III. "On the Expression of the Product of any two Legendre's Coefficients by means of a Series of Legendre's Coefficients." By Professor J. C. ADAMS, M.A., F.R.S. Received November 22, 1877.

The expression for the product of two Legendre's coefficients which is the subject of the present paper, was found by induction on the 13th of February, 1873, and on the following day I succeeded in proving that the observed law of formation of this product held good generally. Having considerably simplified this proof, I now venture to offer it to the Royal Society; and, for the sake of completeness, I have prefixed to it the whole of the inductive process by which the theorem was originally arrived at, although for the proof itself only the first two steps of this process are required. The theorem seems to deserve attention, both on account of its elegance, and because it appears to be capable of useful applications.

As usual let Legendre's nth coefficient be denoted by P_n, then P_n may be defined by the equation

$$P_n=\frac{1}{2^n\underline{|n}}\cdot\frac{d^n}{d\mu^n}(\mu^2-1)^n$$

It is well known that the following relation holds good between three consecutive values of the functions P, viz.:

$$(n+1)P_{n+1}=(2n+1)\mu P_n-nP_{n-1}$$

Now $P_1=\mu$

$$\therefore\ P_1P_n=\frac{n+1}{2n+1}P_{n+1}+\frac{n}{2n+1}P_{n-1}$$

Again, we have $$P_2=\frac{3}{2}\mu P_1-\frac{1}{2}$$

$$\therefore\ P_2P_n=\frac{3}{2}\mu P_1P_n-\frac{1}{2}P_n$$

$$=\frac{3}{2}\frac{n+1}{2n+1}\mu P_{n+1}+\frac{3}{2}\frac{n}{2n+1}\mu P_{n-1}-\frac{1}{2}P_n$$

Substitute for μP_{n+1} and μP_{n-1} their equivalents obtained by writing $n+1$ and $n-1$ successively for n in the above formula

$$\therefore\ P_2P_n=\frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2}$$

$$+\left\{\frac{3}{2}\frac{(n+1)^2}{(2n+1)(2n+3)}-\frac{1}{2}+\frac{3}{2}\frac{n^2}{(2n-1)(2n+1)}\right\}P_n$$

$$+\frac{3}{2}\frac{(n-1)n}{(2n-1)(2n+1)}P_{n-2}$$

By a slight reduction the coefficient of P_n becomes

$$\frac{n(n+1)}{(2n-1)(2n+3)}$$

Hence

$$P_2P_n=\frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2}+\frac{n(n+1)}{(2n-1)(2n+3)}P_n$$

$$+\frac{3}{2}\frac{(n-1)n}{(2n-1)(2n+1)}P_{n-2}$$

Again, putting $n=2$ in our original formula, we have

$$P_3=\frac{5}{3}\mu P_2-\frac{2}{3}P_1$$

$$\therefore\ P_3P_n=\frac{5}{3}\mu P_2P_n-\frac{2}{3}P_1P_n$$

$$=\frac{5}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}\mu P_{n+2}+\frac{5}{3}\frac{n(n+1)}{(2n-1)(2n+3)}\mu P_n$$

$$+\frac{5}{2}\frac{(n-1)n}{(2n-1)(2n+1)}\mu P_{n-2}-\frac{2}{3}\frac{n+1}{2n+1}P_{n+1}-\frac{2}{3}\frac{n}{2n+1}P_{n-1}$$

Substitute for μP_{n+2}, μP_n and μP_{n-2} their equivalents as before

$$\therefore\ P_3P_n=\frac{5}{2}\frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)}P_{n+3}$$

$$+\left\{\frac{5}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}\frac{n+2}{2n+5}+\frac{5}{3}\frac{n(n+1)}{(2n-1)(2n+3)}\frac{n+1}{2n+1}-\frac{2}{3}\frac{n+1}{2n+1}\right\}P_{n+1}$$

$$+\left\{\frac{5}{3}\frac{n(n+1)}{(2n-1)(2n+3)}\frac{n}{2n+1}+\frac{5}{2}\frac{(n-1)n}{(2n-1)(2n+1)}\frac{n-1}{2n-3}-\frac{2}{3}\frac{n}{2n+1}\right\}P_{n-1}$$

$$+\frac{5}{2}\frac{(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)}P_{n-3}$$

