compounds has two values, and he comes to the conclusion that it has the value 3·35 where the oxygen is attached to a carbon atom by a double linking, but 2·76 in hydroxyl and where the oxygen is united to two other atoms.* This is deduced from experimental data: but there are other results which present difficulties. Thus the refraction of no substance is more certainly known than those of water, wood spirit, and alcohol. But the oxygen in H_2O (5·9) appears to have the higher number 3·3, notwithstanding its union to two atoms of hydrogen, while in CH_4O (13·1), C_2H_6O (20·8), as well as higher alcohols, and the diatomic ethene alcohol, $C_2H_4O_2$ (23·7), and the triatomic glycerol, $C_3H_8O_3$ (33·9), the oxygen is not 2·76, but 2·9 or 3·0, the numbers originally assigned to this element.

Nitrogen.—Nitrogen has two values, 4·1 and 5·1, or thereabouts.

The lower value, 4·1, is that originally deduced from cyanogen and metallic cyanides, and it seems to be generally confirmed by the observations on organic cyanides and nitriles. The higher value, 5·1, is deduced from all my observations on organic bases and amides, such as diethylamine (39·4), triethylamine (54·6), formamide (17·4), &c.

The determination of the value of nitrogen in nitro-substitution products presents some peculiar difficulties. The observations are not accordant. Even were the value of NO_2 obtained with certainty, it would not be easy to say how much should be attributed to the oxygen, especially when it is remembered that combination with oxygen alters very materially the refraction of the analogous elements, phosphorus and arsenic.

I hope shortly to submit to the public the data for these calculations, and in fact the whole of my recent observations on the refraction of organic compounds, together with a fuller discussion of the conclusions that may be drawn from them.

II. "On certain Definite Integrals." No. 8. By W. H. L. Russell, F.R.S. Received January 6, 1881.

I commence this paper with some general reflections on the theory of definite integrals. A definite integral may be written thus—

$$\int_\alpha^\beta dx\, f(a,\ b,\ c\ \ldots\ x) = \phi(a,\ b,\ c\ \ldots).$$

If we expand in terms of (a) and equate the coefficients of a^n we shall have

$$\int_\alpha^\beta dx\, f_1(n,\ b,\ c\ \ldots\ x) = \phi,\ (n,\ b,\ c\ \ldots).$$

* These have been calculated for line A.

And again expanding in terms of b, and equating coefficients of b^m, we shall have

$$\int_a^\beta dx f_2(n, m, c \ldots x) = \phi_2(n, m, c \ldots).$$

And thus we may proceed in general until we arrive at a simple definite integral containing only one arbitrary constant and the indices $n, m, \ldots$

Conversely we may obtain a complicated definite integral in many cases from a simple one, by multiplying it by constants raised to the powers of certain quantities contained as indices in the integral, assigning successive values to those indices, and then summing the resulting series. Thus the integral (123),

$$\int_0^{\frac{\pi}{2}} d\theta \cos\theta \cdot \frac{\mu \cos\theta(\beta + \cos\tan\theta) - \lambda \sin\theta \sin\tan\theta}{(1+2\beta\cos\tan\theta+\beta^2)(\lambda^2\sin^2\theta+\mu^2\cos^2\theta)},$$

was obtained from the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^{n-1}\theta d\theta \cos(c\tan\theta + (n-1)\theta)$$

by a process of double summation. These considerations show us why the method of summation is of such great importance in the evaluation of definite integrals.

I now hope to prove, as I stated in the last paper, that every function of an algebraical magnitude may be regarded as a centre from which systems of definite integrals emanate in all directions like rays from a star, in such a manner that the value of each integral is equivalent to the original function transformed by a known symbol.

