we previously observed in our iron tubes, and described in our last communication, was a reversal of part of these lines, though the latter extend much further towards the blue than we had observed the absorption to extend. In fact, the bright lines extend somewhat more than half the distance between *b* and F, from 45 to 50 being visible, and placed at nearly equal distances from each other. They also commence close to the *b* group, *i.e.*, with a wave-length nearly 5,164, but the first two or three lines at that end are not so bright as those which immediately succeed them. The light giving these lines does not extend to more than a short distance from the electrodes, and is generally most conspicuous at the negative electrode. There is a difficulty in consequence of the flickering character of the discharge in getting any accurate measures of them, though they are bright enough, especially at the less refrangible end, to be easily seen. The comparative faintness of the light from the iron tubes appears to us almost sufficient to account for our not having seen the reversed lines so completely as the bright ones; nevertheless, it is quite in accordance with what we in other cases observed, to suppose that some of these lines may be more easily reversed at the temperature of the iron tubes than others.

XII. "An Experimental Investigation into the Velocities of Normal Propagation of Plane Waves in a Biaxal Crystal, with a Comparison of the Results with Theory." By R. T. GLAZEBROOK, B.A., Fellow of Trinity College, Cambridge. Communicated by J. CLERK MAXWELL, M.A., F.R.S. Received June 19, 1878.

(Abstract.)

In his report to the British Association in 1862, Professor Stokes called attention to the desirability of accurate measurements of the velocity of normal propagation of plane waves in a biaxal crystal, with a view to testing by the results Fresnel's theory of double refraction, and suggested then a method to determine this velocity. Let the crystal to be examined be cut into the form of a prism, two or more natural faces being left to determine accurately the position of the cut faces with reference to the axes of elasticity.

"Let us consider a plane wave of light passing through the crystal prism."

"Let V be the velocity in air, v in the crystal, let $\phi\, \phi'\ \psi\ \psi'$ have their usual meanings, let i be the angle of the prism, D the deviation of the wave normal after passing through the prism."

Let us observe the angle of incidence ϕ, and the deviation D.

Then ψ is given by the formula

$$\psi = \mathrm{D} + i - \phi \quad . \quad . \quad . \quad . \quad (1).$$

"But without making any other supposition as to the law of double refraction, or assuming anything beyond the truth of Huyghen's principle, which, following at once from the superposition of small motions, lies at the base of the whole theory of undulations, we may at once deduce from the directions of incidence and emergence the direction and velocity of propagation in the crystal."

For we have—

$$\frac{\sin\phi}{\mathrm{V}} = \frac{\sin\phi'}{v},$$

$$v\sin\phi = \mathrm{V}\sin\phi' \quad . \quad . \quad . \quad . \quad (2),$$

$$v\sin\psi = \mathrm{V}\sin\psi' \quad . \quad . \quad . \quad . \quad (3),$$

$$\phi' + \psi' = i \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Adding and subtracting (2) and (3),

$$v\sin\frac{\phi+\psi}{2}\cos\frac{\phi-\psi}{2} = \mathrm{V}\sin\frac{\phi'+\psi'}{2}\cos\frac{\phi'+\psi'}{2},$$

$$v\cos\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2} = \mathrm{V}\cos\frac{\phi'+\psi'}{2}\sin\frac{\phi'-\psi'}{2}.$$

Dividing and recollecting (4),

$$\tan\frac{\phi+\psi}{2}\tan\frac{\phi-\psi}{2} = \tan\frac{i}{2}\cot\frac{\phi'-\psi'}{2},$$

$$\therefore \tan\frac{\phi'-\psi'}{2} = \tan\frac{i}{2}\cot\frac{\phi+\psi}{2}\tan\frac{\phi-\psi}{2}.$$

This gives $\frac{\phi'-\psi'}{2}$.

Combining with (4) we can get ϕ' and ψ', and then find the value of v from either (2) or (3). In practice the value of μ or $\frac{\mathrm{V}}{v}$ was used.

In accordance with these suggestions I undertook a series of observations at the Cavendish Laboratory, Cambridge, which I propose to describe in the paper, adding moreover a comparison of the results with the theories of Fresnel and Lord Rayleigh. Fresnel is the only experimenter who has attempted to verify his theory by experiment, and his attempt affords no verification, for in applying it he used approximate results, and to the degree of approximation to which he went the theory developed by Lord Rayleigh (Phil. Mag., vol. xli, Series iv, 1871) leads to exactly the same equations to determine the velocity of propagation as were used by Fresnel, so that his results form equally a verification of Lord Rayleigh's theory.

The work was carried out on two pieces of aragonite supplied by A. Hilger, Tottenham Court Road. In both cases two of the faces marked $m\,m'$ (Miller's "Mineralogy," p. 567) gave the best reflexions, and were therefore reserved to afford means for the determination of the position of the artificial faces. We will consider the two separately.

The first crystal was cut at Professor Stokes' suggestion, so as to form two prisms.

