May 29, 1879.

THE PRESIDENT in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

I. "On the Conduction of Heat in Ellipsoids of Revolution." By C. NIVEN, M.A., Professor of Mathematics in Queen's College, Cork. Communicated by J. W. L. GLAISHER, F.R.S. Received May 7, 1879.

(Abstract.)

The object of the present paper is to investigate the expressions which present themselves in the mathematical treatment of the problem of the conduction of heat in an ellipsoid of revolution. The results obtained constitute a generalisation of the corresponding solution for the sphere, and are found, in the first instance, for an ellipsoid whose major axis is the axis of revolution, but a slight alteration will render them applicable also to a planetary ellipsoid. We may effect the transformation to ellipsoidal co-ordinates of the general equation of conduction as follows: the semi-axes of the ellipsoid and hyperboloid through any point confocal to the surface being denoted by $c\cosh\alpha$, $c\sinh\alpha$, and $c\cos\beta$, $c\sin\beta$, the co-ordinates of the point may be expressed by $c\cos\phi\sinh\alpha\sin\beta$, $c\sin\phi\sinh\alpha\sin\beta$, $c\cosh\alpha\cos\beta$, and the general equation of conduction becomes

$$\frac{d^2\mathrm{V}}{d\alpha^2}+\frac{d^2\mathrm{V}}{d\beta^2}+\coth\alpha\,\frac{d\mathrm{V}}{d\alpha}+\cot\beta\,\frac{d\mathrm{V}}{d\beta}+\left(\frac{1}{\sin^2\beta}+\frac{1}{\sinh^2\alpha}\right)\frac{d^2\mathrm{V}}{d\phi^2}$$
$$=\frac{c^2}{k}(\cosh^2\alpha-\cos^2\beta)\frac{d\mathrm{V}}{dt},$$

V being the temperature at any point.

We may satisfy this equation by

$$\mathrm{V}=(\cos m\phi \text{ or } \sin m\phi)e^{-k^2t}\Theta_m^z(\beta)\Omega_m^z(\alpha),$$

where

$$\frac{d^2\theta}{d\beta^2}+\cot\beta\frac{d\theta}{d\beta}-\frac{m^2\theta}{\sin^2\beta}=\lambda^2c^2\cos^2\beta\,\theta-z\theta,$$

$$\frac{d^2\Omega}{d\alpha^2}+\coth\alpha\frac{d\Omega}{d\alpha}-\frac{m^2\Omega}{\sinh^2\alpha}=-\lambda^2c^2\cosh^2\alpha\,\Omega-z\Omega,$$

z being a constant to be determined, the same in both; and we have to determine the appropriate solutions of these equations.

Writing ν for $\cos\beta$, and denoting by $P_m^n(\nu)$ the tesseral harmonic $\frac{(n-m)!}{(2n)!}(\nu^2-1)^{\frac{m}{2}}\frac{d^{m+n}(\nu^2-1)^n}{d\nu^{m+n}}$, it is proved that the equation in θ is satisfied by expression of the form

$$\theta_m^z = a_0 P_m^m - a_1 P_m^{m+2} + a_2 P_m^{m+4} - \ldots$$

or

$$\theta_m^z = b_0 P_m^{m+1} - b_1 P_m^{m+3} + b_2 P_m^{m+5},$$

and the coefficients (a) and (b) are related as follows—

Putting

$$\lambda^2 c^2 = \epsilon, \frac{(n^2-m^2)(\overline{n-1}^2-m^2)}{(4n^2-1)(4.\overline{n-1}^2-1)} = \mu_r \text{ if } n=m+2r, \text{and } =\mu s \text{ if } n=m+2s+1$$

$$\frac{2n^2+2n-2m^2-1}{(2n-1)(2n+3)}.\epsilon + n(n+1) - z = \phi_r \quad \text{if} \quad n=m+2r, \quad \text{and} \quad =\phi s \quad \text{if} \quad n=m+2s+1,$$

we have the systems

$$(1)\ n=m+2r,\ \mu_1 a_1 = \frac{1}{\epsilon}\phi_0 a_0,\ \mu_2 a_2 = \frac{1}{\epsilon}\phi_1 a_1 - a_0,\ \ldots\ \mu_{r+1} a_{r+1} = \frac{1}{\epsilon}\phi_r a_r - a_{r-1},$$

z one of the roots of $a_\infty = 0$.

$$(2)\ n=m+2s+1,\ \mu_1 b_1 = \frac{1}{\epsilon}\phi_0 b_0,\ \mu_2 b_2 = \frac{1}{\epsilon}\phi_1 b_1 - b_0, \ldots \mu_{r+1} b_{r+1} = \frac{1}{\epsilon}\phi_r b_r - b_{r-1}, \ldots$$

z one of the roots of $b_\infty = 0$;

the expressions dividing themselves into two classes, for the former of which the values of θ are symmetrically equal, and for the latter equal and opposite, on opposite sides of the equator.

