tion was not examined by means of the stethometer; but from Mr. Le Gros Clark's experiments it would appear that they have but little influence in forced inspiration.

With the intrinsic thoracic muscles the case is different. It is probable that, with the exception of the *levatores costarum*, each of these muscles has some further power beyond that of simply raising or depressing the ribs.

The *triangularis sterni* is more especially a constrictor of the anterior part of the thorax.

The *intercostals* have very various actions assigned to them; but when all the evidence of Traube, Sibson, and others is considered, it seems most probable that the six upper ribs are raised by the external intercostals, and depressed by the internal muscles; but the ribs are not rigid bars, and hence these muscles have a further action upon them. From careful experiments upon a model with elastic vulcanite ribs, and elastic bands stretched between them in the direction of (*a*) the external intercostals, (*b*) the internal intercostals, and (*c*) with now the upper ribs fixed, now the lower, it was concluded that whilst modifications were introduced into their action by the last-named condition, yet that without doubt the tendency of the external intercostals was, 1st, to draw the ribs upwards; 2nd, to separate their anterior ends; 3rd, to straighten them. On the other hand, the action of the internal intercostals was, 1st, to draw the ribs downwards; 2nd, to bring their anterior extremities nearer together; 3rd, to bend them inwards. These results were explained by resolving the forces of these muscles in the directions (*a*) along the ribs, (*b*) at right angles to them.

The diaphragm may also be considered to have an important action in bending the lower ribs.

Evidence can be adduced showing the value of the stethometer used in these inquiries, both in physiological studies and in medical practice.

III. "On Linear Differential Equations."—No. VI. By W. H. L. Russell, F.R.S. Received July 30, 1872.

We now consider linear differential equations which are satisfied by the roots of an algebraical equation admitting of explicit solution. To determine in what cases the linear differential equation

$$P\frac{d^n y}{dx^n}+Q\frac{d^{n-1}y}{dx^{n-1}}+\ldots+Ry=0$$

is satisfied by assuming

$$y=\sqrt[m]{X+\sqrt[r]{Y+\sqrt[s]{Z+\ldots}}}$$

when X, Y, Z are rational functions of (x). We shall commence by

supposing them also entire functions of (x). At present we shall confine ourselves to the linear differential equation of the second order,

$$\mathrm{P}\frac{d^2y}{dx^2}+\mathrm{Q}\frac{dy}{dx}+\mathrm{R}y=0.$$

The most general form of solution which this equation will admit will be $y=\{\mathrm{X}+\sqrt{\mathrm{Y}}\}^n$. For if we put $y=\{\mathrm{X}+\sqrt[3]{\mathrm{Y}}\}^n$, we should have three particular integrals corresponding to the three cube roots of Y, n being supposed fractional.

Substituting in the proposed equation, we shall find, after making some reductions,

$$n(n-1)\mathrm{P}\{\mathrm{X}'+\tfrac{1}{2}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Y}'\}^2+n\mathrm{P}\{\mathrm{X}+\mathrm{Y}^{\frac{1}{2}}\}\{\mathrm{X}''+\tfrac{1}{2}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Y}''-\tfrac{1}{4}\mathrm{Y}^{-\frac{3}{2}}\mathrm{Y}'^2\}$$
$$+n\mathrm{Q}\{\mathrm{X}+\mathrm{Y}^{\frac{1}{2}}\}\{\mathrm{X}'+\tfrac{1}{2}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Y}'\}+\mathrm{R}\{\mathrm{X}+\mathrm{Y}^{\frac{1}{2}}\}^2=0\,; \quad \ldots \quad (1)$$

from whence we have

$$n(n-1)\mathrm{PYX}'^2+\tfrac{1}{4}n(n-1)\mathrm{PY}'^2+n\mathrm{PXYX}''+\tfrac{1}{2}n\mathrm{PYY}''-\frac{n}{4}\mathrm{PY}'^2$$
$$+n\mathrm{QXYX}'+\tfrac{1}{2}n\mathrm{QYY}'+\mathrm{RYX}^2+\mathrm{RY}^2=0, \quad \ldots \quad (2)$$
$$n(n-1)\mathrm{PX'Y'Y}+\tfrac{1}{2}n\mathrm{PXYY}''-\tfrac{1}{4}n\mathrm{PXY}'^2+n\mathrm{PX''Y}^2$$
$$+\tfrac{1}{2}n\mathrm{QXYY}'+n\mathrm{QX'Y}^2+2\mathrm{RXY}^2=0 \quad \ldots \quad (3)$$

