in the spectra of hydrogen vacua a notable difference in the lines seen in the negative light, sometimes all and sometimes only one of the recognised lines of hydrogen being visible in that, and in many cases not visible in other parts of the tube. I had tried an experiment with a hydrogen vacuum tube of Geissler; but in that the difference was but slight between the positive and negative lights, though it was very great between the light in the narrow central part of the tube and in the wide portions on each side of it, the crimson light in the narrow tube giving a brilliant three-line spectrum, and the blue light, both on the positive and negative side, giving a comparatively dim fluted spectrum of many bands. The difference between the light of narrow and wide parts of the vacuum tubes has, I believe, been noticed; it is in this case the converse of the effects observed by me in the air vacuum.

II. "On the Precession of a Viscous Spheroid, and on the Remote History of the Earth." By George H. Darwin, M.A., Fellow of Trinity College, Cambridge. Communicated by J. W. L. Glaisher, F.R.S. Received July 22, 1878.

(Abstract.)

This paper is a continuation of a previous one on the bodily tides of homogeneous viscous spheroids (read on May 23rd), and it contains the investigation of the rotation of such a body as modified by the tides raised in it by external disturbing bodies. The earth is taken as the type of the rotating body, and the sun and moon as types of the disturbing ones; this plan not only affords a useful vocabulary, but permits an easy transition from questions of abstract dynamics to those of direct applicability to the physical history of the earth.

In the paper on tides it was shown that, if the disturbing influence be expressed as a potential, which is expanded as a series of solid harmonics, each multiplied by a simple time harmonic, then each such term in the expansion corresponds with a tide in a viscous or imperfectly elastic sphere, which is independent of the tides corresponding to all other terms. Also the height of every such tide is expressible as a fraction of the corresponding equilibrium tide of a perfectly fluid spheroid, and the tide is subject to a retardation which is a function of the frequency of the generating term, and of the constants expressive of the physical constitution of the distorted spheroid.

The case of the moon, supposed to move in a circular orbit in the ecliptic, is treated first. The tide generating potential of that body (of the type $\cos^2-\frac{1}{3}$*) has first to be expanded in the desired form;

* Terms of higher orders are shown to be negligeable.

and then a formula expressive of the shape of the distorted spheroid may be at once written down.

The spheroid or earth is found to be distorted by tides of seven different periods; three nearly semi-diurnal, three nearly diurnal, and one fortnightly tide.

Each such tide has a height which is a different fraction of the corresponding equilibrium tide of a perfectly fluid spheroid, and is differently altered in phase. Throughout nearly the whole investigation it is, however, sufficiently accurate, if the three semi-diurnal tides are grouped together, and so also with the three diurnal tides; by this approximation the earth is regarded as distorted by only three tides.

The next process is the formation of the couples acting on the earth, which are caused by the attraction of the moon on the several tidal protuberances. In the development of these couples only those terms are retained which can give rise to secular alterations in the precession, the obliquity to the ecliptic, and the length of day. These expressions are then substituted in the differential equations of motion, and the equations are integrated; whence follow the correction to the uniform precession of the earth considered as a rigid body, and differential equations expressive of the rate of change of obliquity, and the rate of retardation of the earth's diurnal rotation.

It appears that, if the tides do not lag (as with a perfectly fluid or perfectly elastic spheroid), the obliquity and rotation are unaffected, and, whether they lag or not, the correction to the precession is but a small fraction of the whole precession.

Henceforth it is only the changes of obliquity and rotation which are of interest.

A second disturbing body—the sun—is now introduced. A new set of bodily tides are of course raised, and the expressions for the couples are augmented by the addition of solar terms, and also by terms depending on the attraction of the sun on the lunar tides and the moon on the solar tides. It seems paradoxical that there should be these combined effects, for the sun's and moon's periods have no common multiple. But, as far as concerns their interaction, the sun and moon may be conceived to be replaced by two annular satellites of masses equal to those of the two bodies. The combined effects vanish with the obliquity, and depend solely on those tides which run through their periods in a sidereal day, and in twelve sidereal hours. Up to this point all the analysis is conducted so that the solutions may be applied either to a viscous, elastic, or imperfectly elastic spheroid.

In the case where the earth is purely viscous a graphical examination of the equation, giving the rate of change of obliquity, shows that the obliquity sometimes tends to increase and sometimes to diminish, according as the obliquity and viscosity vary. There are also a number of positions of dynamical equilibrium, some stable and some unstable;

but it would be necessary to give a figure, and to go into details, to give the results satisfactorily.

