"On the Theory of the Solution of a System of Simultaneous Non-linear Partial Differential Equations of the First Order." By E. J. NANSON. Communicated by Professor CAYLEY, F.R.S. Received June 5, read June 17, 1875*.

Given an equation of the form

$$z=\phi(x_1, x_2, \ldots x_{n+r}, \ a_1, a_2, \ldots a_r, a_{r+1}),$$

we obtain by differentiation with respect to each of the $n+r$ variables $n+r$ equations, together with the original equation $n+r+1$ equations, from which, eliminating the $r+1$ constants, we have a system of n non-linear partial differential equations.

Conversely, given a system of n non-linear partial differential equations with $n+r$ independent variables, if there exists an equation

$$z=\phi(x_1, x_2, \ldots x_{n+r}, \ a_1, a_2, \ldots a_r, a_{r+1})$$

with $r+1$ constants, giving rise, as above, to the given system of n equations, then this is the "complete primitive" of the given system.

Starting with such a system of partial differential equations, it is in the present paper proposed to determine the conditions which must be satisfied in order that the system may admit of a complete primitive, and also to examine what kind of solution, if any, exists when the conditions above referred to are not satisfied.

The late Professor Boole has given an elegant method of treating a system of linear partial differential equations of the first order; but I am not aware that any one has considered the case of a non-linear system.

Let us begin with the case in which the dependent variable z is not explicitly involved in the proposed system, which can therefore be presented in the form

$$\left.\begin{aligned} &f_1(x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r})=0, \\ &f_2(x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r})=0, \\ &\vdots \\ &f_n(x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r})=0, \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(i)}$$

where $p_1=\dfrac{dz}{dx_1}$, $p_2=\dfrac{dz}{dx_2}$, &c. ..., the number of equations being n, and the number of independent variables being $n+r$. It is assumed that $f_1, f_2, \ldots f_n$, considered as functions of $x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r}$, are mutually independent; if $f_1, f_2, \ldots f_n$ were not mutually independent, then, provided the given system were a consistent one, we could replace it by a new system containing a less number of equations.

Now the existence of a solution involves the supposition that values of $p_1, p_2, \ldots p_{n+r}$ can be found which will satisfy the equations (i) and at

* See Proc. Roy. Soc. vol. xxiii. p. 510.

the same time render the expression

$$p_1dx_1+p_2dx_2+\ldots+p_{n+r}dx_{n+r} \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

a perfect differential, dz.

First suppose the functions $f_1, f_2, \ldots f_n$ are such that for every pair the condition

$$[f_i, f_j]=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(iii)*}$$

is identically satisfied. If we determine r new functions

$$f_{n+1}, f_{n+2}, \ldots f_{n+r} \text{ of } x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r}$$

such that for every pair of the whole series $f_1, f_2, \ldots f_r$ the condition (iii) is satisfied, then the values of $p_1, p_2, \ldots p_{n+r}$ derived from the n equations (i) and the r equations

$$\left.\begin{array}{l} f_{n+1}(x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r})=a_1, \\ \vdots \\ f_{n+r}(x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r})=a_r, \end{array}\right\} \quad \ldots\ldots \quad \text{(iv)}$$

where $a_1, a_2, \ldots a_r$ are any r arbitrary constants, will render the expression (ii) a perfect differential, and the value of z found by integration, viz.

$$z=\phi(x_1, x_2, \ldots x_{n+r}, a_1, a_2, \ldots a_r)+b, \quad \ldots\ldots\ldots \quad \text{(v)}$$

b being a new arbitrary constant, will satisfy the given system (i), and will be a complete primitive in the sense above defined.

Now the determination of $f_{n+1}, f_{n+2}, \ldots f_{n+r}$ under the above conditions is a part of the problem considered in Boole's 'Differential Equations,' Supplementary volume, p. 115: the determination is there shown to be possible, and a method is given for effecting it. Hence we see that when the $\frac{1}{2}n(n-1)$ conditions $[f_i, f_j]=0$ are satisfied, the proposed system has a complete primitive.

When the given system is a linear one, these conditions are identical with those used by Boole, p. 81.