The edge of one of these was nearly parallel to the axis b of the crystal, the mean axis of elasticity, so that the principal plane of the prism almost coincided with the principal plane AOC of the wave surface, and I was able to show that the error arising from supposing the coincidence to be exact would never amount to as much as ·00001.

The axis OA of the crystal almost bisected the angle of the prism, which was 42° 50′ 30″. The observations of deviation and incidence were made with a goniometer by Grubb, lent me by Professor Stokes. The circle, about a foot in diameter, was graduated on silver, and was read to 10″ by verniers.

Each observation was repeated two or three times on different occasions, and the mean of the results taken. It was rarely that two measures of the same quantity differed by 20″.

Careful precautions were taken to ensure the light passing in a principal plane of the prism.

The observations with this prism extended from about 8° on one side of the axis OC to 16° on the other, passing almost through the extremity of an optic axis. Observations were taken at angles of incidence, increasing uniformly by 2°, thus forming an arithmetical progression.

The values of μ for one wave were nearly constant, and varied but little from

$$1{\cdot}68125.$$

Let a, b, c, be the principal refractive indices. The series of values given above corresponds to the circle of radius b. So that

$$b = 1{\cdot}68125.$$

The values corresponding to the other wave varied considerably. According to Fresnel they ought to be radii vectores to an ellipse axes a and c.

a was determined by passing light along the axis OC, which was possible, the crystal having been cut with this object in view. I found

$$a = 1{\cdot}68580.$$

To find c recourse was had to the second prism, which had its edge nearly parallel to the axis of c. I found

$$c = 1{\cdot}53013.$$

The value for the angle between the optic axes, seen in air through a face normal to c is, from these values

$$31^\circ\ 0'\ 0''.$$

Kirchhoff found by experiment—

$$30^\circ\ 54'.$$

The agreement is fairly close, much closer than that given by the values of the principal indices as determined by Rudberg.

Having thus found a and c, we proceed to calculate the theoretical values of μ in different directions and compare with theory, with the following result:—For from about 8° on one side of the axis a to 10° on the other, theory and experiment agree closely. The difference only in two cases amounts to ·0001, and is sometimes positive, sometimes negative. [The error in experiment is certainly not so great as ·00005.]

But for the next 6° through which the observations extended, the differences continually increase, reaching ·00024 for the last observation, the experimental values of μ being uniformly greater than the theoretical.

So that the results of observation would be represented by a circle of radius 1·68125, and an oval curve with the same axes as Fresnel's ellipse, viz.,

$$a = 1{\cdot}68580$$
$$c = 1{\cdot}53013$$

which agrees closely with the ellipse for 10° on either side of the axis a, and for the rest of the arc observed lies outside it, the difference between the radii vectores of the two curves increasing as we recede from a.

The differences between experiment and Lord Rayleigh's theory increase much more rapidly, and amount, at the end of the arc of 16° observed, to ·00202, or about ten times as much as on Fresnel's theory. Thus Lord Rayleigh's theory differs from the truth by considerably more than Fresnel's.

The second prism was cut so as to have its edge nearly parallel to OC.

The parallelism, however, was not sufficiently exact to enable me to treat the principal plane of the prism as coincident with a principal section AOB of the surface of wave slowness. It was necessary, therefore, to determine the values of $v_1\ v_2$, the velocities of normal propagation, from Fresnel's construction. Let the optic axes meet a unit sphere centre at the centre of the surface in OO′ respectively; let P be the part in which any wave normal meets the sphere $OP=\theta\ O'P=\theta'$.

Then we may show that the values of $v_1\ v_2$ are given by—

$$v_1^2=\frac{1}{b^2}-\frac{a^2-c^2}{a^2c^2}\sin\left\{\frac{\theta+\theta'}{2}-\mathrm{AO}\right\}\sin\left\{\frac{\theta+\theta'}{2}+\mathrm{AO}\right\},$$

$$v_2^2=\frac{1}{b^2}+\frac{a^2-c^2}{a^2c^2}\sin\left\{\frac{\theta-\theta'}{2}+\mathrm{AO}\right\}\sin\left\{\mathrm{AO}-\frac{\theta-\theta'}{2}\right\}.$$

The work extended over an arc of about 19°, with the following results :—

For one wave the agreement was close throughout. This section differed but slightly from the circular section of radius 1·53013.

For the other the differences were much greater.

The results of experiment were represented by a curve, which in the neighbourhood of the lesser axis of Fresnel's section lies within that section, cutting it at about the middle of the arc considered, and afterwards lying without it.

The excess of experiment over theory changes in the arc considered from

$$-{\cdot}0002 \text{ to } +{\cdot}0005.$$

This agrees with the result for the first prism in lying outside Fresnel's surface as we approach the major axis.

I then proceeded to estimate the effect of any possible errors made in the determination of a, b, c, or the position of the plane, and showed that no change at all within the limits of experimental error would reconcile theory and experiment more closely. As a test of the accuracy of the experimental work, it may be stated that a series of observations, taken at an interval of some three months previously to those described above, gave results which rarely differed from those results by more than

$${\cdot}00004.$$

To proceed now to the second crystal. The measurements on it were made at an interval of nearly a year after those already described. The crystal was in the form of a hexagonal prism, the base of the prism being nearly perpendicular to c. This base was polished. The other end was cut so as to be inclined to the base at an angle of 35° 2′ 56″, the line of intersection of the faces of the prism thus formed being nearly parallel to that of m and c. One of the faces, m, was cut so as to be inclined to the oblique section at about 37°, the line of junction being again nearly parallel to that of m and c. So that I thus formed two prisms whose principal planes were nearly coincident, having one face in common.