A similar examination of the equation, giving the retardation of the earth's rotation, shows that there is not so much variety of result, for the tidal friction always tends to retard the earth.

This completes the consideration of the instantaneous effects on the earth, and the next point demanding attention is the reaction, which the bodily tides have upon the disturbing bodies.

The problem is solved by the consideration that however the three bodies may interact the resultant moment of momentum of the moon-earth system remains constant, except in so far as it is affected by the sun's action on the earth. The application of this principle results in an equation giving the rate of increase of the square root of the moon's distance in terms of the heights and retardations of the several bodily tides on the earth; it appears that all the tides, except the fortnightly one, tend to make the moon's distance increase with the time, but the fortnightly tide acts in the opposite direction; its effect is, however, in general very small compared with that of the other tides. It is proved, also, that the tidal reaction on the sun, which goes to modify the earth's orbit, has quite insignificant effects, and may be neglected.

I will now show, from geometrical considerations, how some of the results previously stated come to be true. It will not, however, be possible to obtain a quantitative estimate in this way.

The three following propositions do not properly belong to an abstract, since they are not given in the paper itself; they merely partially replace the analytical method pursued therein. The results of the analysis were so wholly unexpected in their variety, that I have thought it well to show that the more important of them were conformable to common sense. These general explanations might doubtless be multiplied by some ingenuity, but it would not have been easy to discover the results, unless the way had been first shown by analysis.

Prop. I. *If the viscosity be small the earth's obliquity increases, the rotation is retarded, and the moon's distance and periodic time increase.*

The figure represents the earth as seen from above the South Pole, so that S is the Pole, and the outer circle the Equator. The earth's rotation is in the direction of the curved arrow at S. The half of the inner circle which is drawn with a full line is a semi-small-circle of S. lat., and the dotted semi-circle is a semi-small-circle in the same N. lat.

Generally dotted lines indicate parts of the figure which are below the plane of the paper.

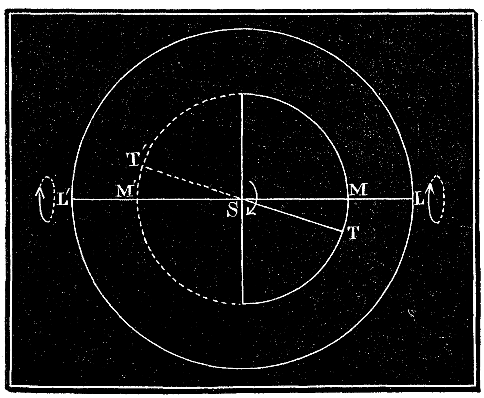
It will make the explanation somewhat simpler, if we suppose the tides to be raised by a moon and anti-moon diametrically opposite to one another. Then let M and M′ be the projections of the moon and anti-moon on to the terrestrial sphere.

If the substance of the earth were a perfect fluid or perfectly elastic, the apices of the tidal spheroid would be at M and M′. If, however, there is internal friction due to any sort of viscosity, the tides will lag, and we may suppose the tidal apices to be at T and T′.

Now, suppose the tidal protuberances to be replaced by two equal heavy particles at T and T′, which are instantaneously rigidly connected with the earth. Then the attraction of the moon on T is greater than on T′; and of the anti-moon on T′ is greater than on T. The resultant of these forces is clearly a pair of forces acting on the earth in the direction of TM, T′M′.

The effect on the obliquity will be considered first.

These forces TM, T′M′, clearly cause a couple about the axis in the equator, which lies in the same meridian as the moon and anti-moon. The direction of the couple is shown by the curved arrows at L, L′.



Now, if the effects of this couple be compounded with the existing rotation of the earth, according to the principle of the gyroscope, it will be seen that the South Pole S tends to approach M, and the North Pole to approach M′. Hence supposing the moon to move in the ecliptic, the inclination of the earth's axis to the ecliptic diminishes, or the obliquity increases.

Next, the forces TM, T′M′, clearly produce a couple about the earth's polar axis, which tends to retard the diurnal rotation.

Lastly, since action and reaction are equal and opposite, and since the moon and anti-moon cause the forces TM, T′M′, on the earth, therefore the earth must cause forces on those two bodies (or on their equivalent single moon) in the directions MT and M′T′. These forces are in the direction of the moon's orbitual motion, and therefore her linear velocity is augmented. Since the centrifugal force of her orbitual motion must remain constant, her distance increases, and with

the increase of distance comes an increase of periodic time round the earth.