Let $$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$$

Then

$$4 \cdot 3A_4 x^2 + 5 \cdot 4 \cdot A_5 x^3 + 6 \cdot 5 \cdot A_6 x^4 + \ldots = f''(x) - 2A_2 - 2 \cdot 3 \cdot A_3 x;$$

or since $$n(n-1) = \frac{2^{n+2}}{\pi}\int_0^{\frac{\pi}{2}} d\theta \cos^n\theta \cos(n-4)\theta,$$

we shall have

$$\frac{2^6 x^2}{\pi}\int_0^{\frac{\pi}{2}} d\theta\{A_4\cos^4\theta + A_5\cos^5\theta \cdot \cos\theta \cdot 2x + A_6\cos^5\theta\cos 2\theta(2x)^2 + \ldots\}$$
$$= f''(x) - 2A_2 - 6A_3 x;$$

or putting $x=\frac{1}{2}$, $$\frac{2^3}{\pi}\int_0^{\frac{\pi}{2}} d\theta\{A_4\cos^4\theta + A_5\cos^5\theta\epsilon^{i\theta} + A_6\cos^6\theta\epsilon^{2i\theta} + \ldots\}$$

$$+\frac{2^3}{\pi}\int_0^{\frac{\pi}{2}} d\theta\{A_4\cos^4\theta + A_5\cos^5\theta\epsilon^{-i\theta} + A_6\cos^6\theta\epsilon^{-2i\theta} + .\} = f''(\tfrac{1}{2}) - 2A_2 - 3A_3.$$

Hence we have

$$\frac{2^3}{\pi}\int_0^{\frac{\pi}{2}} d\theta\{\epsilon^{-4i\theta}f(\cos\theta\epsilon^{i\theta})+\epsilon^{4i\theta}f(\cos\theta\epsilon^{-i\theta})\}$$

$$=\frac{2^4}{\pi}\int_0^{\frac{\pi}{2}} d\theta\{A_0\cos 4\theta+A_1\cos\theta\cos 3\theta+A_2\cos^2\theta\cos 2\theta+A_3\cos^3\theta\cos\theta\}$$

$$+f''\tfrac{1}{2}-2A_2-3A_3=\frac{2^4}{\pi}\left(A_2\,.\,\frac{\pi}{2^3}+A_3\,.\,\frac{3\pi}{2^4}\right)+f''\tfrac{1}{2}-2A_2-3A_3.$$

Hence we shall have :—

$$\int_0^{\frac{\pi}{2}} d\theta\{\epsilon^{-4i\theta}f(\cos\theta\epsilon^{i\theta})+\epsilon^{4i\theta}f(\cos\theta\epsilon^{-i\theta})\}=\frac{\pi}{8}f''(\tfrac{1}{2})\quad . \quad . \quad (133).$$

This formula was obtained by differentiating $f(x)$ twice, but similar formulæ may be obtained by differentiating any number of times.

By analogous processes we may obtain likewise the following integrals :—

$$\int_0^{\frac{\pi}{2}} d\theta\{\epsilon^{-4i\theta}f(\cos^{\frac{1}{2}}\theta\epsilon^{\frac{i\theta}{2}})+\epsilon^{4i\theta}f(\cos^{\frac{1}{2}}\theta\epsilon^{-\frac{i\theta}{2}})\}=\frac{\pi}{8}\left\{\frac{1}{2}f''\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}f'\frac{1}{\sqrt{2}}\right\}$$

$$. \quad . \quad . \quad (134).$$

This integral requires the evaluation of

$$\int_0^{\frac{\pi}{2}} \cos^n\theta\cos\beta\theta d\theta$$

when β is greater than n, and consequently the usual formula does not apply. We may, however, proceed thus; since

$$\int d\theta\cos^n\theta\cos\beta\theta=\tfrac{3}{2}\int d\theta\cos^{n+1}\theta\cos(\beta-1)\theta-\tfrac{1}{2}\int d\theta\cos^{n+1}\theta\cos(\beta-3)\theta$$
$$-\int d\theta\cos^n\theta\cos(\beta-2)\theta+\int d\theta\cos^{n+2}\theta\cos(\beta-2)\theta\quad . \quad . \quad . \quad . \quad (135),$$

we are able by successive reductions to reduce the required integral to known forms.