Next suppose that the condition $[f_i, f_j]=0$ is not satisfied for every pair of the functions $f_1, f_2, \ldots f_n$. Let the expressions of the form $[f_i, f_j]$ which are not zero be denoted by $\phi_1, \phi_2, \phi_3, \ldots$; then it is plain that no relation can be found which will satisfy the proposed system (i) without at the same satisfying the system of equations

$$\phi_1=0, \quad \phi_2=0, \quad \phi_3=0, \ldots;$$

hence the required solution must be sought for as the most general solution of the system

$$f_1=0, f_2=0, \ldots f_n=0, \phi_1=0, \phi_2=0, \phi_3=0, \ldots;$$

* Adopting the notation of Boole, Donkin, and others, the symbol $[f_i, f_j]$ is used as an abbreviation for the expression

$$\Sigma_k\left(\frac{df_i}{dx_k}\frac{df_j}{dp_k}-\frac{df_i}{dp_k}\frac{df_j}{dx_k}\right),$$

the summation extending from $k=1$ to $k=n+r$ inclusive.

if this system is inconsistent no solution exists; if it be consistent, and the functions

$$f_1, f_2, \ldots f_n, \quad \phi_1, \phi_2, \phi_3, \ldots$$

are not mutually independent, let it be replaced by the equivalent system

$$f_1=0, f_2=0, \ldots f_n=0, f_{n+1}=0, \ldots f_{n+s}=0,$$

where $f_1, f_2, \ldots f_{n+s}$ are mutually independent. There are now three cases to be considered.

I. If s be greater than r, then no solution exists.

II. If $s=r$, we have $n+r$ equations to find $p_1, p_2, \ldots p_{n+r}$ in terms of $x_1, x_2, \ldots x_{n+r}$: if the values thus found are such as to make (ii) a perfect differential, that is if the functions $f_1, f_2, \ldots f_{n+r}$ are such that for every pair the condition (iii) is satisfied, then we have an integral of the form

$$z=\phi(x_1, x_2, \ldots x_{n+r})+b, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(vi)}$$

containing the single arbitrary constant b. If the conditions are not satisfied then there is no solution.

III. If s be less than r, we have a system similar to the original one, only containing s more equations.

We may therefore apply the above process to this system, and so either demonstrate the non-existence of a solution, or find a complete primitive of the form

$$z=\phi(x_1, x_2, \ldots x_{n+r}, a_1, a_2, \ldots a_{r-s})+b,$$

that is, an integral of the form (vi), or fall upon a new system analogous to the given system (i), only containing more equations than either of the previous systems.

By continually repeating this process, it is seen that we must either arrive at a solution or prove that a solution does not exist.

We have now to consider the case in which the dependent variable z is explicity involved in the proposed system, which is therefore presentable in the form

$$\left.\begin{array}{l} f_1(z, x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r})=0, \\ \vdots \\ f_n(z, x_1, x_2, \ldots x_{n+r}, p_1, p_2, \ldots p_{n+r})=0. \end{array}\right\} \quad \ldots\ldots\ldots \quad \text{(vii)}$$

Now let

$$\phi(z, x_1, x_2, \ldots x_{n+r})=0 \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(viii)}$$

be any relation between the primitive variables which satisfies the given system (vii). Differentiating (viii) with respect to each of the $n+r$ independent variables $x_1, x_2, \ldots x_{n+r}$, we have

$$\frac{d\phi}{dx_1}+p_1\frac{d\phi}{dz}=0, \ldots \frac{d\phi}{dx_{n+r}}+p_{n+r}\frac{d\phi}{dz}=0.$$

Hence, determining $p_1, \ldots p_{n+r}$, and substituting in the proposed system (vii), we get a system of n equations of which the type is

$$f\left(z, x_1, x_2, \ldots x_{n+r}, -\frac{\frac{d\phi}{dx_1}}{\frac{d\phi}{dz}}, -\frac{\frac{d\phi}{dx_2}}{\frac{d\phi}{dz}}, \ldots -\frac{\frac{d\phi}{dx_{n+r}}}{\frac{d\phi}{dz}}\right)=0. \quad \ldots\ldots \text{(ix)}$$

Hence ϕ considered as a function of the $n+r+1$ independent variables $z_1, x, x_2, \ldots x_{n+r}$ must satisfy a system of n partial differential equations of the first order; and in this system the dependent variable does not appear explicitly. If, then, there be any value of ϕ which satisfies the system, it can be found by the method given above; and provided it involve z in its expression, the value of z found from the equation $\phi=0$ will be a solution of the original system. If z does not occur in the expression for ϕ, then the proposed system can have no solution.