This general explanation remains a fair representation of the state of the case so long as the different harmonic constituents of the aggregate tide-wave do not suffer very different amounts of retardations; and this is the case so long as the viscosity is not great.

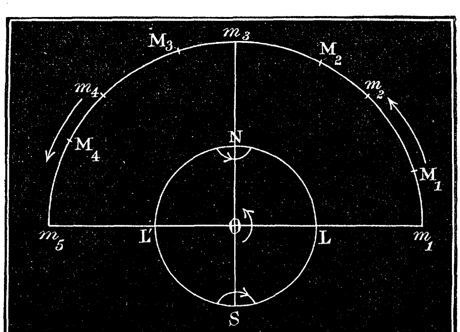
Prop. II. *The attraction of the moon on a lagging fortnightly tide causes the earth's obliquity to diminish, but does not affect the diurnal rotation; the reaction on the moon causes a diminution of her distance, and periodic time.*

The fortnightly tide of a perfectly fluid earth is a periodic increase and diminution of the ellipticity of figure; the increment of ellipticity varies as the square of the sine of the obliquity of the equator to the ecliptic, and as the cosine of twice the moon's longitude from her node. Thus the ellipticity is greatest when the moon is in her nodes, and least when she is 90° removed from them.

In a lagging fortnightly tide the ellipticity is greatest some time after the moon has passed the nodes, and least an equal time after she has passed the point 90° removed from them.

The effects of this alteration of shape may be obtained by substituting for these variations of ellipticity two attractive or repulsive particles, one at the North Pole and the other at the South Pole of the earth. These particles must be supposed to wax and wane, so that when the real ellipticity of figure is greatest they have their maximum repulsive power, and when least they have their maximum attractive power; and their positive and negative maxima are equal.

We will now take the extreme case when the obliquity is 90°; this makes the fortnightly tide as large as possible.



Let the plane of the paper be that of the ecliptic, and let the outer semicircle be the moon's orbit, which she describes in the direction of

the arrows. Let NS be the earth's axis, which lies by hypothesis in the ecliptic, and let LL′ be the nodes of the orbit. Let N be the North Pole; that is to say, if the earth were turned about the line LL′, so that N rises above the plane of the paper, the earth's rotation would be in the same direction as the moon's orbitual motion.

First consider the case where the earth is perfectly fluid, so that the tides do not lag.

Let m_2, m_4 be points in the orbit whose longitudes are 45° and 135°; and suppose that couples acting on the earth about an axis at O perpendicular to the plane of the paper are called positive when they are in the direction of the curved arrow at O. Then, when the moon is at m_1 the particles at N and S have their maximum repulsion. But at this instant the moon is equidistant from both, and there is no couple about O. As, however, the moon passes to m_2 there is a positive couple, which vanishes when the moon is at m_2, because the particles have waned to zero. From m_2 to m_3 the couple is negative; from m_3 to m_4 positive; and from m_4 to m_5 negative. Now, the couple goes through just the same changes of magnitude, as the moon passes from m_1 to m_2, as it does while the moon passes from m_4 to m_5, but in the reverse order; the like may be said of the arcs m_2m_3 and m_3m_4. Hence it follows that the average effect, as the moon passes through half its course, is *nil*, and therefore there can be no secular change in the position of the earth's axis.

But now consider the case when the tide lags. When the moon is at m_1 the couple is zero, because she is equally distant from both particles. The particles have not, however, reached their maximum of repulsiveness; this they do when the moon has reached M_1, and they do not cease to be repulsive until the moon has reached M_2. Hence, during the description of the arc m_1M_2, the couple round O is positive.

Throughout the arc M_2m_3 the couple is negative, but it vanishes when the moon is at m_3, because the moon and the two particles are in a straight line. The particles reach their maximum of attractiveness when the moon is at M_3, and the couple continues to be positive until the moon is at M_4.

Lastly, during the description of the arc M_4m_5 the couple is negative.

But now there is no longer a balance between the arcs m_1M_2 and M_4m_5, nor between M_2m_3 and m_3M_4. The arcs during which the couples are positive are longer and the couples are more intense than in the rest of the semi-orbit. Hence the average effect of the couples is a positive couple, that is to say, in the direction of the curved arrow round O.

It may be remarked that if the arcs m_1M_1, m_2M_2, m_3M_3, m_4M_4 had been 45°, there would have been no negative couples at all, and the positive couples would merely have varied in intensity.