By reduction the coefficient of P_{n+1} in this expression becomes

$$\frac{3}{2}\frac{n(n+1)(n+2)}{(2n-1)(2n+1)(2n+5)}$$

and similarly the coefficient of P_{n-1} becomes

$$\frac{3}{2}\frac{(n-1)n(n+1)}{(2n-3)(2n+1)(2n+3)}$$

Hence we have

$$P_3P_n=\frac{5}{2}\frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)}P_{n+3}$$
$$+\frac{3}{2}\frac{n(n+1)(n+2)}{(2n-1)(2n+1)(2n+5)}P_{n+1}$$
$$+\frac{3}{2}\frac{(n-1)n(n+1)}{(2n-3)(2n+1)(2n+3)}P_{n-1}$$
$$+\frac{5}{2}\frac{(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)}P_{n-3}$$

Again since
$$P_4=\frac{7}{4}\mu P_3-\frac{3}{4}P_2$$

we have
$$P_4P_n=\frac{7}{4}\mu(P_3P_n)-\frac{3}{4}(P_2P_n)$$

Whence by substituting the values found above for $P_3\,P_n$ and $P_2\,P_n$ and again for μP_{n+3}, $\mu\,P_{n+1}$, &c., we obtain

$$P_4P_n=\frac{5\,.\,7}{2\,.\,4}\frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)}\left\{\frac{n+4}{2n+7}P_{n+4}+\frac{n+3}{2n+7}P_{n+2}\right\}$$
$$+\frac{3\,.\,7}{2\,.\,4}\frac{n(n+1)(n+2)}{(2n-1)(2n+1)(2n+5)}\left\{\frac{n+2}{2n+3}P_{n+2}+\frac{n+1}{2n+3}P_n\right\}$$
$$+\frac{3\,.\,7}{2\,.\,4}\frac{(n-1)n(n+1)}{(2n-3)(2n+1)(2n+3)}\left\{\frac{n}{2n-1}P_n+\frac{n-1}{2n-1}P_{n-2}\right\}$$
$$+\frac{5\,.\,7}{2\,.\,4}\frac{(n-2)(n-1)n}{(2n-3)(2n-1)(2n+1)}\left\{\frac{n-2}{2n-5}P_{n-2}+\frac{n-3}{2n-5}P_{n-4}\right\}$$
$$-\frac{3\,.\,3}{2\,.\,4}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2}-\frac{3}{4}\frac{n(n+1)}{(2n-1)(2n+3)}P_n$$
$$-\frac{3\,.\,3}{2\,.\,4}\frac{(n-1)n}{(2n-1)(2n+1)}P_{n-2}$$

By reduction, the coefficient of P_{n+2} in this expression becomes

$$\frac{5}{2}\frac{n(n+1)(n+2)(n+3)}{(2n-1)(2n+1)(2n+3)(2n+7)}$$

Similarly, the coefficient of P_{n-2} becomes

$$\frac{5}{2}\frac{(n-2)(n-1)n(n+1)}{(2n-5)(2n-1)(2n+1)(2n+3)}$$

and finally, the coefficient of P_n becomes

$$\left(\frac{3}{2}\right)^2\frac{(n-1)n(n+1)(n+2)}{(2n-3)(2n-1)(2n+3)(2n+5)}$$

Hence, collecting the terms, we have

$$P_4P_n=\frac{1.3.5.7}{1.2.3.4}\frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)(2n+5)(2n+7)}P_{n+4}$$