The values of Ω also fall into two corresponding classes, and are expressed most appropriately in terms of the function

$$S_n(x) = x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x}.$$

If we write $\xi = \lambda c \sinh\alpha$, $\eta = \lambda c \cosh\alpha$, $\zeta = \cosh\alpha$, we find for Ω these expressions:—

Class I. Any one of the three forms

$$a_0 S_m(\xi) + \sum^\infty (-1)^r a_r \frac{1.3\ldots(2r-1)}{(2m+2r+1)(2m+2r+3)\ldots(2m+4r-1)} \cdot \underset{m+2r}{S}(\xi),$$

$$\frac{(\eta^2-\lambda^2c^2)^{\frac{m}{2}}}{\eta^m}$$

$$\left(a_0 S_m(\eta) + \sum_1^\infty a_r \cdot \frac{2^r.(m+1)(m+2)\ldots(m+r)}{(2m+2r+1)(2m+2r+3)\ldots(2m+4r-1)} \underset{m+2r}{S}(\eta)\right)$$

$$a_0 P_m^m(\zeta) + \sum_1^\infty (-1)^r a_r \underset{m}{\overset{m+2r}{P}}(\zeta).$$

Class II. Any one of the forms

$$\frac{(\xi^2+\lambda^2c^2)^{\frac{m}{2}}}{\xi^m}$$

$$\left(b_0\mathop{\mathrm{S}(\xi)}_{m+1}+\sum_1^\infty(-1)^s b_s\,.\,\frac{1\,.\,3\,.\,5\,.\,.\,.\,(2s+1)}{(2m+2s+3)(2m+2s+5)\,.\,.\,(2m+4s+1)}\,.\,\mathop{\mathrm{S}(\xi)}_{m+2s+1}\right)$$

$$\frac{(\eta^2-\lambda^2c^2)^{\frac{m}{2}}}{\eta^m}$$

$$\left(b_0\mathop{\mathrm{S}(\eta)}_{m+1}+\sum_1^\infty b_s\,.\,\frac{2^s\,.\,(m+1)(m+2)\,.\,.\,.\,(m+s)}{(2m+2s+3)(2m+2s+5)\,.\,.\,(2m+4s+1)}\,.\,\mathop{\mathrm{S}(\eta)}_{m+2s+1}\right),$$

$$b_0\overset{m+1}{\mathrm{P}_m}(\zeta)+\sum_1^\infty(-1)^s b_s\overset{m+2s+1}{\mathrm{P}_m}(\zeta).$$

It is shown that the roots of $a_\infty=0$ are all real and definite in position, and that in the neighbourhood of these roots the series a_r, a_{r+1}, . . . rapidly converge; similar results hold for the roots of $b_\infty=0$.

The general form of the ratios $a_r : a_0$ is ascertained, and the equation $a_\infty=0$ written in the form

$$1-\epsilon^2\Sigma_1+\epsilon^4\Sigma_2-\,.\,.\,.\,=0.$$

where Σ_r denotes the sum of the products of every r of the series $\frac{\mu_0}{\phi_0\phi_1}$, $\frac{\mu_1}{\phi_1\phi_2}$ . . . of which no two in the same product are adjacent.

We then approximate to the values of z by series ascending by powers of ϵ, and the expression for the $\overline{r+1}$th root is, up to ϵ^5, given by

$$\phi_r=\epsilon^2\left(\frac{\mu_r}{\phi''_{r-1}}+\frac{\mu_{r+1}}{\phi''_{r+1}}\right)+\epsilon^4\left(\frac{\mu_{r-1}\mu_r}{\phi'_{r-2}\phi'^2_{r-1}}+\frac{\mu_{r+1}\mu_{r+2}}{\phi'^2_{r+1}\phi'_{r+2}}\right),$$

in which ϕ'_p represents the result of putting in ϕ_p a first approximation to z, up to ϵ given by $\phi_r=0$, and ϕ_p'' the resulting of substituting a second approximation up to ϵ^3 given by $\phi_r=\epsilon^2\left(\frac{\mu_r}{\phi'_{r-1}}+\frac{\mu_{r+1}}{\phi'_{r+1}}\right)$.

These expressions are then used to expand z in powers of ϵ; and, more especially, the first roots of each equation for any given value of m, are found. For the $\overline{r+1}$th root the term a_r is the leading term of the series, and expansions are found for $a_{r+1} : a_r$, $a_{r-1} : a_r$, $a_{r+2} : a_r$ . . . . as far as ϵ^3.

In the case of Class I a few of the lower roots are found for the smaller numbers, $m=0$, 1, 2, and the numerical expansions given for one or two of the coefficients adjacent to the leading one to (in general) two terms. In Class II the numerical calculations have been performed for a few of the roots only. These expansions represent the solution found by M. Mathieu.