Suppose the system of equations of which (ix) is the type to have a complete primitive of the form

$$\phi=\psi(z, x_1, x_2, \ldots x_{n+r}, a_1, a_2, \ldots a_{r+1})+a_{r+2}$$

containing the $r+2$ arbitrary constants $a_1, a_2, \ldots a_{r+2}$. Then the equation $\phi=0$ gives us the value of z in the form

$$z=\theta(x_1, x_2, \ldots x_{n+r}, a_1, a_2, \ldots a_{r+2}). \quad \ldots\ldots\ldots\ldots \text{(x)}$$

And it is to be observed that this value appears to contain one more than the number of arbitrary constants indicated by the theory of the genesis of the system (vii) as the proper number. But from the fact that (x) satisfies the system (vii) of n equations, it follows that the constants a_1, $a_2, \ldots a_{r+2}$ must be virtually equivalent to $r+1$ constants only. An instance of this occurs in the first example given below.

The results of the preceding inquiry may be collected into the following rules.

Given a system of n non-linear partial differential equations of the first order in $n+r$ independent variables, and in which the dependent variable does not explicitly appear, to find the nature of the possible solution.

Rule. Let the equations by algebraical reduction be brought to the form

$$\left.\begin{array}{l} f_1 \equiv p_1+F_1(x_1, \ldots x_{n+r}, p_{n+1}, \ldots p_{n+r})=0, \\ \vdots \\ f_n \equiv p_n+F_n(x_1, \ldots x_{n+r}, p_{n+1}, \ldots p_{n+r})=0, \end{array}\right\} \ldots\ldots\ldots \text{(xi)}$$

and examine whether the condition

$$[f_i, f_j] \equiv \frac{dF_i}{dx_j}-\frac{dF_j}{dx_i}+\sum_{k=1}^{k=r}\left(\frac{dF_i}{dx_{n+k}}\cdot\frac{dF_j}{dp_{n+k}}-\frac{dF_j}{dx_{n+k}}\cdot\frac{dF_i}{dp_{n+k}}\right)=0 \,. \text{(xii)}$$

is *identically* satisfied for every pair of the functions $F_1, \ldots F_n$. If it be so, then the system will have a complete primitive of the form

$$z=\phi(x_1, x_2, \ldots x_{n+r}, a_1, a_2, \ldots a_r)+b, \quad \ldots\ldots\ldots \text{(xiii)}$$

which may be formed by the method of Boole referred to. But if any

one of the equations of condition (xii) be not satisfied *identically*, it will constitute a new partial differential equation to be combined with the given system (xi). Let such combination be presented in the form

$$\left.\begin{array}{l} f_1 \equiv p_1 \quad + F_1 \quad (x_1, \ldots x_{n+r}, p_{n+2}, \ldots p_{n+r}) = 0, \\ \vdots \\ f_{n+1} \equiv p_{n+1} + F_{n+1}(x_1, \ldots x_{n+r}, p_{n+2}, \ldots p_{n+r}) = 0. \end{array}\right\} \ldots\ldots \text{(xiv)}$$

Treating this system in the same way as we have just done (xi), we either get a solution of the form

$$z = \phi(x_1, \ldots x_{n+r}, a_1, \ldots a_{r-1}) + b,$$

or fall upon a system of $n+2$ equations analogous to (xi) and (xiv). Proceeding in this way we must finally arrive at a solution of the form

$$z = \phi(x_1, x_2, \ldots x_{n+r}, a_1, a_2, \ldots a_s) + b,$$

where n is less than r, or else we shall have the system

$$\begin{array}{l} p_1 \quad + F_1 \quad (x_1, x_2, \ldots x_{n+r}) = 0, \\ \vdots \\ p_{n+r} + F_{n+r}(x_1, x_2, \ldots x_{n+r}) = 0\,; \end{array}$$

and if the $\frac{1}{2}(n+r)(n+r-1)$ conditions

$$\frac{dF_i}{dx_j} - \frac{dF_j}{dx_i} = 0$$

are satisfied, we have a solution

$$z = \phi(x_1, x_2, \ldots x_{n+r}) + b\,;$$

but if these conditions are not satisfied, then there is no solution.