Let

$$\mathrm{P}=\alpha+\beta x+\gamma x^2+\delta x^3,\ \mathrm{Q}=\alpha'+\beta' x+\gamma' x^2,\ \mathrm{R}=\alpha''+\beta'' x.$$

This will give us

$$\mathrm{X}=p+qx,\ \mathrm{Y}=r+x.$$

For if we assumed either X or Y to be of higher dimensions, we should obtain, on equating the coefficients of the powers of (x) in 2 and 3 to zero, more equations than the total number of constants in the differential equation and assumed solution. The number of disposable constants is eleven. If we assumed X of two dimensions or Y of two dimensions, we should have more than eleven equations.

Now equate the coefficient of the highest power of (x) in (2) to zero, and we have

$$n^2\delta+n(\gamma'-\delta)+\beta''=0,$$

which determines n independently of the constants in X and Y.

Similarly, from (3),

$$n(n-1)\delta-\frac{n}{4}\delta+\frac{n\gamma'}{2}+n\gamma'+2\beta''=0.$$

This of course becomes an equation of condition between the coefficients of the given differential equation, when we substitute the value of (n) we have found from the last equation.

To determine p, q, r, we proceed as follows:—In equation (3) put x successively equal to the three roots of the equation

$$\alpha+\beta x+\gamma x^2+\delta x^3=0.$$

Then we have three equations, linear in $\frac{q}{p}$, $\frac{q}{p}r$, r, from which these quan-

tities may be determined. But putting $x=0$ in equation (1), the equation becomes

$$n(n-1)aq^2r+na'rpq+\frac{na'r}{2}+a''rp^2+\frac{n^2a}{4}-\frac{na}{2}+a''r^2=0.$$

Hence, substituting for $\frac{q}{p}$ and r the values we have just obtained, we have a linear equation to determine p^2; hence p, q, r are determined.

Now, suppose P, Q, R to be such that we may assume

$$X=p+qx+x^2,\quad Y=m+rx+sx^2;$$

then the terms of equation (2), which contain the two highest powers of x, are linear in Y, or we may divide it out in equating their coefficients to zero. Hence we may determine q at once.

Put for x successively in (2) four roots of the equation $P=0$; then we have four equations linear in p^2, p, m, r, s, whence we have a quadratic equation to determine p, and then m, r, s are known. We have beside a number of equations of condition.

Now suppose P, Q, R to be such that we may have

$$X=p+qx+x^2,\quad Y=r+sx;$$

then we shall find that the terms which contained the three highest powers of (x) are linear in Y; consequently p, q, and therefore X, may be determined at once. X being known, we may obtain a series of equations easily by which the constants in Y may be determined*. But we may adopt a somewhat different method, which will be useful in many cases of this nature. Resuming the equation

$$(\alpha+\beta x+\gamma x^2+\delta x^3)\frac{d^2y}{dx^2}+(\alpha'+\beta' x+\gamma' x^2)\frac{dy}{dx}+(\alpha''+\beta'' x)y=0,$$

equation (1) shows that $\alpha+\beta x+\gamma x^2+\delta x^3$ must vanish for every quantity which causes $X+\sqrt{Y}$ to vanish, and equation (3) that the same quantity must also vanish for every quantity which causes Y to vanish.

Let

$$\alpha+\beta x+\gamma x^2+\delta x^3=\delta(\mu_1+x)(\mu_2+x)(\mu_3+x).$$

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* If

$$X=p_0+p_1x+p_2x^2+\ldots+x^\mu,$$
$$Y=q_0+q_1x+q_2x^2+\ldots+q_\mu x^\mu,$$

the terms which contain the μ highest powers are linear in Y; and therefore

$$p_1p_2\ldots p_{\mu-1}$$

can be determined at once, commencing with $p_{\mu-1}$.

If

$$X=p_0+p_1x+p_2x^2+\ldots x^\mu,$$
$$Y=q_0+q_1x+q_2x^2+\ldots q_{\mu-1}x^{\mu-1},$$

the terms which contain the $(\mu+1)$ highest powers are linear in Y; and therefore

$$p_0p_1\ldots p_{\mu-1}$$

may be determined at once.—W. H. L. R., Nov. 21, 1872.

