V. "Mechanical Integration of the Linear Differential Equations of the Second Order with Variable Coefficients." By Prof. Sir William Thomson, LL.D., F.R.S. Received January 28, 1876.

Every linear differential equation of the second order may, as is known, be reduced to the form

$$\frac{d}{dx}\left(\frac{1}{\mathrm{P}}\frac{du}{dx}\right)=u, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P is any given function of x.

On account of the great importance of this equation in mathematical physics (vibrations of a non-uniform stretched cord, of a hanging chain, of water in a canal of non-uniform breadth and depth, of air in a pipe of non-uniform sectional area, conduction of heat along a bar of non-uniform section or non-uniform conductivity, Laplace's differential equation of the tides, &c. &c.), I have long endeavoured to obtain a means of facilitating its practical solution.

Methods of calculation such as those used by Laplace himself are exceedingly valuable, but are very laborious, too laborious unless a serious object is to be attained by calculating out results with minute accuracy. A ready means of obtaining approximate results which shall show the general character of the solutions, such as those so well worked out by Sturm*, has always seemed to me a desideratum. Therefore I have made many attempts to plan a mechanical integrator which should give solutions by successive approximations. This is clearly done now, when we have the instrument for calculating $\int\phi(x)\,\psi(x)\,dx$, founded on my brother's disk-, globe-, and cylinder-integrator, and described in a previous communication to the Royal Society; for it is easily proved† that if

$$\left.\begin{aligned} u_2&=\int_0^x \mathrm{P}\,(\mathrm{C}-\int_0^x u_1dx)\,dx,\\ u_3&=\int_0^x \mathrm{P}\,(\mathrm{C}-\int_0^x u_2dx)\,dx, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

&c.,

where u_1 is any function of x, to begin with, as for example $u_1=x$; then u_2, u_3, &c. are successive approximations converging to that one of the solutions of (1) which vanishes when $x=0$.

Now let my brother's integrator be applied to find $\mathrm{C}-\int_0^x u_1dx$, and let its result feed, as it were, continuously a second machine, which shall find the integral of the product of its result into P dx. The second machine

* "Mémoire sur les équations différentielles linéaires du second ordre," Liouville's Journal, vol. i. 1836.

† Cambridge Senate-House Examination, Thursday afternoon, January 22nd, 1874.

will give out continuously the value of u_2. Use again the same process with u_2 instead of u_1, and then u_3, and so on.

After thus altering, as it were, u_1 into u_2 by passing it through the machine, then u_2 into u_3 by a second passage through the machine, and so on, the thing will, as it were, become refined into a solution which will be more and more nearly rigorously correct the oftener we pass it through the machine. If u_{i+1} does not sensibly differ from u_i, then each is sensibly a solution.

So far I had gone and was satisfied, feeling I had done what I wished to do for many years. But then came a pleasing surprise. Compel agreement between the function fed into the double machine and that given out by it. This is to be done by establishing a connexion which shall cause the motion of the centre of the globe of the first integrator of the double machine to be the same as that of the surface of the second integrator's cylinder. The motion of each will thus be necessarily a solution of (1). Thus I was led to a conclusion which was quite unexpected; and it seems to me very remarkable that the general differential equation of the second order with variable coefficients may be rigorously, continuously, and in a single process solved by a machine.

Take up the whole matter *ab initio*: here it is. Take two of my brother's disk-, globe-, and cylinder-integrators, and connect the fork which guides the motion of the globe of each of the integrators, by proper mechanical means, with the circumference of the other integrator's cylinder. Then move one integrator's disk through an angle $=x$, and simultaneously move the other integrator's disk through an angle always $=\int_0^x \mathrm{P}\,dx$, a given function of x. The circumference of the second integrator's cylinder and the centre of the first integrator's globe move each of them through a space which satisfies the differential equation (1).

To prove this, let at any time g_1, g_2 be the displacements of the centres of the two globes from the axial lines of the disks; and let dx, $\mathrm{P}\,dx$ be infinitesimal angles turned through by the two disks. The infinitesimal motions produced in the circumferences of two cylinders will be

$$g_1 dx \text{ and } g_2 \mathrm{P}\,dx.$$

But the connexions pull the second and first globes through spaces respectively equal to those moved through by the circumferences of the first and second cylinders. Hence

$$g_1 dx = dg_2, \text{ and } g_2 \mathrm{P}\,dx = dg_1;$$

and eliminating g_2,

$$\frac{d}{dx}\left(\frac{1}{\mathrm{P}}\frac{dg_1}{dx}\right) = g_1,$$

which shows that g_1 put for u satisfies the differential equation (1).

The machine gives the complete integral of the equation with its two arbitrary constants. For, for any particular value of x, give arbitrary values G_1, G_2. [That is to say mechanically; disconnect the forks from the cylinders, shift the forks till the globes' centres are at distances G_1, G_2 from the axial lines, then connect, and move the machine.]

We have for this value of x,

$$g_1 = G_1, \text{ and } \frac{dg_1}{dx} = G_2 P;$$

that is, we secure arbitrary values for g_1 and $\frac{dg_1}{dx}$ by the arbitrariness of the two initial positions G_1, G_2 of the globes.

VI. "Mechanical Integration of the general Linear Differential Equation of any Order with Variable Coefficients." By Prof. Sir WILLIAM THOMSON, LL.D., F.R.S. Received January 28, 1876.

Take any number i of my brother's disk-, globe-, and cylinder-integrators, and make an integrating chain of them thus:—Connect the cylinder of the first so as to give a motion equal to its own* to the fork of the second. Similarly connect the cylinder of the second with the fork of the third, and so on. Let g_1, g_2, g_3, up to g_i, be the positions† of the globes at any time. Let infinitesimal motions P_1dx, P_2dx, P_3dx, ... be given simultaneously to all the disks (dx denoting an infinitesimal motion of some part of the mechanism whose displacement it is convenient to take as independent variable). The motions ($d\kappa_1$, $d\kappa_2$, ... $d\kappa_i$) of the cylinders thus produced are

$$d\kappa_1 = g_1 P_1 dx,\ d\kappa_2 = g_2 P_2 dx, \ldots d\kappa_i = g_i P_i dx. \quad . \quad . \quad . \quad (1)$$

But, by the connexions between the cylinders and forks which move the globes, $d\kappa_1 = dg_2$, $d\kappa_2 = dg_3$, ... $d\kappa_{i-1} = dg_i$; and therefore

and

$$\left.\begin{array}{l} dg_2 = g_1 P_1 dx,\ dg_3 = g_2 P_2 dx, \ldots dg_i = g_{i-1} P_{i-1} dx \\ d\kappa_1 = g_1 P_1 dx,\ d\kappa_2 = g_2 P_2 dx, \ldots d\kappa_i = g_i P_i dx. \end{array}\right\} \quad . \quad . \quad (2)$$

Hence

$$g_1 = \frac{1}{P_1}\frac{d}{dx}\frac{1}{P_2}\frac{d}{dx}\cdots\frac{1}{P_{i-1}}\frac{d}{dx}\frac{1}{P_i}\frac{dx_i}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Suppose, now, for the moment that we couple the last cylinder with the

* For brevity, the motion of the circumference of the cylinder is called the cylinder's motion.

† For brevity, the term "position" of any one of the globes is used to denote its distance, positive or negative, from the axial line of the rotating disk on which it presses.