$$\int_0^{\frac{\pi}{2}} d\theta\log_\epsilon\cos\theta\cos^2\theta\{\epsilon^{4i\left(\frac{\pi}{2}+\theta\right)}f\epsilon^{2i\left(\frac{\pi}{2}+\theta\right)}+\epsilon^{-4i\left(\frac{\pi}{2}+\theta\right)}f\epsilon^{-2i\left(\frac{\pi}{2}+\theta\right)}\}$$

$$=\frac{\pi}{4}\phi(1)\text{ where }\phi(x)=\iiint f(x)dx^3\quad . \quad . \quad . \quad . \quad (136).$$

$$\int_0^{\pi} d\theta\,\Theta\sin\frac{\theta}{2}(\epsilon^{\frac{3}{2}i}f\epsilon^{\theta i}-\epsilon^{-\frac{3\theta i}{2}}f\epsilon^{-\theta i})=i\pi\phi(1),\text{ where }\phi(x)=\iiint f(x)dx^3,$$

and Θ is the quantity defined in the fifth paper of this series . (137).

$$\int_0^{\frac{\pi}{2}} d\theta \cos^2\theta \{ f\epsilon^{\cos^3\theta \epsilon^{3i\theta}} + f\epsilon^{\cos^3\theta\epsilon^{-3i\theta}} \} = \frac{\pi}{4}\left\{ 2f\epsilon^{\frac{1}{8}} + \frac{3\epsilon^{\frac{1}{8}}}{8} f'\epsilon^{\frac{1}{8}} \right\} \quad . \quad . \quad . \quad (138).$$

$$\int_0^{\frac{\pi}{2}} d\theta \cos^2\theta \{ f\epsilon^{\cos\theta\epsilon^{2\theta}} + f\epsilon^{\cos\theta\epsilon^{-2\theta}} \} = \frac{\pi}{2}\left\{ f\epsilon^{\frac{1}{8}} + \frac{\epsilon^{\frac{1}{8}}}{4} f'\epsilon^{\frac{1}{8}} \right\} \quad . \quad . \quad . \quad . \quad . \quad (139).$$

$$\int_0^{\infty} dx \frac{\epsilon^{2irx} f\left(2\sin x\epsilon^{i\left(\frac{\pi}{2}+x\right)}\right) + \epsilon^{-2irx} f\left(2\sin x\epsilon^{-i\left(\frac{\pi}{2}+x\right)}\right)}{(a^2+x^2)(b^2+x^2)(c^2+x^2)\ldots(e^2+x^2)}$$

$$= \frac{\pi}{a(a^2-b^2)(a^2-c^2)\ldots(a^2-e^2)} f(\epsilon^{-2a}-1)\,.\,\epsilon^{-2ar}$$

$$+ \frac{\pi}{b(b^2-a^2)(b^2-c^2)\ldots(b^2-e^2)} f(\epsilon^{-2b}-1)\,.\,\epsilon^{-2br} + \ldots \quad . \quad . \quad (140).$$

Similarly we may find

$$\int_0^{\infty} x dx \frac{\epsilon^{2irx} f\left(2\sin x\,\epsilon^{i\left(\frac{\pi}{2}+x\right)}\right) - \epsilon^{-2irx} f\left(2\sin x\epsilon^{-i\left(\frac{\pi}{2}+x\right)}\right)}{(a^2+x^2)(b^2+x^2)(c^2+x^2)\ldots(e^2+x^2)} \quad . \quad . \quad . \quad . \quad (141).$$

$$\int_0^{\pi} d\theta \frac{\epsilon^{ir\theta} f\left(2\sin\frac{\theta}{2}\epsilon^{i\left(\frac{\pi+\theta}{2}\right)}\right) + \epsilon^{-ir\theta} f\left(2\sin\frac{\theta}{2}\epsilon^{-i\left(\frac{\pi+\theta}{2}\right)}\right)}{1-2\alpha\cos\theta+\alpha^2} = \frac{2\pi f(\alpha-1)\,.\,\alpha^r}{1-\alpha^2} \quad (142).$$