Then r must be one of the quantities μ_1, μ_2, μ_3; we will suppose μ_1. Moreover, we must have

$$p-q\mu_2+\sqrt{r-\mu_2}=0,\quad p-q\mu_3+\sqrt{r-\mu_3}=0,$$

or

$$p-q\mu_2+\sqrt{\mu_1-\mu_2}=0,\ p-q\mu_3+\sqrt{\mu_1-\mu_3}=0;$$

whence

$$p=\frac{\mu_2\sqrt{\mu_1-\mu_3}-\mu_3\sqrt{\mu_1-\mu_2}}{\mu_3-\mu_2},$$

$$q=\frac{\sqrt{\mu_1-\mu_3}-\sqrt{\mu_1-\mu_2}}{\mu_3-\mu_2}.$$

Generally, if

$$P\frac{d^ny}{dx^n}+Q\frac{d^{n-1}y}{dx^{n-1}}+\ldots+Ry=0$$

be a linear equation, admitting a solution of the form

$$y=\sqrt[n]{X+\sqrt[r]{Y+\sqrt[s]{Z+\ldots}}},$$

every quantity which makes $X+\sqrt[r]{Y+\sqrt[s]{Z-\ldots}}$ to vanish will make P vanish; every quantity which makes $Y+\sqrt[s]{Z+\ldots}$ to vanish will make P vanish; every quantity which makes $Z+\ldots$ to vanish will make P vanish. This principle is of course of the greatest use in determining the form of this function.

The most general form of irrational solution for a linear differential equation of the third order will be $\{X+\sqrt{Y}\}^n$ or $\{X+\sqrt[3]{Y}\}^n$; for one of the fourth order we shall have

$$\{X+\sqrt{Y}\}^n,\ \{X+\sqrt[3]{Y}\}^n,\ \{X+\sqrt{Y}+\sqrt{Z}\}^n,\ \{X+\sqrt{Y+\sqrt{Z}}\}^n,$$
$$\{X+\sqrt[4]{Y}\}^n,$$

where (n) is supposed to be fractional.

In all these cases the determination of (n) will be independent of the constants in X, Y, Z. For if we expand the proposed forms of solution in descending powers of (x), substitute in the differential equation, and equate the coefficient of the highest power of (x) to zero, we obtain an equation to determine (n) not involving these constants.

If X, Y, Z, ... are rational fractions, this method will not apply; but since the factors of the denominators of these fractions, not regarding their powers, must also be factors of P, we may proceed as follows:— Let $x-a=z$ be one of the factors of P, substitute $x=a+z$ in the given differential equation, and let $-m$ be the greatest negative value of μ, obtained by substituting $y=z^\mu+Bz^{\mu+1}\ldots$ in the differential equation, and equating the lowest term of the result to zero. Again, let $x-b=z$, and let $-r$ be the greatest negative value of μ, obtained by substituting $x=b+z$ in like manner. Then, if we put $v=(x-a)^m(x-b)^r\ldots y$, we may obtain an equation whose irrational solution will be free from negative factors.

We have seen that if $P\epsilon^{\omega}$, when substituted for y, satisfies a linear differential equation, ω is an invariant, that is, it remains the same function of x, and the constants involved in the equation whatever numerical value we are able to assign to these constants, on the supposition that P and ω are rational functions of (x). We shall call this the "principle of index invariance," and proceed to show that it is true when ω is irrational. Let

$$(a+\beta x+\gamma x^2)\frac{d^2y}{dx^2}+(a'+\beta'x+\gamma'x^2+\delta'x^3)\frac{dy}{dx}+(a''+\beta''x+\gamma''x^2+\delta''x^3)y=0$$

be a differential equation; to find the condition that it may be satisfied by $y=P\epsilon^{\int\omega dx}$, where ω is irrational. The irrational quantity in ω must be of the form $\sqrt{X}$, otherwise the equation would admit of more than two particular integrals. Moreover the factors of X must be factors of $a+\beta x+\gamma x^2$. We will suppose $X=a+\beta x+\gamma x^2$. Hence we may assume

$$\omega=\mu+\nu x+\rho\sqrt{a+\beta x+\gamma x^2};$$

for if we expand ω in descending powers of (x), and substitute the resulting value of y in the differential equation, we shall easily see that it can have no power higher than the first. μ, ν, and ρ are of course constants to be determined, and the double sign of the radical will give rise to four equations instead of two, which will imply an equation of condition.