Now, a couple round O in the direction of the arrow, when combined with the earth's rotation, would, according to the principle of the gyroscope, cause the pole N to rise above the plane of the paper, that is to say, the obliquity of the ecliptic would diminish. The same thing would happen, but to a less extent, if the obliquity had been less than 90°; it would not, however, be nearly so easy to show this from general considerations.

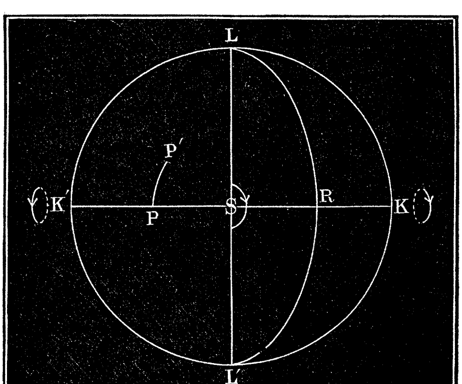
Since the forces which act on the earth always pass through N and S, therefore there can be no moment about the axis NS, and the rotation about that axis remains unaffected. This can hardly be said to amount to strict proof that the diurnal rotation is unaffected by the fortnightly tide, because it has not been rigorously shown that the two particles at N and S are a complete equivalent to the varying ellipticity of figure.

Lastly, the reaction on the moon must obviously be in the opposite direction to that of the curved arrow at O; therefore there is a force retarding her linear motion, the effect of which is a diminution of her distance and of her periodic time.

The fortnightly tidal effect must be far more efficient for very great viscosities than for small ones, for, unless the viscosity is very great, the substance of the spheroid has time to behave sensibly like a perfect fluid, and the tide hardly lags at all.

Prop. III. *An annular satellite not parallel to the planet's equator attracts the lagging tides raised by it, so as to diminish the inclination of the planet's equator to the plane of the ring, and to diminish the planet's rotation. The effects of the joint action of sun and moon may be explained from this.*

Suppose the figure to represent the planet as seen from vertically over the South Pole S; let LL′ be the nodes of the ring, and LRL′ the projection of half the ring on to the planetary sphere.



If the planet were perfectly fluid the attraction of the ring would

produce a ridge of elevation all along the neighbourhood of the arc LRL′, together with a compression in the direction of an axis perpendicular to the plane of the ring. This tidal spheroid may be conceived to be replaced by a repulsive particle placed at P, the pole of the ring, and an equal repulsive particle at its antipodes, which is not shown in the figure.

Now suppose that the spheroid is viscous, and that the tide lags; then since the planet rotates in the direction of the curved arrow at S, the repulsive particle is carried past its place, P, to P′. The angle PSP′ is a measure of the lagging of the tide.

We now have to consider the effect of the repulsion of the ring on a particle which is instantaneously and rigidly connected with the planet at P′.

Since P′ is nearer to the half L of the ring, than to the half L′, the general effect of the repulsion must be a force somewhere in the direction P′P.

Now this force P′P must cause a couple in the direction of the curved arrows K, K′ about an axis, KK′, perpendicular to LL′, the nodes of the ring. The effects of this couple, when compounded with the planet's rotation, is to cause the pole S to recede from the ring LRL′. Hence the inclination of the planet's equator to the ring diminishes.

Secondly, the force P′P produces a couple about S, adverse to the planet's rotation about its axis S. If the obliquity of the ring be small, this couple will be small, because P′ will lie close to S.

Lastly, it may be shown analytically that the tangential force on the ring in the direction of the planet's rotation, corresponding with the tidal friction, is exactly counterbalanced by a tangential force in the opposite direction, corresponding with the change of the obliquity. Thus the diameter of the ring remains constant. It would not be very easy to prove this from general considerations.

It may be shown that, as far as concerns their joint action, the sun and moon may be conceived to be replaced by a pair of rings, and these rings may be replaced by a single one; hence the above proposition is also applicable to the explanation of the joint action of the two bodies on the earth, and numerical calculation shows that these joint effects exercise a very important influence on the rate of variation of obliquity.

To return to the paper: the retardation of the earth's rotation would cause an apparent acceleration of the moon, if that body were unaffected, but this is partly counterbalanced by the true retardation of the moon. We thus have the means of connecting an apparent acceleration of the moon with the heights and retardations of the several bodily tides. I have applied this idea to the supposition that the moon is subject to an

apparent acceleration of 4″ per century, and I found that, if the earth were purely viscous, the moon must be undergoing a secular retardation of 3·6″ per century, while the earth (considered as a clock) must be losing fourteen seconds in the same time. The obliquity also must be diminishing at the rate of 1° in 470 million years.

