73465 66314 96500 75868 11665 04160 31880 36102......

For $S_{10,000}$ the sum of the + Bernoulli terms is

+·00000 00008 33333 33333 33333 37301 58730 15873 02344 87734 48773 45710 67821 06782 13732 16500 84808 26100 49665 77626 52262 19358......

For $S_{10,000}$ the sum of the − Bernoulli terms is

−·00000 00000 00000 00083 33333 33333 33337 50000 00000 00002 10927 96092 79613 71220 73188 24992 34000 15344 13659 87023 61119 37034......

Suppose n, in the harmonic series, $=1$ followed by 1000 ciphers; then,

since n is very large, we may disregard $-\frac{1}{2n}+\frac{B_1}{2n^2}$ &c. We thus have $E=Sn-\log_\epsilon n$, or $Sn=\log_\epsilon n+E$; but $\log_\epsilon n=\log_\epsilon 10\times 1$ followed by 100 ciphers, therefore

S_n=2·30258 50929 94045 68401 79914 54684 36420 76011 01488 62877
29760 33327 90096 75726 09677 35248 02359 97205 08959 82983
99889 35053 24395 34694 06073 44733 23049 86088 22209 63091
14157 00596 30696 81232 73586 10266 98852 09395 27703 06845
63633......

August 28th, 1871.

III. "An Experimental Determination of the Velocity of Sound." By E. J. Stone, M.A., F.R.S., Astronomer Royal at the Cape of Good Hope. Received August 21, 1871.

(Abstract.)

A galvanic current passes from the batteries at the Royal Observatory, Cape Town, at 1 o'clock, and discharges a gun at the Castle, and through relays drops a time-ball at Port Elizabeth. It appeared to the author that a valuable determination of the velocity of sound might be obtained by measuring upon the chronograph of the Observatory the interval between the time of the sound reaching some point near the gun and that of its arrival at the Observatory. As there is only a single wire between the Observatory and Cape Town, some little difficulty was experienced in making the necessary arrangements, without any interference with the 1 o'clock current to Port Elizabeth; but this difficulty was overcome by a plan which the author describes, and which was brought into successful operation on Feb. 27, 1871. The experiments could not have been carried out, on account of the encroachment they would have made on the time of the Observatory staff, had it not been for the assistance of J. Den, Esq., the acting manager of the Cape Telegraph Company, to whom the author is indebted for the preparation of a good earth-connexion near the gun, for permission to Mr. Kirby, a gentleman attached to the telegraph office, to assist in the experiments, and for a general superintendence of the arrangements at Cape Town.

The observed times of hearing the sound were recorded on the chronograph by two observers, situated one (Mr. Kirby) at a distance of 641 feet from the gun, the other (Mr. Mann) at the Observatory, at a distance of 15,449 feet from the gun. The former distance was sufficient to allow the connexion of the main wire to be broken at the telegraph office after the gun had been fired, but before the sound reached the first observer.

As there were no reciprocal signals, a correction was made by calculation for the effect of the wind, its velocity being measured by a set of Robin-