$$+\frac{1.3.5}{1.2.3}\cdot\frac{1}{1}\frac{n(n+1)(n+2)(n+3)}{(2n-1)(2n+1)(2n+3)(2n+7)}P_{n+2}$$

$$+\frac{1.3}{1.2}\cdot\frac{1\cdot3}{1.2}\frac{(n-1)n(n+1)(n+2)}{(2n-3)(2n-1)(2n+3)(2n+5)}P_n$$

$$+\frac{1}{1}\cdot\frac{1.3.5}{1.2.3}\frac{(n-2)(n-1)n(n+1)}{(2n-5)(2n-1)(2n+1)(2n+3)}P_{n-2}$$

$$+\frac{1.3.5.7}{1.2.3.4}\frac{(n-3)(n-2)(n-1)n}{(2n-5)(2n-3)(2n-1)(2n+1)}P_{n-4}$$

where the law of the terms is obvious, except perhaps as regards the succession of the factors in the several denominators.

With respect to this it may be observed that the factors in the denominator of any term P_p are obtained by omitting the factor $2p+1$ from the regular succession of five factors $(n+p-3)$ $(n+p-1)$ $(n+p+1)$ $(n+p+3)$ $(n+p+5)$.

For instance, where $p=n+4$, $2p+1=2n+9$, so that the factor $2n+9$ is to be omitted, and we have $2n+1$, $2n+3$, $2n+5$ and $2n+7$, as the remaining factors, and so of the rest.

Hence by induction we may write, supposing to fix the ideas that m is not greater than n

$$P_mP_n=\frac{1.3.5..(2m-1)}{1.2.3...\quad m}\cdot\frac{(n+1)(n+2)....(n+m)}{(2n+1)(2n+3)...(2n+2m+1)}$$
$$\times[(2n+2m+1)P_{n+m}]$$

$$+\frac{1.3.5..(2m-3)}{1.2.3..(m-1)}\cdot\frac{1}{1}\cdot\frac{n(n+1)...(n+m-1)}{(2n-1)(2n+1)...(2n+2m-1)}$$
$$\times[(2n+2m-3)P_{n+m-2}]$$

$$+\ \&c.\qquad \&c.$$

$$+\frac{1.3.5...(2m-2r-1)}{1.2.3...(m-r)}\cdot\frac{1.3.5...(2r-1)}{1.2.3....\quad r}$$
$$\times\frac{(n-r+1)(n-r+2)...(n-r+m)}{(2n-2r+1)(2n-2r+3)...(2n-2r+2m+1)}$$
$$\times[(2n+2m-4r+1)P_{n+m-2r}]$$

$$+\ \&c.\qquad \&c.$$

$$+\frac{1}{1}\cdot\frac{1.3.5..(2m-3)}{1.2.3..(m-1)}\cdot\frac{(n-m+2)(n-m+3)...(n+1)}{(2n-2m+3)(2n-2m+5)...(2n+3)}$$
$$\times[(2n-2m+5)P_{n-m+2}]$$

$$+\frac{1.3.5..(2m-1)}{1.2.3...\ m}\cdot\frac{(n-m+1)\,(n-m+2)\ldots n}{(2n-2m+1)\,(2n-2m+3)\ldots(2n+1)}$$
$$\times\left[(2n-2m+1)\,P_{n-m}\right]$$

And it remains to verify this observed law by proving that if it holds good for two consecutive values of m, it likewise hold good for the next higher value.

If the function $\frac{1.3.5\ldots(2m-1)}{1.2.3\ldots m}$ be denoted by $A(m)$, the general term of the above expression for P P_n may be very conveniently represented by

$$\frac{A(m-r)\,A(r)\,A(n-r)}{A(n+m-r)}\left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right)P_{n+m-2r}$$

r being an integer which varies from 0 to m.