The values of λ must be determined from the condition satisfied at the surface of the solid. If the surface be maintained at constant

(zero) temperature, λ is given either by $0=\Omega^{z}_{m}(\lambda b)$ or $0=\Omega^{z}_{m}(\lambda a)$, $2a$ and $2b$ being the major and minor axes of the ellipsoid. The latter is the more convenient for expressing λ in powers of the eccentricity $e=\frac{c}{a}$. It is shown how the values of λ may be found, and in particular for the first root of $m=0$, the equation becomes $\Omega^{z_0}_{0}=0$, which gives for λ,

$$\lambda a=i\pi+\frac{2}{3}i\pi e^2+\frac{i\pi}{135}\left(\frac{122}{7}i^2\pi^2+613\right)e^4+\ \ldots$$

When the solid cools by radiation the boundary condition is

$$\frac{dV}{d\xi}+\frac{h}{\lambda}\sqrt{1-e^2\xi^2}V=0(\xi=\lambda b)\,;$$

and, to deal with this condition, it is necessary to show that any function of ν, at least every function capable of expansion in the form $^{n}\sum\limits_{m}^{\infty}A_nP^{n}_{m}(\nu)$, may be expanded in the form $\Sigma B_n\theta^{z}_{m}$. The possibility of this depends on the theorem $\int_{-1}^{+1}\theta^{z}_{m}\theta^{z}_{m}{}'d\nu=0$, z not $=z'$, which is proved true.

In particular the coefficients of the expansion of $\nu^2\theta_m$ in this form are fully worked out.

To satisfy all the conditions of the problem we must now assume for V an expression of the form ($\cos m\phi$ or $\sin m\phi$) $e^{-\lambda^2kt}\Sigma C_r\theta^{zr}_{m}\Omega^{zr}_{m}$; and general equations are given whereby both the values of λ and the coefficients (C) are determined. If we require the expansion of λ only as far as e^4, the equation which finds it can be readily given. If we wish only λ up to e^2, this equation breaks up into a system of the form

$$\frac{d\Omega^{zr}_{m}}{d\xi}+g^{r}_{r}\Omega^{zr}_{m}=0,\ (\xi=\lambda b),$$

in which g^{r}_{r} are constant coefficients whose values are found. The course of the subsequent approximation is free from difficulty.

To calculate the coefficients which depend on the initial state of the ellipsoid, we have to evaluate the integral

$$\int_{-1}^{+1}d\nu\int_{0}^{e^{-1}}v^2(\zeta^2-\nu^2)d\zeta,$$

in which v may be either $\theta^{z}_{m}\Omega^{z}_{m}$ or $\Sigma Cr\theta^{zr}_{m}\Omega^{zr}_{m}$, as the case may be; and to do so we must determine the following single integrals:—

$$\int_{-1}^{+1}(\theta^{z}_{m})^2d\nu,\quad \int_{-1}^{+1}\nu^2\theta^{z}_{m}\theta^{z}_{m}{}'d\nu,\quad z'=\text{or not}=z,$$

$$\int_{0}^{e^{-1}}\Omega^{z}_{m}\Omega^{z}_{m}{}'d\zeta,\quad \int_{0}^{e^{-1}}\zeta^2\Omega^{z}_{m}\Omega^{z}_{m}{}'d\zeta,\quad z'=\text{or not}=z.$$

These integrals are given in the first three cases, and the formula for finding the last, which is somewhat complicated, furnished.

Finally, it is shown that the results may be transferred to the case of a planetary ellipsoid by changing, in the formulæ, ϵ into $-\epsilon$.

II. "On a New Method of Investigating the Magnetic Lines of Force in Magnets, demonstrating the Obliquity of the Equator and Axis of Bar Magnets." By RICHARD C. SHETTLE, M.D. Communicated by Dr. ROYSTON-PIGOTT, F.R.S. Received April 24, 1879.

It was not until some thousand observations had been made in the manner about to be described, that I was rewarded with the discovery of the obliquity of the bar magnetic equator, which is the subject of the present communication.

On a former occasion the Royal Society did me the honour of accepting a similar research on the magnetic condition of arterial and venous blood.

My present object is to detail the manner in which I have been led up to the result now obtained.

It appeared to me that the usual method of displaying magnetic currents by means of the curves assumed by steel filings were only rough approximations.

Long after I had been occupied with these observations I became aware, through the kindness of Professor Stokes, Secretary to the Royal Society, that Sir George Airy, the Astronomer Royal, had in a short paper in the "Transactions," investigated (with a pocket compass furnished with a magnetic inch-needle) these lines of force. The method which I have adopted is essentially different, and greatly exceeds in delicacy.

1st.—Inch magnetized steel needles were suspended and nicely balanced on a single silk fibre 6 inches long.

These needles carried over the magnets assumed rapidly changing positions, and readily demonstrated, as in the paper alluded to, the varying force and direction as usually portrayed in treatises on magnetism. But these long needles utterly failed in the niceties of research on which I was employed.

2nd.—Needles varying from half an inch to the 1–16th of an inch were mounted and suspended in a similar manner.

I found the latter thoroughly competent to trace sudden changes in