Under these circumstances the earth must have so great an effective rigidity that the bodily semi-diurnal and diurnal tides would be quite insensible; the bodily fortnightly tide would however be so considerable that the oceanic fortnightly tide would be reduced to one-seventh of its theoretical value on a rigid nucleus, and the time of high water would be accelerated by three days.

The supposition that the earth is a nearly perfectly elastic spheroid leads to very different results in this respect, which, however, I will now pass over.

From this and other considerations, I conclude that a secular acceleration of the moon's motion affords no datum for determining the present amount of tidal friction.

Sir W. Thomson has discussed the probable age of the earth from considering the tidal friction, and he derived his estimate of the rate at which the earth's diurnal rotation is slackening, principally from the secular acceleration of the moon. He fully admitted that his data did not admit of precise results, but, if I am correct, it certainly appears that his argument loses some of its force.

The differential equations, which have to be solved in order to investigate the secular changes in the configuration of the three bodies, are exceedingly complex, and I was only able to solve them by a laborious method, depending partly on analysis and partly on numerical quadratures.

The solution was only applicable to the case where the earth is a purely viscous body, and the numerical value chosen for the coefficient of viscosity was such that the changes proceed with about the maximum rapidity. Starting with the present values of the obliquity, day, month, and year, the changes were traced backwards in time. As we go backwards we find the year sensibly constant, but the obliquity, day and month all diminishing—the last with far the greatest rapidity. The changes proceed at a rapidly increasing rate, as in the retrospect the moon approaches the earth.

At the point where I found it convenient to stop in the first method of solution, about 56 million years have been traversed backwards, and the obliquity is found to have diminished by 9°, the day is found to have fallen to 6 hrs. 50 mins., and the sidereal month to only 1 day 14 hrs.

It is a question of great interest to geologists to determine whether any part of changes of this kind can have taken place during geological history; and I conclude that it might be so. The physical meaning

of the coefficient of viscosity which is used in this solution is as follows:—If a slab of the materials of the earth an inch thick have one face held fixed, and if the other face be subjected to a tangential stress of $13\frac{1}{2}$ tons to the square inch for 24 hours, then the two faces have been displaced relatively to one another through one-tenth of an inch. Such a material would in ordinary parlance be called a solid, and in the tidal problem this must be regarded as a moderately small viscosity, whence I conclude that the earth may have been habitable, and yet have undergone these changes.

Amongst the conclusions of interest to geologists is the following: namely, that the amount of heat generated in the interior of the earth by internal friction, during these 56 million years, would be sufficient, if applied all at once, to heat the whole earth's mass 1,755° F., supposing the earth to have the specific heat of iron. If then it is permissible to suppose that any considerable part of these changes has taken place during geological history, the estimate of the age of the earth, which is founded on the assumption that the earth is simply a cooling sphere, would have to undergo modification.

A second solution of the differential equations is next given, adapted to the hypothesis that the earth stiffened as it cooled; but no definite law of stiffening is assumed. This solution follows a line closely similar to that of the last up to the point where the day has fallen to 6 hrs. 50 mins. The obliquity is, however, found to decrease slightly more than in the previous solution.

At this point it was found necessary to abandon the approximation by which the three semi-diurnal and the three diurnal tides are classified together. The problem then becomes much more complex, and a new method of solution is required.

It is found that in the retrospect the obliquity will only continue to diminish a little beyond the point already reached; for when the month has become equal to twice the day there is no longer a tendency to diminution, and for smaller values of the month the tendency is reversed. This shows that for values of the month less than twice the day, the position of zero obliquity of the earth's axis is dynamically stable. The whole diminution of obliquity, from the initial state back to the critical point of relationship between the month and day, is found to be 10°.

After considering the various discrepancies between the ideal problem solved and the real case of the earth, I conclude that while a large part of the obliquity may be probably referred to these causes, yet that there probably remains an outstanding part which is not so explicable.

The obliquity to the ecliptic is now set on one side, and from a consideration of the equation of conservation of moment of momentum, the initial state is determined, towards which the solution has been

running back. It is found that the initial condition is one in which moon and earth rotate, as though fixed together, in 5 hrs. 40 mins.; and that this condition is one of dynamical instability, so that the moon must either have fallen into the earth, or have receded from it, and have then gone through the changes which were traced backwards.