The fundamental property of the function A is that

$$A\,(m+1)=\frac{2m+1}{m+1}A(m)$$

$$\text{or}\quad A(m)=\frac{m+1}{2m+1}\ A(m+1)$$

We may interpret $A(m)$ when m is zero or a negative integer, by supposing this relation to hold good generally, so that putting $m=0$, we have

$$A(0)=A(1)=1$$

Similarly $$A(-1)=\frac{0}{-1}A(0)=0$$

and hence the value of $A(m)$ when m is a negative integer will be always zero.

We will now proceed to the general proof of the theorem stated above.

Let $Q_{m,n}$ or Q_m simply, denote the quantity of which the general term is

$$\frac{A(m-r)A(r)A(n-r)}{A(n+m-r)}\left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right)P_{n+m-2r}$$

In this expression r is supposed to vary from 0 to m, but it may be remarked that if r be taken beyond those limits, for instance if $r=-1$, or $r=m+1$, then in consequence of the property of the function A above stated, the coefficient of the corresponding term will vanish. Hence practically we may consider r to be unrestricted in value.

Similarly, let Q_{m-1} denote the quantity of which the general term is

$$\frac{A(m-r)\,A(r-1)\,A(n-r+1)}{A(n+m-r)}\left(\frac{2n+2m-4r+3}{2n+2m-2r+1}\right)P_{n+m-2r+1}$$

writing $m-1$ for m and $r-1$ for r in the general term given above. Also let Q_{m+1} denote the quantity of which the general term is

$$\frac{A(m-r+1)\,A(r)\,A(n-r)}{A(n+m-r+1)}\left(\frac{2n+2m-4r+3}{2n+2m-2r+3}\right)P_{n+m-2r+1}$$

writing $m+1$ for m in the general term first given. In consequence of the evanescence of $A(m)$ when m is negative, we may in all these general terms suppose r to vary from 0 to $m+1$.

Let us assume that $Q_{m-1}=P_{m-1}P_n$, and also that $Q_m=P_mP_n$, then we have to prove that $Q_{m+1}=P_{m+1}P_n$.

As before, $(m+1)P_{m+1}+mP_{m-1}-(2m+1)\mu P_m=0$

$$\therefore\ (m+1)P_{m+1}P_n+mP_{m-1}P_n-(2m+1)\mu P_mP_n=0$$

Hence our theorem will be established if we prove that

$$(m+1)Q_{m+1}+mQ_{m-1}-(2m+1)\mu Q_m=0$$

Now
$$Q_m=\ \ldots\ldots$$
$$+\frac{A(m-r+1)A(r-1)A(n-r+1)}{A(n+m-r+1)}\left(\frac{2n+2m-4r+5}{2n+2m-2r+3}\right)P_{n+m-2r+2}$$
$$+\frac{A(m-r)\,A(r)\,A(n-r)}{A(n+m-r)}\left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right)P_{n+m-2r}$$
$$+\ \ldots\ldots$$

Multiplying by μ and substituting for $\mu P_{n+m-2r+2}$ and μP_{n+m-2r} &c., in terms of $P_{n+m-2r+1}$ &c., we find the coefficient of $P_{n+m-2r+1}$ in μQ_m to be

$$\frac{A(m-r+1)\,A(r-1)\,A(n-r+1)}{A(n+m-r+1)}\left(\frac{n+2m-2r+2}{2n+m-2r+3}\right)$$
$$+\frac{A(m-r)\,A(r)\,A(n-r)}{A(n+m-r)}\left(\frac{n+m-2r+1}{2n+2m-2r+1}\right)$$