Example 1. Required the integral of the simultaneous equations

$$z^2 = \left(\frac{dz}{dx_1}\right)^2 + \cdots + \left(\frac{dz}{dx_n}\right)^2,$$

$$z = b_1 \frac{dz}{dx_1} + \ldots b_n \frac{dz}{dx_n}.$$

Let

$$\phi(z, x_1, \ldots x_n) = 0$$

be any integral, and let

$$p = \frac{d\phi}{dz}, \quad p_r = \frac{d\phi}{dx_r}\,;$$

then

$$\begin{array}{l} 2f_1 \equiv \ \ p_1^2 + \ p_2^2 + \ldots + \ \ p_n^2 - z^2 p^2 = 0, \\ f_2 \equiv b_1 p_1 + b_2 p_2 + \ldots + b_n p_n + zp = 0, \end{array}$$

and the condition $[f_1, f_2] = 0$ is satisfied. Also it is easily seen that the functions $f_3 = p_1, f_4 = p_2, \ldots f_{n+1} = p_{n-1}$ satisfy the requisite conditions. Hence we take

$$p_1 = a_1, \quad p_2 = a_2, \quad \ldots p_n = a_n,$$

$$p = \frac{\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}}{z},$$

$a_1, a_2, \ldots a_{n-1}$ being arbitrary constants, and a_n determined by the conditions

$$a_1b_1+a_2b_2+\ \ldots\ +a_nb_n+\sqrt{a_1^2+a_2^2+\ldots+a_n^2}=0.$$

These values of $p_1 \ldots p_n, p$ give

$$d\phi=a_1dx_1+\ldots+a_ndx^2+\sqrt{\Sigma a^2}.\frac{dz}{z};$$

$$\therefore\ \phi=a_1x_1+\ldots+a_nx_n+\sqrt{\Sigma a^2}.\log z+C;$$

whence the solution required is

$$a_1x_1+\ldots+a_nx_n+\sqrt{\Sigma a^2}.\log z+C=0,$$

which is equivalent to

$$\log z=c_1x_1+\ldots+c_nx_n+D,$$

D being arbitrary, and $c_1 \ldots c_n$ connected by the equations

$$\begin{aligned} c_1^2+\ c_2^2+\ldots+\ c_n^2&=1,\\ b_1c_1+b_2c_2+\ldots+b_nc_n&=1. \end{aligned}$$

Since there are $n-1$ arbitrary constants, we have a "complete primitive" as defined above.

Example 2. Have the simultaneous equations

$$z^2=\left(\frac{dz}{dx_1}\right)^2+\ldots+\left(\frac{dz}{dx_n}^2\right),$$

$$1=b_1\frac{dz}{dx_1}+\ldots+b_n\frac{dz}{dx_n}$$

any solution?

Proceeding as in the last example, we find

$$\begin{aligned} 2f_1&\equiv p_1^2+p_2^2+\ldots+p_n^2-z^2p^2=0,\\ f_2&\equiv b_1p_1+b_2p_2+\ldots+b_np_n+p=0, \end{aligned}$$

and the condition $[f_1,f_2]=0$ is not satisfied. Accordingly we write

$$f_3\equiv[f_2,f_1]\equiv zp^2=0.$$

From these we find

$$[f_1,f_3]=z^2p^3,\quad [f_2,f_3]=p^2,$$

and are thence led to the equivalent system,

$$\begin{aligned} 2f_1&\equiv p_1^2+p_2^2+\ldots+p_n^2=0,\\ f_2&\equiv b_1p_1+b_2p_2+\ldots+b_np_n=0,\\ f_3&\equiv p=0. \end{aligned}$$