Similarly we may obtain :—

$$\int_0^{\pi} d\theta \sin\theta \frac{\epsilon^{ir\theta} f\left(2\sin\frac{\theta}{2}\epsilon^{i\left(\frac{\pi+\theta}{2}\right)}\right) - \epsilon^{-ir\theta} f\left(2\sin\frac{\theta}{2}\epsilon^{-i\left(\frac{\pi+\theta}{2}\right)}\right)}{1-2\alpha\cos\theta+\alpha^2} \quad . \quad . \quad . \quad . \quad (143).$$

$$\int_0^{\infty} dx \frac{\epsilon^{2irax} f\left(2\sin ax\epsilon^{i\left(\frac{\pi}{2}+ax\right)}\right) + \epsilon^{-2irax} f\left(2\sin ax\epsilon^{-i\left(\frac{\pi}{2}ax\right)}\right)}{(\alpha^2+x^2)(\beta^2+x^2)(\gamma^2+x^2)\ldots(\lambda^2+x^2)\,.\,\cos ax} \quad . \quad . \quad . \quad (144).$$

$$\int_0^{\infty} x dx\,.\,\frac{\epsilon^{2irax} f\left(2\sin ax\epsilon^{i\left(\frac{\pi}{2}+ax\right)}\right) + \epsilon^{-2irax} f\left(2\sin ax\epsilon^{-i\left(\frac{\pi}{2}ax\right)}\right)}{(\alpha^2+x^2)(\beta^2+x^2)(\gamma^2+x^2)\ldots(\lambda^2+x^2)\,.\,\sin ax} \quad . \quad . \quad . \quad (145).$$

$$\int_0^{\frac{\pi}{2}} d\theta\,.\,\frac{f(\cos\theta\epsilon^{i\theta}) + f(\cos\theta\epsilon^{-i\theta})}{x^2\cos^2\theta+\alpha^2\sin^2\theta} = \frac{\pi}{ax} f\frac{x}{x+a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146).$$

$$\int_0^{\infty} dx \frac{f(\epsilon^{2x}) - f(\epsilon^{-ix})}{x} = \pi i(f(1) - f(0)) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (147).$$

These formulæ may be greatly extended. I add a few examples of their application to particular cases :—

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\epsilon^{\cos^2\theta\cos2\theta}\cos(\cos^2\theta\sin2\theta)}{x^2\cos^2\theta+\alpha^2\sin^2\theta} = \frac{\pi}{2\alpha x}\epsilon^{\frac{x^2}{(x+\alpha)^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148).$$

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\log_\epsilon (1+2\alpha\cos^2\theta\cos 2\theta+\alpha^2\cos^4\theta)}{x^2\cos^2\theta+a^2\sin^2\theta}=\frac{\pi}{ax}\log_\epsilon\frac{(x+\alpha)^2+\alpha x^2}{(x+a)^2} \quad (149).$$

$$\int_0^\infty \frac{d\theta}{\theta}\cdot\frac{\sin\theta}{1+2\alpha\cos\theta+\alpha^2}=\frac{\pi}{2}\cdot\frac{1}{1+\alpha} \quad \ldots\ldots\ldots \quad (150).$$

$$\int_0^\infty d\theta\cdot\frac{\epsilon^{\alpha\cos\theta}\sin(\alpha\sin\theta)}{\theta}=\frac{\pi}{2}(\epsilon^\alpha-1) \quad \ldots\ldots\ldots \quad (151).$$

$$\int_0^\infty \frac{d\theta}{\theta}\sqrt{\sqrt{(1+2\alpha\cos\theta+\alpha^2)}-(1+\alpha\cos\theta)}=\frac{\pi}{\sqrt{2}}(\sqrt{1+\alpha}-1) \quad . \quad (152).$$

In my last paper I gave the integral

$$\int_0^\pi d\theta\frac{\sin r\theta}{1-2\alpha\cos\theta+\alpha^2} \quad \ldots\ldots \quad (132).$$

It is obvious that by a similar process we can find

$$\int_0^\pi d\theta\frac{\cos r\theta}{1-2\alpha\sin\theta+\alpha^2} \quad . \quad (153). \qquad \int_0^\pi d\theta\frac{\sin r\theta}{1-2\alpha\sin\theta+\alpha^2} \quad . \quad (154),$$

remembering that $1-2\alpha\cos\left(\frac{\pi}{2}+\theta\right)+\alpha^2=1+2\alpha\sin\theta+\alpha^2$.