Hence the coefficient of $P_{n+m-2r+1}$ in $(m+1)Q_{m+1}+mQ_{m-1}-(2m+1)\mu Q_m$ will be

$$\frac{A(m-r+1)A(r)A(n-r)}{A(n+m-r+1)}\,(m+1)\left(\frac{2n+2m-4r+3}{2n+2m-2r+3}\right)$$
$$-\frac{A(m-r+1)A(r-1)A(n-r+1)}{A(n+m-r+1)}\,(2m+1)\left(\frac{n+m-2r+2}{2n+2m-2r+3}\right)$$
$$-\frac{A(m-r)A(r)A(n-r)}{A(n+m-r)}(2m+1)\left(\frac{n+m-2r+1}{2n+2m-2r+1}\right)$$
$$+\frac{A(m-r)A(r-1)A(n-r+1)}{A(n+m-r)}\ m\left(\frac{2n+2m-4r+3}{2n+2m-2r+1}\right)$$

The sum of the first two lines of this expression is

$$\frac{A(m-r+1)\,A(r-1)\,A(n-r)}{A(n+m-r+1)\,(2n+2m-2r+3)}$$

$$\times\left\{\frac{2r-1}{r}(m+1)(2n+2m-4r+3)-\frac{2n-2r+1}{n-r+1}(2m+1)(n+m-2r+2)\right\}$$

Suppose for a moment that $n-r+1=q$, then the quantity within the brackets becomes

$$\frac{2r-1}{r}(m+1)\,(2m+1+2q-2r)-\frac{2q-1}{q}(2m+1)\,(m+1+q-r)$$

Now this quantity evidently vanishes when $q=r$, and therefore it is divisible by $q-r$. It also vanishes when $m+1=r$, and therefore it is likewise divisible by $m-r+1$.

Hence it is readily found that this quantity

$$=-\frac{q-r}{qr}(m-r+1)\,(2m+2q+1)$$

or $$=-\frac{n-2r+1}{r(n-r+1)}(m-r+1)\,(2n+2m-2r+3)$$

So that the sum of the first two lines of the expression for the coefficient of $P_{n+m-2r+1}$ is

$$-\frac{A(m-r+1)\,A(r-1)\,A(n-r)}{A(n+m-r+1)}\left\{\frac{(m-r+1)\,(n-2r+1)}{r(n-r+1)}\right\}$$

Again, the sum of the other two lines of the expression for the coefficient of $P_{n+m-2r+1}$, is

$$\frac{A(m-r)\,A(r-1)\,A(n-r)}{A(n+m-r)\,(2n+2m-2r+1)}$$

$$\times\left\{-\frac{2r-1}{r}(2m+1)\,(n+m-2r+1)+\frac{2n-2r+1}{n-r+1}m(2n+2m-4r+3)\right\}$$

As before suppose $n-r+1=q$, and the quantity within the brackets becomes

$$-\frac{2r-1}{r}(2m+1)\,(m+q-r)+\frac{2q-1}{q}m(2m+1+2q-2r)$$

Now this quantity evidently vanishes when $q=r$, so that it is divisible by $q-r$. It also vanishes when $m=-q$, and therefore it is likewise divisible by $m+q$.

Hence it is readily found that this quantity

$$=\frac{q-r}{qr}(q+m)\,(2m-2r+1)$$

or $$=\frac{n-2r+1}{r(n-r+1)}(n+m-r+1)\,(2m-2r+1)$$

and therefore the sum of the last two lines of the expression for the coefficient of $P_{n+m-2r+1}$ is

$$\frac{A(m-r)\,A(r-1)\,A(n-r)}{A(n+m-r)}$$
$$\times\left\{\frac{(n-2r+1)}{r(n-r+1)}\cdot\frac{(n+m-r+1)\,(2m-2r+1)}{2n+2m-2r+1}\right\}.$$

Hence the whole coefficient of $P_{n+m-2r+1}$ is

$$\frac{A(m-r)\,A(r-1)\,A(n-r)}{A(n+m-r+1)}\cdot\frac{n-2r+1)}{r(n-r+1}$$
$$\times\{2m-2r+1)-(2m-2r+1)\} = 0.$$

And the same holds good for the coefficient of every term. Hence we finally obtain

$$(m+1)Q_{n+1}+mQ_{m-1}-(2m+1)\mu Q_m=0,$$

which establishes the theorem above enunciated.