Also it is easily seen that the functions

$$f_4=p_1,\quad f_5=p_2,\ \ldots f_{n+1}=p_{n-2}$$

satisfy the conditions

$$[f_i, f_j]=0.$$

Hence we take

$$p_1=a_1, \quad p_2=a_2, \ldots, p_n=a_n;$$

$a_1, a_2, \ldots a_{n-2}$ being arbitrary constants, and a_{n-1}, a_n being determined by the conditions

$$a_1^2 + a_2^2 + \ldots + a_n^2 = 0,$$

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = 0;$$

and we find

$$d\phi = a_1 dx_1 + a_2 dx_2 + \ldots + a_n dx_n,$$

and therefore

$$\phi = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n + C.$$

As this value of ϕ does not involve z, there can be no solution of the given equations. But the work has shown that the simultaneous equations

$$\left.\begin{aligned} \left(\frac{d\phi}{dx_1}\right)^2 + \left(\frac{d\phi}{dx_2}\right)^2 + \ldots + \left(\frac{d\phi}{dx_n}\right)^2 - z^2\left(\frac{d\phi}{dz}\right)^2 &= 0, \\ b_1\frac{d\phi}{dx_1} + b_2\frac{d\phi}{dx_2} + \ldots + b_2\frac{d\phi}{dx_n} + \frac{d\phi}{dz} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (\alpha)$$

have an "integral"

$$\phi = a_1 x_1 + \ldots + a_n x_n + C$$

containing $n-1$ arbitrary constants; and that the system of simultaneous equations formed by (α) and

$$\frac{d\phi}{dz} = 0$$

has the same equation for a "complete primitive."

Example 3. Find the nature of the solution of the simultaneous equations

$$x\frac{du}{dx}\left(\frac{du}{dy} + y\right) - \frac{du}{dz} - z = 0,$$

$$\left(\frac{du}{dy} - z\right)^2 + \left(\frac{du}{dz} - y\right)^2 - 2xz\frac{du}{dx} - 2y = 0.$$

Let

$$p = \frac{du}{dx}, \quad q = \frac{du}{dy}, \quad r = \frac{du}{dz},$$

$$\left.\begin{aligned} f_1 &\equiv xp(q+y) - r - z = 0, \\ \tfrac{1}{2}f_2 &\equiv (q-z)^2 + (r-y)^2 - 2xzp - 2y = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots (\beta)$$

Then we find

$$[f_1, f_2] = (px-1)(q+r-y-z).$$

Accordingly the condition $[f_1, f_2]=0$ requires that

$$px - 1 = 0,$$

or else that

$$q + r - y - z = 0.$$

Now the first of these taken along with the equations (β) leads to the equivalent system

$$f_1 \equiv px-1=0,$$
$$f_2 \equiv q-z-r+y=0,$$
$$f_3 \equiv (q-z)^2-y-z=0.$$

For this system the conditions

$$[f_2, f_1]=0, \quad [f_3, f_1]=0, \quad [f_1, f_2]=0$$

are satisfied; and the values of p, q, r drawn from it being

$$p=\frac{1}{x}, \quad q=z\pm\sqrt{y+z}, \quad r=y\pm\sqrt{y+z},$$

we find an integral

$$u=\log x+yz\pm\tfrac{2}{3}(y+z)+C.$$

The second equation,

$$q+r-y-z=0,$$

does not lead to any solution.

Example 4. Find the possible solution of the two simultaneous equations

$$f_1 \equiv x^2p+y^2+zr^2-x+y=0,$$
$$f_2 \equiv x^2py^2+x^2yp-xy^2-zr^2-xy=0.$$

Here the condition $[f_1, f_2]=0$ is satisfied; we have then to find a function f_3 such that

$$[f_1, f_3]=0, \quad [f_2, f_3]=0,$$

and it will be easily found that

$$f_3=q^2+y$$

is a common integral. We have then

$$q^2+y=a;$$

and p, r must be found from

$$x^2p-x+zr^2=-a,$$
$$(x^2p-x)a-zr^2=0;$$

whence

$$p=-\frac{a}{(1+a)x^2}+\frac{1}{x},$$
$$q=\sqrt{a-y},$$
$$r=-\sqrt{\frac{-a^2}{(1+a)z}};$$

and we get for a complete primitive

$$u=\frac{a}{(1+a)x}+\log x-\tfrac{2}{3}(a-y)^{\frac{3}{2}}+2\sqrt{\frac{-a^2z}{1+a}}+C.$$