III. "A Memoir on the Transformation of Elliptic Functions." By Professor CAYLEY, F.R.S. Received November 14, 1873.

(Abstract.)

The theory of Transformation in Elliptic Functions was established by Jacobi in the 'Fundamenta Nova' (1829); and he has there developed, transcendentally, with an approach to completeness, the general case, n an odd number, but algebraically only the cases $n=3$ and $n=5$; viz. in the general case the formulæ are expressed in terms of the elliptic functions of the nth part of the complete integrals, but in the cases $n=3$ and $n=5$ they are expressed rationally in terms of u and v (the fourth roots of the original and the transformed moduli respectively), these quantities being connected by an equation of the order 4 or 6, the modular equation. The extension of this algebraical theory to any value whatever of n is a problem of great interest and difficulty. The general case should admit of being treated in a purely algebraical manner; but the difficulties are so great that it was found necessary to discuss it by means of the formulæ of the transcendental theory, in particular by means of the expressions involving Jacobi's q (the exponential of $-\frac{\Pi K'}{K}$), or, say, by means of the q-transcendants. Several important contributions to the theory have since been made:—Sohnke, "Equationes Modulares pro transformatione functionum Ellipticarum," Crelle, t. xvi. (1836), pp. 97–130 (where the modular equations are found for the cases $n=3$, 5, 7, 11, 13, 17, & 19); Joubert, "Sur divers équations analogues aux équations modulaires dans la théorie des fonctions elliptiques," Comptes Rendus, t. xlvii. (1858), pp. 337–345 (relating among other things to the multiplier equation for the determination of Jacobi's M); and Königsberger, "Algebraische Untersuchungen aus der Theorie der elliptischen Functionen," Crelle, t. lxxii. (1870), pp. 176–275; together with other papers by Joubert and by Hermite in later volumes of the 'Comptes Rendus,' which need not be more particularly referred to. In the present Memoir I carry on the theory, algebraically as far as I am able; and I have, it appears to me, put the purely algebraical question in a clearer light than has hitherto been done; but I still find it necessary to resort to the transcendental theory. I remark that the case $n=7$ (next succeeding those of the 'Fundamenta Nova'), on account of the peculiarly simple form of the modular equation $(1-u^8)(1-v^8)=(1-uv)^8$, presents but little difficulty; and I give the complete formulæ for this case, obtaining them as well algebraically as transcendentally; I also to a considerable extent discuss algebraically the case of the next succeeding prime value $n=11$. For the sake of completeness I reproduce Sohnke's modular equations, exhibiting them for greater clearness in a square form, and adding to them those for the non-prime cases $n=9$ and $n=15$; also a valuable table given by him for the powers of $f(q)$; and I give other tabular results which are of assistance in the theory.