Also since

$$\int\frac{d\theta\cos(r+1)\theta}{(a+b\cos\theta)^n}=\frac{2}{b}\int\frac{d\theta\cos r\theta}{(a+b\cos\theta)^{n-1}}-\frac{2a}{b}\int\frac{d\theta\cos r\theta}{(a+b\cos\theta)^n}-\int\frac{d\theta\cos(r-1)\theta}{(a+b\cos\theta)^n} \quad \ldots \quad (155),$$

$$\int\frac{d\theta\sin(r+1)\theta}{(a+b\cos\theta)^n}=\frac{2}{b}\int\frac{d\theta\sin r\theta}{(a+b\cos\theta)^{n-1}}-\frac{2a}{b}\int\frac{d\theta\,.\sin r\theta}{(a+b\cos\theta)^n}-\int\frac{d\theta\sin(r-1)\theta}{(a+b\cos\theta)^n} \quad \ldots \quad (156),$$

$$\int\frac{d\theta\sin(r+1)\theta}{(a+b\sin\theta)^n}=\frac{2}{b}\int\frac{d\theta\cos r\theta}{(a+b\sin\theta)^{n-1}}-\frac{2a}{b}\int\frac{d\theta\,.\cos r\theta}{(a+b\sin\theta)^n}+\int\frac{d\theta\sin(r-1)\theta}{(a+b\sin\theta)^n} \quad \ldots \quad (157),$$

$$\int\frac{d\theta\cos(r+1)\theta}{(a+b\sin\theta)^n}=\frac{2a}{b}\int\frac{d\theta\sin r\theta}{(a+b\sin\theta)^n}-\frac{2}{b}\int\frac{d\theta\sin r\theta}{(a+b\sin\theta)^{n-1}}+\int\frac{d\theta\cos(r-1)\theta}{(a+b\sin\theta)^n} \quad \ldots \quad (158),$$

it is manifest that:—

$$\int_0^\pi d\theta\frac{\cos r\theta}{(1-2\alpha\cos\theta+\alpha^2)^n} \quad . \quad (159), \qquad \int_0^\pi d\theta\frac{\sin r\theta}{(1-2\alpha\cos\theta+\alpha^2)^n} \quad . \quad (160),$$

$$\int_0^\pi d\theta \,.\, \frac{\sin r\theta}{(1-2\alpha\sin\theta+\alpha^2)^n} \;.\; (161), \qquad \int_0^\pi d\theta \frac{\cos r\theta}{(1-2\alpha\sin\theta+\alpha^2)^n} \;.\; (162),$$

may be reduced to integrals 132, 153, 154, and other known forms, and that consequently (resolving into partial fractions)

$$\int_0^\pi d\theta \frac{\cos r\theta}{(1-2a\cos\theta+\alpha^2)^m(1-2\beta\cos\theta+\beta^2)^n\dots(1-2\lambda\cos\theta+\lambda^2)^r} \; (163),$$

$$\int_0^\pi d\theta \frac{\sin r\theta}{(1-2\alpha\cos\theta+\alpha^2)^m(1-2\beta\cos\theta+\beta^2)^n\dots(1-2\lambda\cos\theta+\lambda^2)^r} \; (164),$$

$$\int_0^\pi d\theta \frac{\cos r\theta}{(1-2\alpha\sin\theta+\alpha^2)^m(1-2\beta\sin\theta+\beta^2)^n\dots(1-2\lambda\sin\theta+\lambda^2)^r} \; (165),$$

$$\int_0^\pi d\theta \,.\, \frac{\sin r\theta}{(1-2\alpha\sin\theta+\alpha^2)^m(1-2\beta\sin\theta+\beta^2)^n\dots(1-2\lambda\sin\theta+\lambda^2)^r} \; (166),$$

may be ascertained. We may also find

$$\int_0^\pi \log_\epsilon{}^2\left(2\cos\frac{\theta}{2}\right)d\theta = \frac{\pi^2}{12} \quad . \; . \; . \; . \; . \quad (167).$$

$$\int_1^\infty \frac{dx}{x^2}\left(\log_\epsilon\frac{x}{\epsilon}\right)^3\left(\log_\epsilon x\right)^n = (n^4+3n^3+5n^2+2n)\Gamma(n) \quad . \quad (168).$$