By this means I was able to work over an arc which extended from the neighbourhood of the principal section AOC to more than 70° on the other side of it.

The planes cut the principal plane, AOC, in two points, L L′, such that—

$$
\begin{aligned}
CL &= 1^\circ\ 21'\ 42'' \\
CL' &= 1^\circ\ 15'\ 40'' \\
ALP &= 59^\circ\ 20'\ 11'' \\
AL'P &= 59^\circ\ 13'\ 2''
\end{aligned}
$$

while the position of P, the normal to the common face of the two prisms, was given by—

$$
\begin{aligned}
LP &= 35^\circ\ 0'\ 19'' \\
\text{or } L'P &= 35^\circ\ 3'\ 14''
\end{aligned}
$$

Let $\theta\ \theta'$ be the angles between any wave normal and the optic axes. The formulæ used to calculate $v_1\ v_2$ were—

$$v_1^2 = \frac{a^2+c^2}{a^2c^2} - \frac{a^2-c^2}{a^2c^2}\cos(\theta+\theta'),$$

$$v_2^2 = \frac{a^2+c^2}{a^2c^2} - \frac{a^2-c^2}{a^2c^2}\cos(\theta-\theta').$$

The values used for a, b, c, were—

$$
\begin{aligned}
a &= 1{\cdot}68560 \\
b &= 1{\cdot}68115 \\
c &= 1{\cdot}53013
\end{aligned}
$$

These values differ slightly from those used in the first part of the paper, though not by nearly so large amounts as Rudberg found between the values for two different specimens of aragonite. He had differences of more than ·0004 ("Pogg. Annalen," xvii, 1). For the outer sheet, which differs least from a sphere, the results of theory and experiment agree fairly closely. But for the inner sheet the curve given by theory agreeing with that given by experiment at the extremity of an axis, lies outside of it throughout the whole of its course away from that axis. The difference is greatest at about 35° away from the principal section, AOC being there as great as ·0009.

From this point the differences decrease, and at the end of the arc considered, or 74° away from the same axis, the value is

·0003.

I found also that no changes in the values of a, b, c, would produce closer agreement, but that if we suppose the arcs LP, L'P, to be increased by 17', and the angles ALP, AL'P, by about 1°, the differences between theory and experiment were reduced about ·00006, taken throughout the arc, being sometimes positive, sometimes negative.

I have shown, however, that this change in the position of P involves alterations in the measures by which its position relative to the faces mm' was determined, which are far in excess of any possible experimental error, and that the only way of accounting for them is by supposing the axes of elasticity of aragonite to be slightly variable in position relatively to the faces of the crystal.

On the whole, however, I prefer to regard Fresnel's theory as a close first approximation to the truth, and to look to the phenomena of dispersion to explain the variation from it.

I may perhaps be allowed to close with a suggestion, which, judging from the results of a few experiments I have already made, appears to have some basis of truth.

Let us suppose that in a crystal

$$\mu = a + \frac{b}{\lambda^2} + \frac{c}{\lambda^4} + \ .$$

where a, b, c, &c., are functions of the directions of vibration and propagation.

Let us suppose that for waves of infinite length Fresnel's construction is true, so that a is a radius vector of Fresnel's surface of wave slowness, and can therefore be calculated, and suppose we neglect the terms $\frac{c}{\lambda^4}$, &c. Observing the values of μ in this direction for different rays, we get

$$\mu_1 - a = \frac{b}{\lambda_1^2},$$

$$\mu_2 - a = \frac{b}{\lambda_2^2},$$

$$\mu_3 - a = \frac{b}{\lambda_3^2}, \text{ \&c.}$$

So that

$$\frac{\mu_1 - a}{\mu_2 - a} = \frac{\lambda_2^2}{\lambda_1^2}, \text{ \&c.}$$

The results of experiments on the rays C, D, and F, in two different directions give

$$\frac{\lambda_C^2}{\lambda_D^2} = 1{\cdot}2403.$$

$$\frac{\mu_D - a}{\mu_C - a} = \begin{cases} 1{\cdot}2875 \text{ (1st experiment).} \\ 1{\cdot}2770 \text{ (2nd direction).} \end{cases}$$

$$\frac{\lambda_D^2}{\lambda_F^2} = 1{\cdot}46978.$$

$$\frac{\mu_F - a}{\mu_D - a} = \begin{cases} 1{\cdot}47208 \text{ (1st direction).} \\ 1{\cdot}47348 \text{ (2nd direction).} \end{cases}$$

These numbers, especially the last, are sufficiently close to make it worth while continuing the observations.

The Society adjourned over the Long Vacation to Thursday, November 21.