From this and other considerations it is concluded that, if the moon and earth were ever molten viscous masses, then it is highly probable that they once formed parts of a common mass.

The rest of the paper is occupied with a number of miscellaneous propositions, and with a discussion of the physical significance of the results obtained.

I will here only mention that the case of the Martian satellites appears to me a very striking corroboration of the applicability of these views to the solar system, whilst the Uranian system of satellites is, at first sight, unfavourable.

A whole series of problems, some of them of great difficulty, still await solution; and not until they are solved will it be possible either decisively to accept or reject the modified form of the nebular hypothesis, to which my results obviously point.

(Postscript.) Added November 8th, 1878.

A subsequent investigation has shown that, although the amount of heat which might be generated by internal friction in the earth might be very great, yet its distribution would be such that it could scarcely sensibly affect Sir W. Thomson's investigation of the secular cooling of the earth.

III. "Problems connected with the Tides of a Viscous Spheroid." By G. H. DARWIN, M.A., Trinity College, Cambridge. Communicated by J. W. L. GLAISHER, F.R.S. Received November 14, 1878.

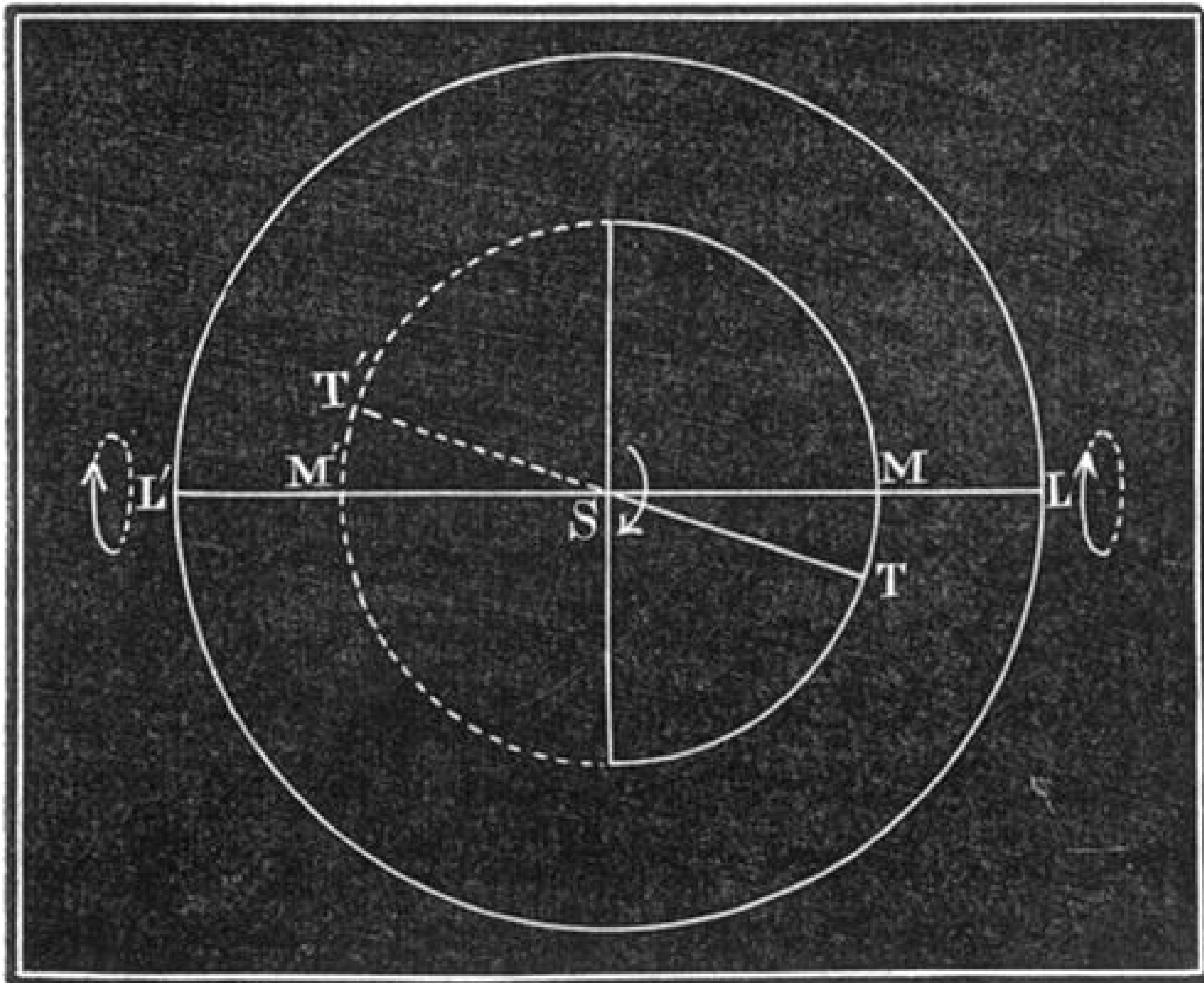
(Abstract.)