We now substitute $y=P\epsilon^{\int\omega dx}$, where P is an algebraical function of (x), in the linear differential equation, and thus obtain

$$\begin{aligned}(a+\beta x+\gamma x^2)\Big\{&\frac{d^2P}{dx^2}+2(\mu+\nu x+\rho\sqrt{a+\beta x+\gamma x^2})\frac{dP}{dx}\\ &+(\mu+\nu x+\rho\sqrt{a+\beta x+\gamma x^2})^2P+(\mu+\nu x+\rho\sqrt{a+\beta x+\gamma x^2})'P\Big\}\\ &+(a'+\beta'x+\gamma'x^2+\delta'x^3)\Big\{\frac{dP}{dx}+(\mu+\nu x+\rho\sqrt{a+\beta x+\gamma x^2})P\Big\}\\ &+(a''+\beta''x+\gamma''x^2+\delta''x^3)P=0.\end{aligned}$$

The coefficients of the two highest powers of (x) must be equated to zero, that is to say, the coefficients of the two highest powers of the multiplier of P must be equated to zero.

The following terms of that multiplier will be sufficient for our purpose:—

$$\begin{aligned}&\delta''x^3+(\gamma'x^2+\delta'x^3)(\mu+\nu x+\rho\sqrt{a+\beta x+\gamma x^2})\\ &\quad+(\beta x+\gamma x^2)(\mu+\nu x+\rho\sqrt{a+\beta x+\gamma x^2})^2,\end{aligned}$$

or

$$\delta''x^3+(\gamma'x^2+\delta'x^3)(\mu+\nu x+\rho x\sqrt{\gamma}(1+\frac{\beta}{2\gamma x}+\ldots))+(\beta x+\gamma x^2)$$
$$\Big\{\mu^2+2\mu\nu x+\nu^2x^2+2\rho(\mu+\nu x)x\sqrt{\gamma}(1+\frac{\beta}{2\gamma x}+\ldots)+\rho^2(a+\beta x+\gamma x^2)\Big\}.$$

Hence, remembering the double sign of $\sqrt{\gamma}$,

$$\nu\delta'+\gamma\nu^2+\rho^2\gamma^2=0, \qquad (1)$$
$$\delta'+2\gamma\nu=0, \qquad (2)$$
$$\delta''+\gamma'\nu+\delta'\mu+2\mu\nu\gamma+\beta\nu^2+2\beta\gamma\rho^2=0, \qquad (3)$$
$$\gamma'\rho+\frac{\delta'\beta\rho}{2\gamma}+3\rho\beta\nu+2\rho\mu\gamma=0.$$

From the first two equations,

$$\nu=-\frac{\delta'}{2\gamma},\ \rho^2=\frac{\delta'^2}{4\gamma^3};$$

and, substituting in the fourth,

$$\mu=\frac{\beta\delta'}{2\gamma^2}-\frac{\gamma'}{2\gamma}.$$

Consequently we shall have

$$y=\mathrm{P}\epsilon^{-\frac{1}{2\gamma^2}\int dx(\gamma\gamma'-\beta\delta'+\gamma'\delta'x+\delta'\sqrt{\gamma}\sqrt{\alpha+\beta x+\gamma x^2})};$$

and the constants in the differential equation must satisfy the condition derived from the third equation,

$$\delta''=\frac{\gamma'\delta'}{2\gamma^2}-\frac{3\beta\delta'^2}{4\gamma^2}.$$

It is easy to see that a similar proof will apply in other cases. We will now recapitulate our results.

The solutions of differential equations, when they can be obtained in finite terms, must be either algebraical, exponential, or logarithmic. If algebraical, they must be either rational or irrational. The method of finding the rational algebraical solutions has long been known. We have endeavoured in the present paper to show how the irrational solutions may be found for equations of the second order. We have proved that a linear differential equation will not admit of a solution of the form $f(\epsilon^x)$; still less will it admit of a solution of the form $f(\epsilon^\omega)$, when ω is a function of (x). The exponential solution must therefore be of the form $\mathrm{P}\epsilon^\omega$, where ω is an invariant, that is, a function which remains unchanged whatever algebraical value we may be able to assign to P. This invariant enables us to reduce linear differential equations containing exponential functions to other linear equations involving algebraical functions only. We have, moreover, shown that logarithmic solutions of linear differential equations must in general be of the form $\mathrm{P}\log_\epsilon \mathrm{Q}$, when P is an algebraical function satisfying the proposed linear equation, and Q can be deduced from another linear equation, which we have shown how to construct. A great deal remains to be done; nevertheless I think we see not obscurely the true foundations on which the solution of linear differential equations in finite terms must of necessity rest.