If we expand the denominator of the integral

$$\int_0^\pi d\theta \frac{\cos\rho\theta}{1-2\alpha\cos\theta+\alpha^2} \text{ where } \rho=\frac{m}{n} \quad . \; . \; . \; . \quad (169),$$

and integrate the terms in succession, we shall have to determine the integral series of the form

$$\frac{\alpha}{n+m}+\frac{\alpha^2}{2n+m}+\frac{\alpha^3}{3n+m}+\dots$$

which may always be found, when the values of (m) and (n) are assigned, from the expanded form of $\log_\epsilon(1+x)$ by the method of summation of the equidistant terms of series. Similar reasoning will apply to

$$\int_0^\pi d\theta \frac{\sin\rho\theta}{1-2\alpha\cos\theta+\alpha^2} \quad . \quad (170). \qquad \int_0^\pi d\theta \,.\, \frac{\cos\rho\theta}{1-2\alpha\sin\theta+a^2} \quad . \quad (171).$$

$$\int_0^\pi d\theta \frac{\sin\rho\theta}{1-2\alpha\sin\theta+\alpha^2} \quad . \; . \; . \; . \; . \; . \; . \quad (172).$$

This method of summing the equidistant terms of series may be applied to the determination of the values of other integrals, as for instance

$$\int_0^\pi \frac{d\theta\,\theta \sin\theta \cos^{2r}\theta}{1-k\cos^8\theta} \quad . \ (173), \qquad \int_0^\pi d\theta \log_\epsilon \cos\theta \frac{\cos 2\theta + \alpha\cos 6\theta}{1-2\alpha\cos 8\theta+\alpha^2} \quad . \ (174),$$

$$\int_0^1 \frac{dx\,.\,x^m}{1-kx^n} \quad . \quad . \quad (175), \text{ with many others.}$$

(Received February 4, 1881.)

I have received permission to write down formula (132) thus amended:—

$$\int_0^\pi \frac{\sin r\theta d\theta}{1-2\alpha\cos\theta+\alpha^2} = \frac{1}{1-\alpha^2}\left(\alpha^r - \frac{1}{\alpha^r}\right)\log_\epsilon \frac{1-\alpha}{1+\alpha}$$

$$+\frac{2}{1-\alpha^2}\left\{\left(\alpha^{r-1}-\frac{1}{\alpha^{r-1}}\right)+\tfrac{1}{3}\left(\alpha^{r-3}-\frac{1}{\alpha^{r-3}}\right)+\ \ldots \frac{1}{r-1}\left(a-\frac{1}{\alpha}\right)\right\}.$$

III. "*Polacanthus Foxii*, a large undescribed Dinosaur from the Wealden Formation in the Isle of Wight." By J. W. Hulke, F.R.S. Received January 3, 1881.

(Abstract.)

A description of the remains of a large Dinosaur, discovered in 1865 by the Rev. W. Fox, in a bed of shaly clay between Barnes and Cowleaze Chines, in the Isle of Wight. Head, neck, shoulder-girdle, and foreribs were missing, but the rest of the skeleton was almost entire. Some of the præsacral vertebræ recovered show a double costal articulation. In the trunk and loins the centrum is cylindroid, relatively long and slender, with plano-concave, or gently biconcave ends. Several lumbar centra are ankylosed together, and the hindmost to the sacrum. The sacrum comprises five relatively stout and short ankylosed centra of a depressed cordiform cross-sectional figure. The front sacral vertebræ have a stout short centrum.

The limb bones are short, their shafts slender, and their articular ends very expanded. The femur has a third trochanter, and the distal end of the tibia has the characteristic dinosaurian figure.

The back and flanks were stoutly mailed with simple, keeled, and spined scutes, and the tail was also sheathed in armour.

The animal indicated by these remains was of low stature, great strength, and probably slow habits. It is manifestly a Dinosaur, and is considered to be very nearly related to *Hylæosaurus*.