The principle of the process employed in the above proof may be thus stated:

Every term in the value of Q_m gives rise to two terms in the value of μQ_m or in that of $(2m+1)\mu Q_m$; one of these terms is to be subtracted from the corresponding term in $(m+1)Q_{m+1}$, and the other from the corresponding term in mQ_{m-1}, and it will be found that the two series of terms thus formed identically destroy each other.

Hence we can find at once the value of the definite integral

$$\int_{-1}^{1} P_m P_n P_p d\mu$$

for if $p=n+m-2r$ we have

$$P_m P_n = \ldots\ldots + \frac{A\left(\frac{m+p-n}{2}\right)A\left(\frac{n+m-p}{2}\right)A\left(\frac{n+p-m}{2}\right)}{A\left(\frac{n+m+p}{2}\right)}\cdot\frac{2p+1}{n+m+p+1}P_p$$
$$+\ \&c.$$

Hence
$$\int_{-1}^{1} P_m P_n P_p d\mu$$

$$A=\frac{\left(\frac{m+p-n}{2}\right)A\left(\frac{n+m-p}{2}\right)A\left(\frac{n+p-m}{2}\right)}{A\left(\frac{n+m+p}{2}\right)}\ \frac{2p+1}{n+m+p+1}\int_{-1}^{1}(P_p)^2 d\mu$$

$$=\frac{2}{n+m+p+1}\,A\frac{\left(\frac{m+p-n}{2}\right)A\left(\frac{n+m-p}{2}\right)A\left(\frac{n+p-m}{2}\right)}{A\left(\frac{n+m+p}{2}\right)}$$

or if $\frac{n+m+p}{2}=s$.

$$\int_{-1}^{1} P_m P_n P_p d\mu = \frac{2}{2s+1} \frac{A(s-m)A(s-n)A(s-p)}{A(s)}$$

where as above

$$A(m)=\frac{1 \,.\, 3 \,.\, 5 \,\ldots\, (2m-1)}{1 \,.\, 2 \,.\, 3 \,\ldots\, m} = 2^m \,.\, \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \left(m-\frac{1}{2}\right)}{1 \,.\, 2 \,.\, 3 \ldots m}$$

It is clear that, in order that this integral may be finite, no one of the quantities m, n, and p must be greater than the sum of the other two, and that $m+n+p$ must be an even integer.

I learn from Mr. Ferrers that, in the course of the year 1874, he likewise obtained the expression for the product of two Legendre's coefficients, by a method very similar to mine. In his work on "Spherical Harmonics," recently published, he gives, without proof, the above result for the value of the definite integral $\int_{-1}^{1} P_m P_n P_p d\mu$.

IV. "Experiments on the Colours shown by thin liquid Films under the Action of Sonorous Vibrations." By SEDLEY TAYLOR, M.A., late Fellow of Trinity College, Cambridge. Communicated by J. W. L. GLAISHER, M.A., F.R.S. Received December 12, 1877.

(Plates 5 and 6.)

Professor Helmholtz remarks, at page 603 of the fourth edition of his "Tonempfindungen," that a film of soapsuds and glycerine forms, when caused to occupy the orifice of one of his "resonators," an extremely sensitive means by which to make visible the vibrations of the air within its cavity.

While I was engaged in verifying this observation, my notice was attracted to the parallel bands of colour which traversed the film, and it occurred to me to examine whether the forms of these bands were affected by the sonorous vibrations which agitated the film. A few rough trials having convinced me that they were so affected, I at once proceeded to submit the phenomena which presented themselves to a closer examination.

Having caused a film to adhere to the circular aperture of a Helmholtz resonator, and allowed the fluid to drain off until the interference-colours became visible, I set the resonator, nipple downwards, in a stand, so that the film was exactly horizontal, and then stroked with a resined bow a tuning-fork of the same pitch mounted on its resonance