In this paper certain problems are treated, which were alluded to in two previous papers on the Tides and Precession of a viscous spheroid.* For brevity the spheroid is spoken of as the earth, and the disturbing body as the moon.

I. *Secular Distortion of the Spheroid, and certain Tides of the second order.*

The distortion arises from the unequal distribution of the tidal frictional couple over the surface of the spheroid.

* Read before the Royal Society on May 23 and December 19 respectively.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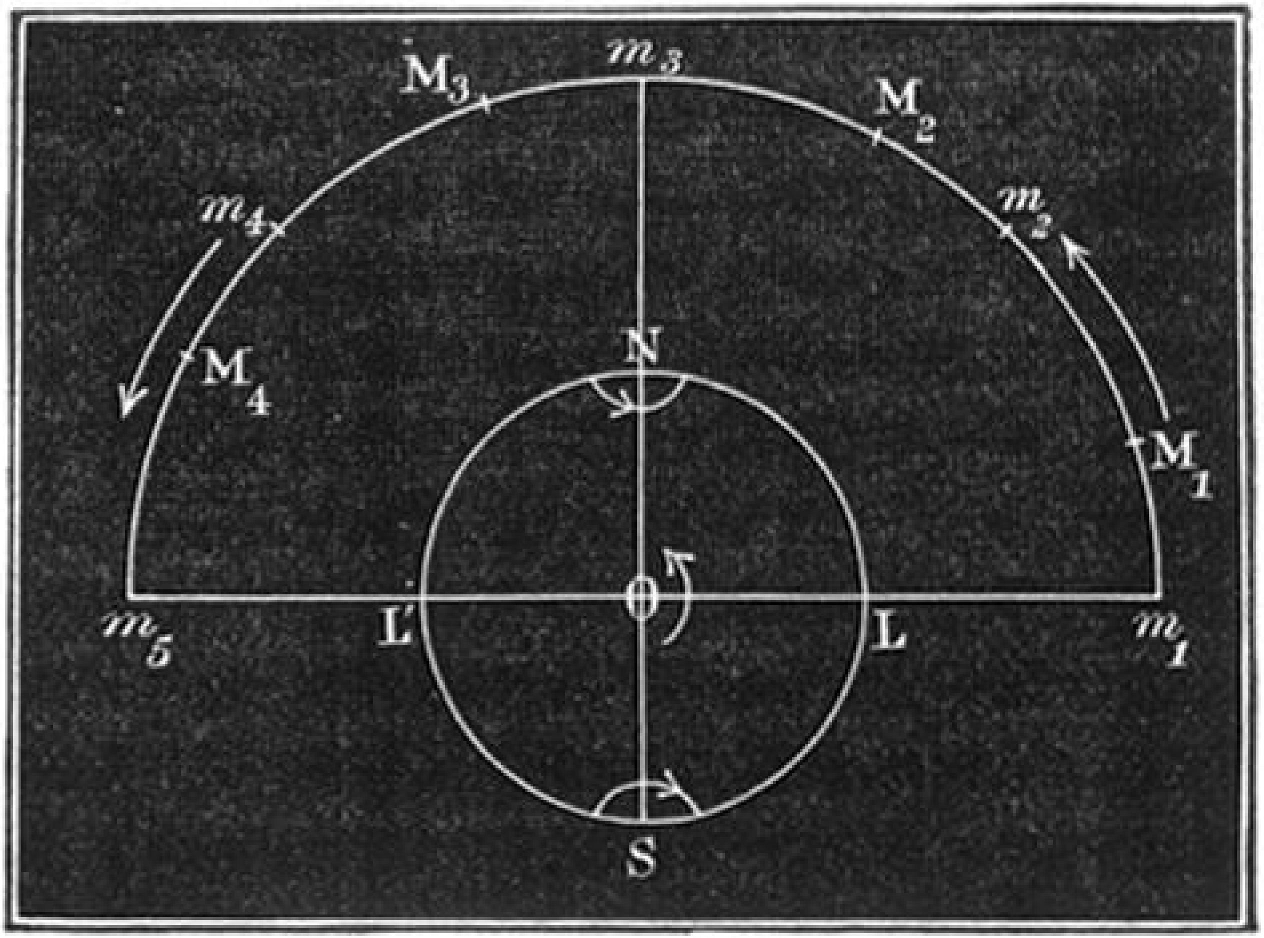
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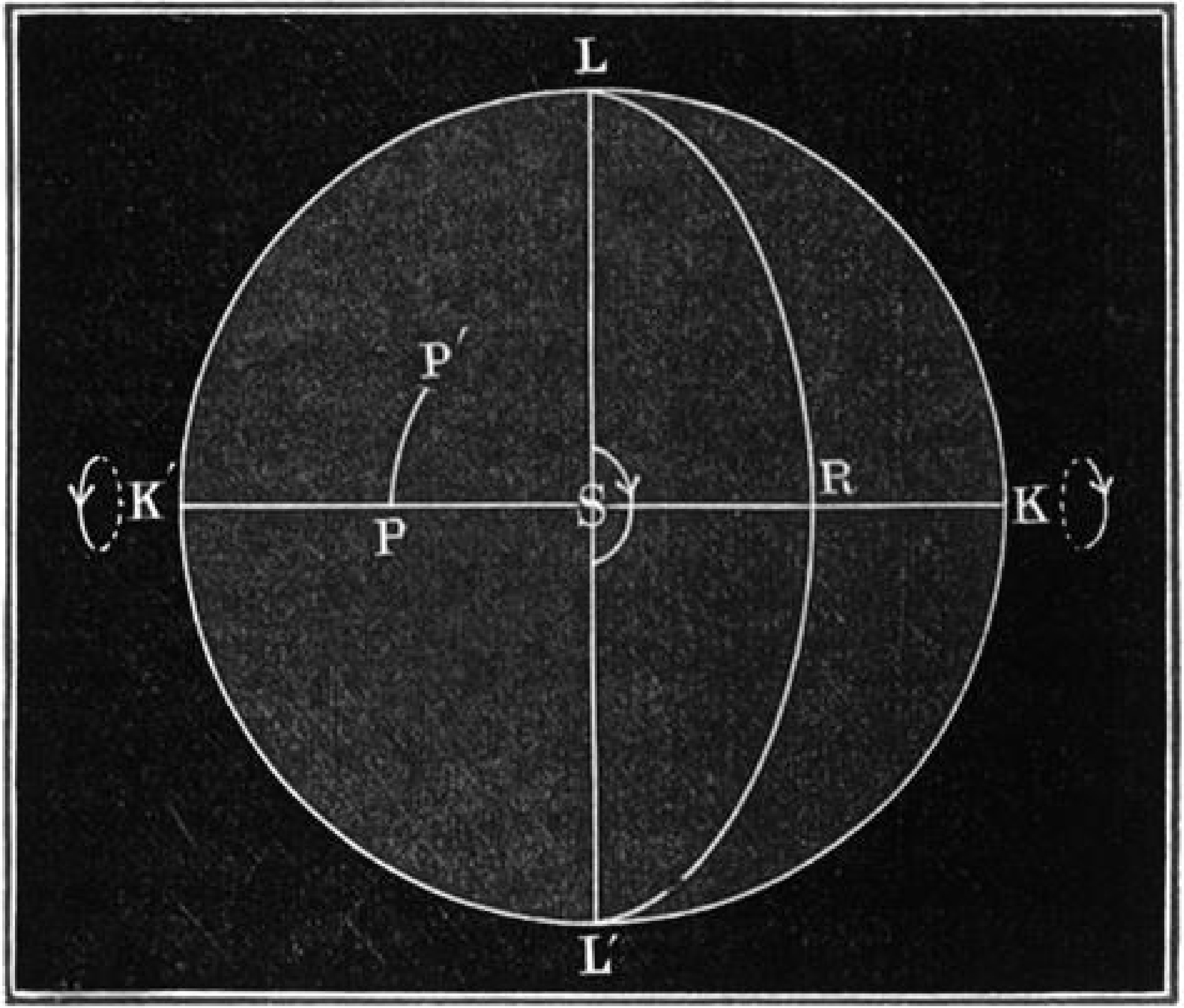

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